Mapping Class Groups
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Lectures and slides by Lee Mosher
Additional notes and diagrams by Yael Algom Kfir

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Subgroups of mapping class groups

Here are three theorems about subgroups of mapping class groups. They are due independently to Ivanov and to a combination of works of McCarthy and of Birman/McCarthy/Lubotzky.

Let S be a finite type surface and let G be any subgroup of $\mathcal{MCG}(S)$.

Theorem 1 (The Tits Alternative for $\mathcal{MCG}$)**.** *One of the following holds:*

- *G contains a free subgroup of rank ≥ 2.*
- *G contains a finitely generated abelian subgroup of finite index.*

(The following was stated incorrectly in the lecture)

Theorem 2 (Classification of abelian subgroups)**.** *If G is abelian then there exists:*

- $\Phi_1, \ldots, \Phi_K \in \mathrm{Homeo}_+(S)$*, representing* $\phi_1, \ldots, \phi_K \in \mathcal{MCG}(S)$*,*
- *an essential subsurface $F \subset S$ with components $F = F_1 \cup \cdots \cup F_K$*

such that for all $k = 1, \ldots, K$:

- $\Phi_k \,\big|\, S - F_k = Id$
- *If F_k is an annulus then $\Phi_k \,\big|\, F_k$ is a nonzero power of a Dehn twist*
- *If F_k is not an annulus then $\Phi_k \,\big|\, F_k$ is pseudo-Anosov.*

and G has a finite index subgroup contained in the free abelian group $\langle\phi_1\rangle \oplus \ldots \oplus \langle\phi_K\rangle$.

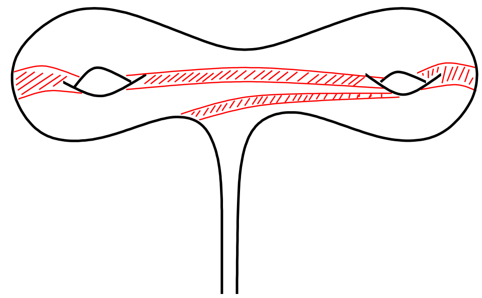

Figure 1: The maximum number of components of an essential suburface is $3g-3+p$

Corollary. *The rank of any free abelian subgroup of $\mathcal{MCG}(S)$ is at most*

$$3 \cdot genus(S) - 3 + \#punctures(S)$$

which is the maximum number of components of an essential subsurface F of S.

When is the maximum rank $3g-3+p$ acheived?

- F = regular neighborhood of a pants decomposition.
- If an annulus A has two different 3-holed spheres on either side, can replace A by a 4-holed sphere (see figure)
- If an annulus A has the same 3-holed sphere on either side, can replace A by a 1-holed torus (see figure).
- In general, the maximum is acheived if and only if F has the following structure:
  - Each component of F is an annulus, a one-holed torus, or a 4-holed sphere
  - F has an annulus component parallel to each boundary curve of a one-holed torus or 4-holed sphere component
  - Each component of $S-F$ is an annulus or 3-holed sphere.

Recall the classification of elements of $\mathcal{MCG}(S)$: every infinite order, irreducible element is pseudo-Anosov.

Theorem 3 (Subgroup Trichotomy)**.** *For any subgroup $G < \mathcal{MCG}(S)$, one of the following holds:*

- *G is finite*
- *There exists an essential curve system $\mathcal{C}$ which is a reducing system for each element of G (by definition, G is* reducible*).*
- *G contains a pseudo-Anosov homeomorphism.*

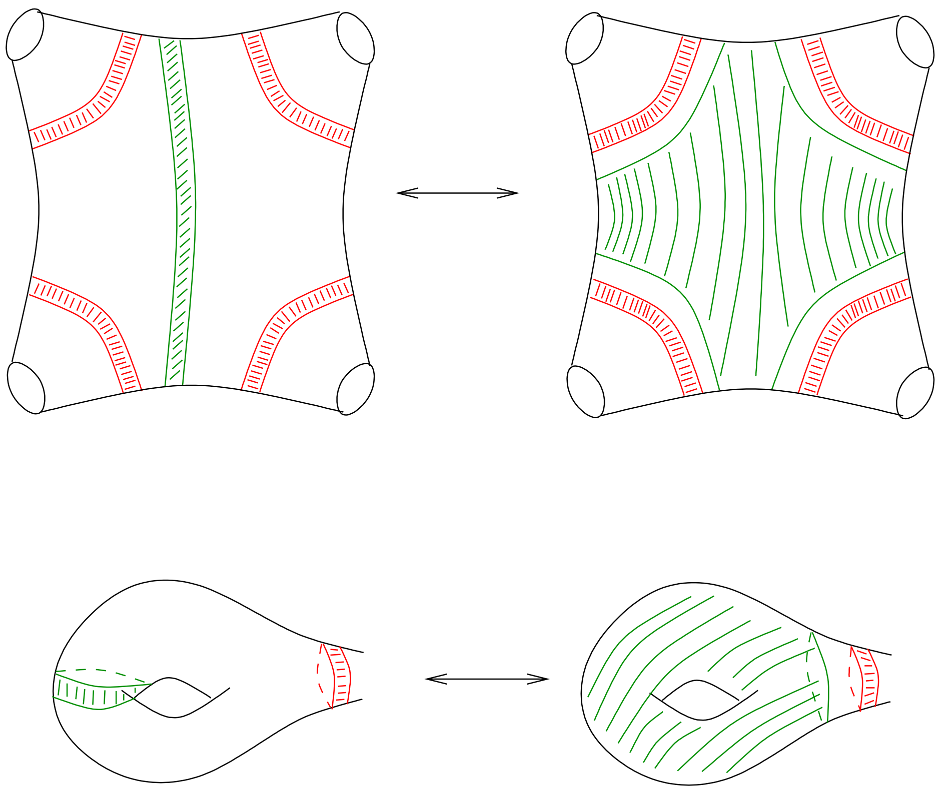

Figure 2: An essential subsurface $F \subset S$ has the maximum number of components when we cannot enlarge it by adding a disjoint component $F' \subset S \setminus F$ (hence all complementary components are 3-holed spheres or annuli) and F cannot be enlarged by replacing a component by a subsurface consisting of more components (so the components of F are annuli, 1-holed tori and 4-holed spheres, with an annulus parallel to each boundary circle of each 1-holed torus or 4-holed sphere). The moves above show how one can replace such a maximal surface by one consisting only of annuli, without changing the number of components. Thus, the maximal rank is achieved by a surface consisting of disjoint annuli and the number of components of such a surface is $3g - 3 + p$

In other words: every infinite, irreducible subgroup contains a pseudo-Anosov element.

Remark. This theorem has an important application in the study by Bestvina-Fujiwara of the 2nd bounded cohomology of a subgroup G of $\mathcal{MCG}(S)$: either G is virtually abelian or $H^2_b(G; \mathbf{Z})$ has uncountable dimension.

The proofs of these three theorems can be found among Ivanov's book, McCarthy's thesis (published version), and the paper of Birman/Lubotzky/McCarthy. The techniques for these three theorems are broadly the same, but different in detail.

Goal: Unify the proofs with the "Omnibus Subgroup Theorem" (joint w/ Michael Handel) which has the above three theorems as quick corollaries.

Motivating the statement of Omnibus Subgroup Theorem:

See how far we can go in proving the Subgroup Trichotomy Theorem without knowing anything.

Suppose that $G < \mathcal{MCG}(S)$ is *not* finite order and does *not* have a simultaneous reducing system. Want to produce a pseudo-Anosov element in G. G must have an infinite order element, since it is not finite and $\mathcal{MCG}(S)$ is virtually torsion free.

Example: Suppose that $\phi \in G$ is a product of Dehn twists about curves in a pants decomposition $\mathcal{C}$.

There exists $\psi \in G$ such that $\psi(\mathcal{C}) \neq \mathcal{C}$ (because G is irreducible).

Let $\phi' = \psi\phi\psi^{-1}$, a product of Dehn twists about the pants decomposition

$$\mathcal{C}' = \psi(\mathcal{C}) \neq \mathcal{C}$$

Since $\mathcal{C}$ and $\mathcal{C}'$ are nonisotopic pants systems, some component of $\mathcal{C}$ crosses some component of $\mathcal{C}'$. (see figure)

$\implies \exists$ connected subsurface $F \subset S$, invariant under ϕ and ϕ', more complicated than an annulus, and filled by components of $\mathcal{C}$ and $\mathcal{C}'$.

$\implies \phi\phi' \big|_F$ is pseudo-Anosov, by a recipe due to Penner.

So $\phi\phi'$ is "less simple" than ϕ.

General method of proof of the trichotomy theorem is by induction on the "simplicity" of mapping classes in G.

Base case is the "least simple" case: when G contains a pseudo-Anosov mapping class. In that case, *DONE*.

Putting off the details of the induction step, the Omnibus Subgroup Theorem states the conclusion of the induction, for any subgroup.

It tells you about the "least simple" element of the subgroup.

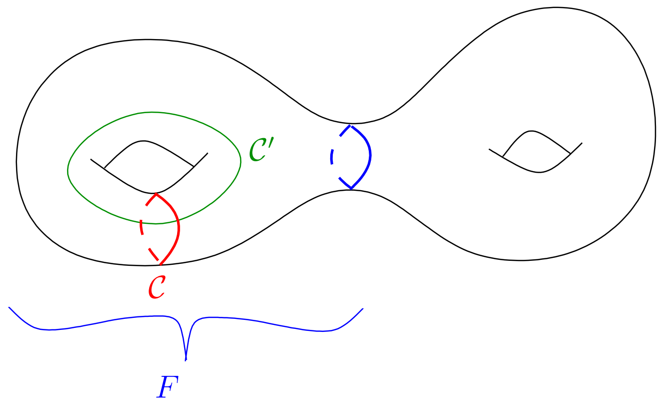

Figure 3: Given a multi-curve $\mathcal{C}$, invariant under ϕ and $\mathcal{C}'$ invariant under ϕ' which intersect, one can find a subsurface F which they fill and which is invariant under both ϕ and ϕ'.

Consider $\phi \in G$.
$\mathcal{C}_\phi$ = canonical reducing system ($\neq \emptyset \iff \phi$ is infinite order and reducible).
N_ϕ = regular neighborhood of $\mathcal{C}_\phi$.
$\mathcal{A}_\phi$ = the *active subsurface* of ϕ is defined to be the union of the following:

- Components of $S - N_\phi$ on which the first return mapping class is pseudo-Anosov
- Components A of N_ϕ such that the components of $S - N_\phi$ on either side of A have first return mapping class of finite order.

Features of the active subsurface $\mathcal{A}_\phi$:

- $\mathcal{A}_\phi$ is an essential subsurface, in particular no two annulus components are isotopic.
- No annulus component of A_ϕ is isotopic into a distinct component of A_ϕ.
- $\mathcal{A}_\phi = \emptyset$ if and only if ϕ has finite order
- $\mathcal{A}_\phi = S$ if and only if ϕ is pseudo-Anosov.

Method of proof for subgroup trichotomy theorem: if there is an element ϕ such that $\mathcal{A}_\phi \neq S$ must find an element with a larger $\mathcal{A}_\phi$, then induct.

Statement of the Omnibus Subgroup Theorem: there is an element with maximal $\mathcal{A}_\phi$.

Theorem 4 (Omnibus Subgroup Theorem (Handel-M))**.** *Every subgroup contains an element whose active subsurface is maximal.*

More precisely, for every subgroup $G < \mathcal{MCG}(S)$ there exists $\phi \in G$ such that for every $\psi \in G$, the subsurface $\mathcal{A}_\psi$ is isotopic into the subsurface $\mathcal{A}_\phi$.

Today, and next time (3 weeks hence), will apply this theorem to prove the three subgroup theorems earlier. Then prove the theorem.

Proof of subgroup trichotomy.
Applying the theorem, choose $\phi \in G$ such that $\mathcal{A}_\phi$ is maximal.

Case 1: $\mathcal{A}_\phi = \emptyset$**.**
Then for all $\psi \in G$, $\mathcal{A}_\psi = \emptyset$, so ψ has finite order.
Every element of G has finite order, and $\mathcal{MCG}(S)$ has a torsion free subgroup of finite index, $\implies$ G is finite. $\diamondsuit$

Case 2: $\mathcal{A}_\phi = S$**.** Then $\phi \in G$ is pseudo-Anosov. $\diamondsuit$

Case 3: $\mathcal{A}_\phi \neq \emptyset, S$**.**

$\implies \partial\mathcal{A}_\phi \neq \emptyset$.
For each $\psi \in G$, shall show that $\psi(\mathcal{A}_\phi)$ is isotopic to $\mathcal{A}_\phi$,
$\implies \partial\mathcal{A}_\phi$ is a reducing system for ψ.

Let $\phi' = \psi\phi\psi^{-1}$, $\quad \implies \mathcal{A}_{\phi'} = \psi(\mathcal{A}_\phi)$.
The subsurfaces $\mathcal{A}_\phi, \mathcal{A}_{\phi'}$ are homeomorphic.
By maximality of $\mathcal{A}_\phi$, may isotope $\mathcal{A}_{\phi'}$ so that

$$\mathcal{A}_{\phi'} \subset \text{int}(\mathcal{A}_\phi)$$

But they are homeomorphic, and no annulus component of A_ϕ (resp. $A_{\phi'}$) is isotopic into A_ϕ ($A_{\phi'}$).
$\implies \mathcal{A}_\phi - \mathcal{A}_{\phi'}$ is a collar neighborhood of $\partial\mathcal{A}_\phi$.
$\implies \mathcal{A}_\phi, \mathcal{A}_{\phi'}$ are isotopic.
$\implies \partial\mathcal{A}_\phi$ is preserved by ψ. $\diamondsuit$

The Tits alternative in the pseudo-Anosov case.

We give the same proof as Ivanov and McCarthy. Once a subgroup is known to be infinite order and irreducible, you have to get your hands dirty (the Omnibus Subgroup Theorem will not help you here).

To prove the Tits alternative when the subgroup G contains a pseudo-Anosov element, need three theorems about pseudo-Anosov mapping classes.

Theorem 5 (Source–sink dynamics)**.** *The action of a pseudo-Anosov mapping class $\phi \in \mathcal{MCG}(S)$ on the sphere of projective measured foliations $\mathbf{P}\mathcal{M}F$ has "source–sink" or "north–south" dynamics:*

$\exists \xi^\pm_\phi \in \mathbf{P}\mathcal{M}F$*, such that* $\xi^+_\phi \neq \xi^-_\phi$ *and such that*
$\forall \eta \in \mathbf{P}\mathcal{M}F$*,*

- *If* $\eta \neq \xi^+_\phi$ *then* $\lim_{n\to+\infty} \phi^n(\eta) = \xi^-_\phi$

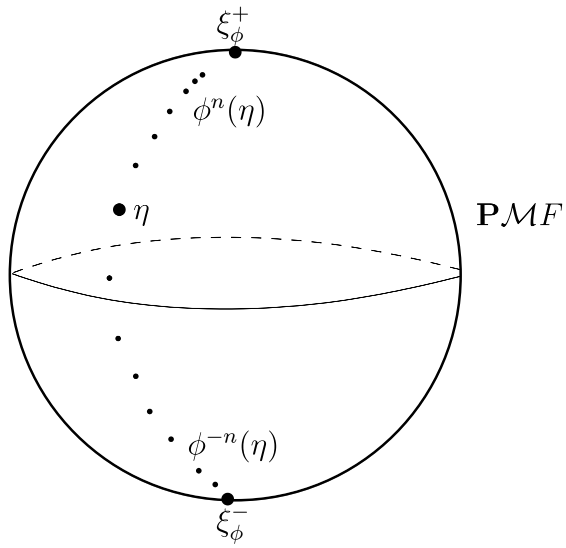

Figure 4: A pseudo-Anosov element acting on $\mathbf{P}\mathcal{M}F$

- *If* $\eta \neq \xi_\phi^-$ *then* $\lim_{n\to+\infty} \phi^{-n}(\eta) = \xi_\phi^+$

Theorem 6. *If* $\phi, \psi \in G$ *are pseudo-Anosov then* $\{\xi_\phi^\pm\}$ *and* $\{\xi_\psi^\pm\}$ *are either equal or disjoint.*

Given a group acting on a space, and a point x in the space, $\mathrm{Stab}(x)$ is the subgroup of elements that fix x.

Theorem 7. *For any pseudo-Anosov* $\phi \in \mathcal{MCG}(S)$ *with source and sink* $\xi_\phi^\pm \in \mathbf{P}\mathcal{M}F$*,* $\mathrm{Stab}(\xi_\phi^-) = \mathrm{Stab}(\xi_\phi^+)$ *and this subgroup contains the infinite cyclic subgroup* $\langle\phi\rangle$ *with finite index.*

This subgroup is the *virtual centralizer* of ϕ, consisting of all $\psi \in \mathcal{MCG}(S)$ such that for some $n \neq 0$ we have $\phi^n\psi = \psi\phi^n$.

Sketch of proof of source–sink dynamics

- pseudo-Anosov homeomorphism ϕ.
- Using train tracks, find:
  - neighborhood U^+ of ξ_ϕ^+ in $\mathbf{P}\mathcal{M}F$ which is attracted to ξ_ϕ^+ under iteration of ϕ
  - neighborhood U^- of ξ_ϕ^- in $\mathbf{P}\mathcal{M}F$ which is attracted to ξ_ϕ^- under iteration of ϕ^{-1}

This is an application of the Perron-Frobenius theorem, although there are subtleties of "diagonal extensions" as one knows from reading, for example, Casson-Bleiler or Masur-Minsky 1.

- Note a corollary of this: ξ_ϕ^+ and ξ_ϕ^- are *uniquely ergodic*.
- Let μ be an xy-structure for ϕ.
- For any measured foliation $\mathcal{F}$, can pull $\mathcal{F}$ tight in the singular Euclidean metric μ. Might squeeze leaves of $\mathcal{F}$ together, but so what?
- Show $\exists\delta > 0$ such that
  - if all leaves of $\mathcal{F}$ have $|\text{slope}| < \delta$ then $\mathcal{F} \in U^+$.
  - if all leaves of $\mathcal{F}$ have $|\text{slope}| > 1/\delta$ then $\mathcal{F} \in U^-$.
- Suppose $\mathcal{F} \neq \xi_\phi^-$
- $\implies$ $\mathcal{F}$ has no vertical leaves (unique ergodicity of ξ_ϕ^-).
- $\implies$ $|\text{slope}(\mathcal{F})|$ bounded away from ∞ (compactness)
- $\implies$ for sufficiently large n, $|\text{slope}(\phi^n(\mathcal{F}))| < \delta$.
- $\implies$ $\phi^n(\mathcal{F})$ converges to ξ_ϕ^+ as $n \to +\infty$.