

# Mapping Class Groups

## MSRI, Fall 2007

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**Reducible mapping classes** Review terminology:

- An *essential curve*  $\gamma$  on  $S$  is a simple closed curve  $\gamma$  such that:
  - no component of  $S - \gamma$  is a disc with 0 or 1 puncture
- An *essential curve system*  $\Gamma$  on  $S$  is a pairwise disjoint union of essential curves.
  - Components of  $\Gamma$  are allowed to be isotopic.
  - $\Gamma$  is *pairwise nonisotopic* if no two components are isotopic.
- An *essential surface* in  $S$  is a subsurface-with-boundary  $F \subset S$  such that:
  - $F$  is a closed subset
  - $\partial F$  is an essential curve system
  - No two components of  $F$  are isotopic.

It is possible for an annulus component of  $F$  to be isotopic into one or two other components.

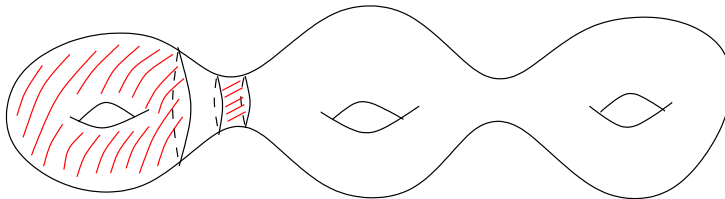


Figure 1: This is an example of an essential surface. The boundary of the subsurface may contain isotopic curves, but no two components of the subsurface are isotopic.

- $\mathcal{MCG}(S)$  acts on isotopy classes of
  - essential curves;
  - essential curve systems;
  - essential subsurfaces
- $\phi \in \mathcal{MCG}(S)$  is *reducible* if there exists a (pairwise nonisotopic) essential curve system  $\Gamma$  whose isotopy class is invariant under  $\phi$ .
  - Such a  $\Gamma$  is a *reduction system* for  $\phi$ .
  - Can choose a representative  $\Phi$  of  $\phi$  such that  $\Phi(\Gamma) = \Gamma$ .
  - Can furthermore choose  $\Phi$  so that  $\Phi(N(\Gamma)) = N(\Gamma)$ .
- For each component  $F$  of  $\overline{S - N(\Gamma)}$  (We think of the interior of  $F$  as a punctured surface):
  - Least  $n \geq 1$  such that  $\Phi^n(F) = F$  is the *first return time* of  $F$ .
  - The mapping class of  $\Phi^n$  is a *component mapping class* of  $F$ .

### Well-definedness of component mapping classes

- Slight problem:
  - $\Gamma$  only well-defined up to isotopy,
  - $\overline{S - N(\Gamma)}$  only well-defined up to isotopy,
- Action of  $\phi$  on set of isotopy classes of components of  $\Gamma$  (and of  $N(\Gamma)$ ) is well-defined
- Action of  $\phi$  on set of isotopy classes of components of  $\overline{S - N(\Gamma)}$  is well-defined.
- First return times are well-defined.
- First return mapping classes are well-defined up to “conjugacy-by-isotopy”.

### Canonical reduction system

- Reduction systems for  $\phi$  need not be unique (up to isotopy):
  - A component mapping class might be reducible  $\implies$  the reduction system can be enlarged.
  - A reduction system might have more than one orbit under action of  $\phi$   $\implies$  the reduction system can be shrunk (see figure).

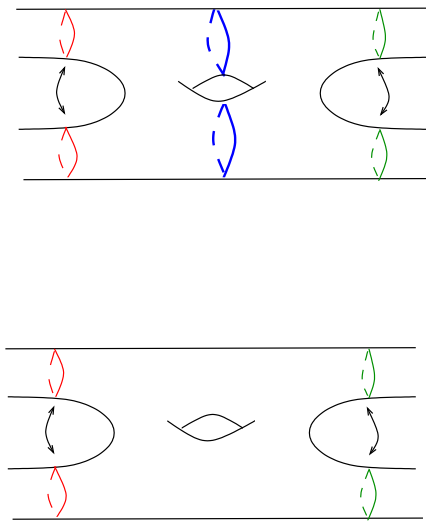


Figure 2: If on the depicted surface the map is a hyperelliptic involution then we can remove the two thick blue curves to get a smaller reducing system.

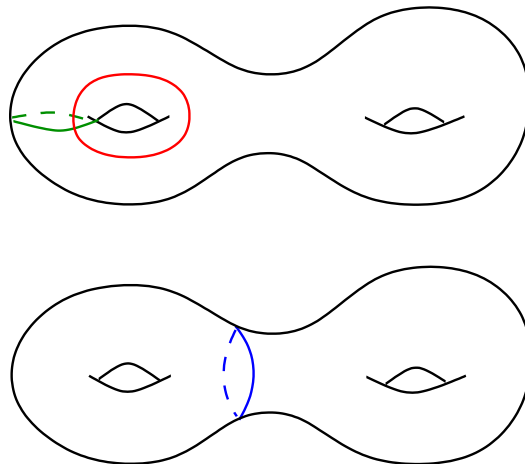


Figure 3: If  $\gamma_1 \in \Gamma$  intersects  $\gamma'_1 \in \Gamma'$  then can find a curve  $\beta_1 = \partial N(\gamma_1 \cup \gamma'_1)$  which is disjoint from both and is still a reducing curve, and the automorphism restricted to  $N(\gamma_1 \cup \gamma_2)$  is periodic. Can continue this way to cut up the surface into periodic/pseudo-Anosov pieces.

**Theorem 1.** *For each  $\phi \in \mathcal{MCG}(S)$  there is a reduction system  $\Gamma$ , possibly empty, which is uniquely characterized up to isotopy by the following:*

1. *For each reduction system  $\Gamma'$  for  $\phi$ , each  $\gamma \in \Gamma$ , and each  $\gamma' \in \Gamma'$ ,  $\langle \gamma, \gamma' \rangle = 0$*
2.  *$\Gamma$  is maximal with respect to previous property.*

$\Gamma$  is also uniquely characterized up to isotopy by the following:

3. *Each component mapping class is either finite order or irreducible - in fact, pseudo-Anosov.*
4.  *$\Gamma$  is minimal with respect to the previous property.*

Corollary:  $\Gamma \neq \emptyset$  if and only if  $\phi$  has infinite order and is not pseudo-Anosov.

### Remarks

- Original source of (3) and (4) is hard to pin down... appears in print in more than one place... probably first known to Thurston.
- (1) and (2) is in paper of Handel–Thurston.
- (1) and (2) is an analogue of the JSJ decomposition of a 3-manifold.
- The analogy is (no coincidence?) very strong: the canonical JSJ decomposition of the mapping torus of  $\phi$  corresponds exactly to the canonical reduction system of  $\phi$ .

### Reducible conjugacy invariants: The reduction graph

Let  $\phi \in \mathcal{MCG}(S)$  be reducible and of infinite order. Define:

$\Gamma_\phi = \{\gamma_i \mid i = 1, \dots, I\}$  = canonical reduction system.

$\{F_j \mid j = 1, \dots, J\}$  = components of  $S - \Gamma_\phi$ .

$\mathcal{G}_\phi$  = reduction graph:

- Vertex  $V_j$  for each component  $F_j$  of  $\overline{S - N(\Gamma)}$ ,
  - labelled with the integer genus( $F_j$ ) and the number of punctures of  $S$  in  $F_j$ .
  - also need to keep track of  $|\partial F_j|$ , but this is equal to the valence of  $V_j$  in the reduction graph.
- Edge  $E_i$  for each component  $\gamma_i$  of  $\Gamma$

Let  $\text{Stab}(\Gamma) = \{\theta \in \mathcal{MCG} \mid \theta(\Gamma) = \Gamma\}$ .

**Lemma 2.** *The labelled isomorphism type of  $\mathcal{G}_\phi$  is a conjugacy invariant of  $\phi$ .  $\diamond$*

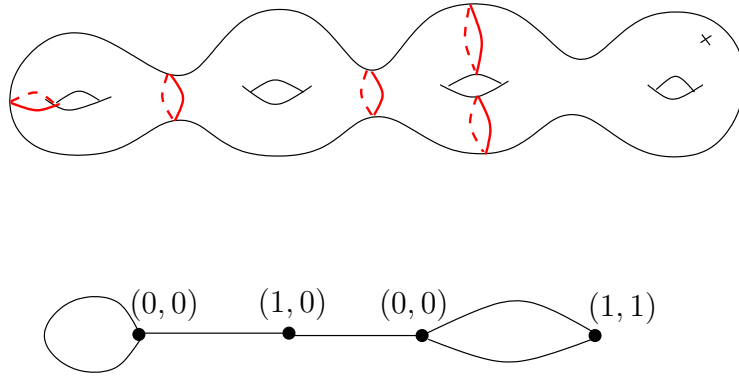


Figure 4: To a reduction system we associate a labeled graph with labels (genus,punctures).

Consider two reducible mapping classes  $\phi, \phi' \in \mathcal{MCG}(S)$ . In order to describe the next level of conjugacy invariants, let us *assume*  $\mathcal{G}_\phi, \mathcal{G}_{\phi'}$  are isomorphic as labelled graphs.

- Choose an isomorphism  $\mathcal{G}_\phi \mapsto \mathcal{G}_{\phi'}$ .
- Lift it to  $\psi \in \mathcal{MCG}(S)$  taking  $\Gamma_\phi$  to  $\Gamma_{\phi'}$  (by the classification of surfaces)
- Both  $\phi$  and  $\phi'' = \psi^{-1}\phi'\psi$  are in the subgroup

$$\text{Stab}(\Gamma_\phi) < \mathcal{MCG}(S)$$

- TFAE:
  1.  $\phi, \phi'$  are conjugate in  $\mathcal{MCG}(S)$
  2.  $\phi, \phi''$  are conjugate in  $\mathcal{MCG}(S)$
  3.  $\phi, \phi''$  are conjugate in  $\text{Stab}(\Gamma_\phi)$
- Proof of 2  $\implies$  3: Recall that  $\Gamma_{\phi''} = \Gamma_\phi$ . If  $\theta\phi\theta^{-1} = \phi''$  then  $\theta\Gamma_\phi = \Gamma_{\phi''} = \Gamma_\phi$  so  $\theta \in \text{Stab}(\Gamma_\phi)$ .

### Structure of $\text{Stab}(\Gamma)$

Given essential curve system  $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ , let  $F_1, \dots, F_k$  be components of  $S - N(\Gamma)$ .

1.  $\text{Stab}(\Gamma)$  acts on  $\mathcal{G}_\phi$ . Let  $\text{Stab}_0(\Gamma)$  be the kernel:

$$1 \rightarrow \text{Stab}_0(\Gamma) \rightarrow \text{Stab}(\Gamma) \rightarrow (\text{finite group}) \rightarrow 1$$

2.  $\text{Stab}_0(\Gamma)$  acts (up to isotopy) on each  $F_i$ ; let  $T(\Gamma)$  be the intersection of the kernels of these actions:

$$1 \rightarrow T(\Gamma) \rightarrow \text{Stab}_0(\Gamma) \rightarrow \prod_{i=1}^k \mathcal{MCG}_0(F_i) \rightarrow 1$$

where  $\mathcal{MCG}_0$  means punctures are fixed.

3.  $T(\Gamma)$  is a free abelian group having as basis the Dehn twists

$$\tau_{\gamma_1}, \dots, \tau_{\gamma_m}$$

Each of items 1,2,3 yields reducibility invariants, and we will go through them one-by-one.

### 1. Action on $\mathcal{G}_\phi$ .

- $\phi$  has well-defined action on  $\mathcal{G}_\phi$ , preserving genus and puncture labels.
- Follows from:
  - uniqueness of  $\Gamma$ ,
  - well-definedness up to isotopy of action on  $\Gamma$
  - and on  $\overline{S - N(\Gamma)}$ .

**Fact.** *The conjugacy type of the  $\phi$  action on  $\mathcal{G}_\phi$  is a conjugacy invariant.* ◇

### 2. Action on components $F_i$ of $S - N(\Gamma_\phi)$ .

- For each vertex  $V_F$ , augment its label with the conjugacy invariants of the first return mapping class of  $F$ .

**Fact.**  $\mathcal{G}_\phi$ , with genus labels, action, and augmented labels, is a conjugacy invariant of  $\phi$ .

BUT, there is still hidden conjugacy information in the action on the components  
...

### Pairing punctures

- Each  $\gamma \in \Gamma$  corresponds to TWO punctures of  $S - N(\Gamma)$ .

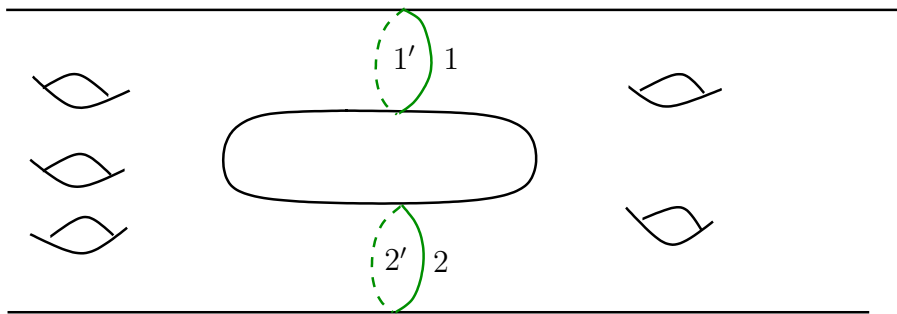


Figure 5: So far, the conjugacy invariants will not distinguish between two mapping classes which are conjugate on each subsurface separately (but not as homeomorphisms of the whole surface). Pairing the punctures gets rid of this discrepancy.

- Each puncture has a certain “identity” in the conjugacy invariants for the first return mapping classes.
- The pairing of punctures, or more strictly speaking the pairing of their “identities” in the conjugacy invariants of first return mapping classes, is itself a conjugacy invariant.
- Formally:
  - Let  $D$  be the collection of unordered pairs of punctures of  $S - N(\Gamma)$ , one pair for each  $\gamma \in \Gamma$ .
  - All pseudo-Anosov and finite order conjugacy classes, on the components of  $S - N(\gamma)$ , need to be lifted to conjugacy invariants in the group  $\mathcal{MCG}(S - N(\gamma), D)$ .
- Extra finite amount of bookkeeping: add finite amount of data to pseudo-Anosov and to finite order conjugacy invariants . . .
- Call this the “puncture gluing” invariants.

**Fact.**  $\mathcal{G}_\phi$ , with previous action and labels, and with added information of puncture gluing invariants, is a conjugacy invariant.

**3. Twist invariants** So far we can’t distinguish a Dehn twist from its square, up to conjugacy.

- To each  $\gamma \in \Gamma$  we shall associate a *twist invariant*, a rational number

$$r(\gamma) \in \mathbf{Q}$$

- For example, given a Dehn twist power  $\tau_\gamma^k$  we will have  $r(\gamma) = k$ .

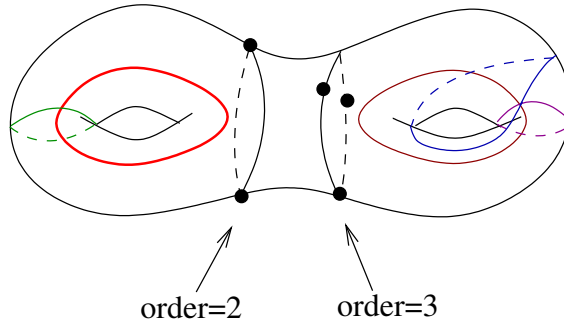


Figure 6: The order of the mapping class on the left is 2, and the boundary circle is rotated by  $1/2$ . The order of the mapping class on the right is 3, and the boundary circle is rotated by  $1/3$ . This yields a twist invariant of the form  $n + \frac{1}{6}$  in the middle curve, for some integer  $n$ .

- Note:  $\partial \left( \overline{S - N(\gamma)} \right) = \partial \bigcup_{j=1}^J F_j$
- On each component  $c$  of each  $F_j$  there is a natural periodic set:
  - All of  $c$  (if  $F_j$  is finite order)
  - Endpoints of stable and unstable leaves (if  $F_j$  is pseudo-Anosov)
- Choose product structure  $N(\gamma_i) = S^1 \times [0, 1]$
- Require action of  $\Phi$  on  $S^1 \times 0$  and  $S^1 \times 1$  to be either:
  - Rigid rotation (on finite order boundary)
  - Alternating source-sink with evenly spaced periodic points (on pseudo-Anosov boundary)
- Lift (first return of)  $\Phi$  to universal covering map

$$\mathbf{R} \times [0, 1] \quad \text{of} \quad S^1 \times [0, 1] = N(\gamma_i)$$

- Define the twist to be  $r(\gamma) = \frac{1}{\text{slope of } \tilde{\Phi}(0 \times [0, 1])}$

**Theorem 3.** *The following data gives a complete conjugacy invariant of a reducible  $\phi \in \mathcal{MCG}(S)$ :*

- *Reduction graph  $\mathcal{G}_\phi$*
- *Label for each vertex: genus and number of punctures*
- *Label for each vertex: conjugacy invariant of first return*

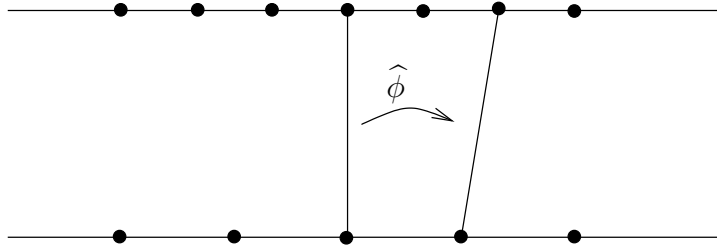


Figure 7: To find the twist parameter - lift to the universal cover and find the reciprocal of the slope of the image of a vertical line.

- *Label for each edge: puncture pairing data*
- *Label for each edge: twist invariant*
- *Action of  $\phi$  on  $\mathcal{G}_\phi$*

### Computing the conjugacy invariant

- Bestvina-Handel paper gives an algorithm for finding either:
  - Invariant train track
  - Reducing system
- Even in the finite order case, the algorithm will produce an invariant graph.
- When the algorithm finds a reducing system  $\Gamma$ , *don't stop*:
  - Obtain a graph having the reducing system  $\Gamma$  as a subgraph.
  - Continue the algorithm, relative to  $\Gamma$ .
- Continue inductively through smaller and smaller subsurfaces
- In the end, on each subsurface, have either train track or invariant graph.
- Pseudo-Anosov invariants are computable from train track
- Finite order invariants are computable from invariant graphs
- Puncture pairing data: evident from incidence of reducing curves, train tracks, invariant graphs.
- Also have: Extra branch coming off the side of each reducing curve.
- Twist: can compute using extra branches.