Capturing, examining, and responding to mathematical thinking through writing

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Introduction

As educators and mathematics teachers, we yearn for direct communication with students, with their thought processes, with their thinking about mathematics. However, natural barriers exist that make it difficult for us to capture, examine, and respond to the inner workings of our students' minds while engaged with mathematics—their mathematical thinking. A non-intrusive technology does not exist that allows us to view in real time the workings of our students' minds as they work on mathematics problems.

Students have similar limitations. Although they can be aware of their thinking, most often students do not have the habit or realize the usefulness of doing so. When they write about their feelings and thoughts concerning particular mathematical ideas, however, their prose offers a vehicle for us and for them to examine, reflect deeply on, and respond to their mathematical thinking.

Writing not only captures mathematical thinking but also facilitates learning in powerful ways. Since this latter proposition was conjectured by psychologists and composition theorists,[1] both composition and mathematics teachers have developed a variety of writing activities and experimented with approaches to using these activities in the classroom.[2] What's the evidence for the assertion in this paragraph's first sentence? Indeed, it may not be immediately obvious that writing can be a powerful tool for learning mathematics and for capturing, examining, and responding to thinking. In this article, I examine a common writing activity known as journals or learning logs[3] to illustrate its usefulness for learning and then elaborate reasons why writing is essential in the mathematics classroom. Finally, I develop a number of theoretical ideas by examining a less-common writing activity, which I call multiple-entry logs.[4]

Using journals in mathematics classes

It is not uncommon for students to explore techniques for calculating the greatest common factor (GCF) and lowest common multiple (LCM) of a group of integers. In an urban school setting, after exploring techniques for finding the prime factorization of positive integers, students worked in small groups on problems and discussed how to find the greatest common factor (GCF) of a group of positive integers. Afterward, each student wrote down what she or he understood about the idea. Let's examine what one student wrote:

> I found that I could find the greatest common factor of two integers by first finding common factors of both integers and then by taking the largest common to both.
>
> E g (24,30) 1, 2, 3, 6 GCF = 6 or $2^1 \times 3^1$

In this journal excerpt, this student describes and illustrates his understanding of how to find the GCF. He lists the common factors of two integers, 24 and 30, and represents their GCF in prime factored form, $2^1 \times 3^1$. Moreover, the style of his writing suggests that he has clarified and claimed

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[1] Cognitive psychologist Jerome Bruner (1968) noted that both writing and mathematics were major tools for thought since they are "devices for ordering thoughts about things and thoughts about thoughts" (p 112) Even more to the point, the well-known composition theorist Janet Emig (1977) argued that writing is a unique, multi-representational, and bispheral "languaging process" that also corresponds to other powerful learning strategies and, therefore, ought to be incorporated as a central academic process

[2] In composition theory, important early work can be gleaned in Britton *et al*, (1975) and Emig (1977) Hairston (1982) provides a useful history on the recognition of writing as a tool for learning in composition theory In mathematics education, as far as I am aware, Geeslin (1977) is the first teacher to have published on the use of writing as a teaching technique Subsequently, an analysis and annotated bibliography of the use of writing to learn in mathematics appeared in Powell, Pierre, and Ramos (1993) For a practical guide to activities for use in grades K through 12, see Countryman (1992), in middle-school classes, for example, see McIntosh (1997) and Di Pillo Sovchik, and Moss (1997), and in various college and university courses see Sterrett (1990). Finally, for a book-length critical analysis of writing as an instrument to assess the learner's ability and achievement as well as of teachers communication of the nature of writing required, see Morgan (1998)

[3] For further discussions of journal writing, see Powell (1985) and Powell and López (1989)

[4] Multiple-entry logs (Hoffman & Powell, 1989, Frankenstein & Powell, 1989) are variations, mainly in content and form, but similar in purpose, of what some call double-entry logs (Jones, 1988), divided pages (Tobias, 1989), or dialectical notebooks (Berthoff, 1982, 1987)

ownership of the process.

In a subsequent journal entry, after exploring ideas concerning the lowest common multiple (LCM) of a group of positive integers, the same student attempts to internalize both concepts and coordinates them with his understanding of prime factors and prime factorization:

The way one goes about finding the LCM of a group of integers is by looking at the prime factorization of the integers in the group, then picking out the common prime factorizations thus giving one the LCM

E.g., LCM(28,36) $=2^2$, since $28 = 2^2 \times 7^1$ and $36 = 2^2 \times 3^2$ In this case 2^2 is the LCM.

Unlike the first excerpt, this entry indicates that this student needs to develop his thinking about LCM Not only do we have a record of how he finds the LCM of a two integers, we have also captured a non-ephemeral, verbal representation of his thinking. If we examine his prose carefully and respond appropriately to this written indication of his thinking, we could turn his journal writing into a dynamic vehicle for challenging and, thereby, augmenting his mathematical awareness.

It appears that this student's confusion concerns a rather simple matter of using the wrong group of three letters—LCM for GCF—not one of misconceptualization. Responding to this journal entry, the teacher pointed out the problem and posed a question. When the student responded in writing, he first reiterates the teacher's question, then answers it, and illustrates his response with a few examples. Here's what the student wrote:

Today, I looked at the prime factorization of a group of numbers to see if I could determine their GCF and LCM just from their prime factorization I found that both of these answers can in fact be determined by the prime factorizations The way one goes about determining the GCF of a group of integers is to first see what prime factors the group has in common The common prime factors of the group is the GCF Note, if there are no common prime factors among the group their GCF is (1) one

E g GCF(60,12) $= 2^2 \times 3^1$ GCF(5,12) = 1
$60 = 2^2 \times 3^1 \times 5^1$ $12 = 2^2 \times 3^1$ $5 = 5^1$ $12 = 2^2 \times 3^1$

In the above portion of the student's journal entry, he describes correctly how to determine the GCF of a group of positive integers and even discusses a special case. However, below, in the second portion of the entry, below, he provides evidence of a lingering mathematical misunderstanding or perhaps a linguistic misconception:

The LCM can be determined in a similar fashion However, when trying to determine the LCM of a group of integers one must take the prime factorizations common to the group

E g LCM(6,12,15) $= 2^2 \times 3^1 \times 5^1 = 60$
$6 = 2^1 \times 3^1$ $12 = 2^1 \times 3^1$ $15 = 3^1 \times 5^1$

Significantly, the verbal representation of this student's thinking captured in his writing is precisely what we rarely have access to when students merely respond to mechanical homework assignments or examination problems. On the one hand, in this portion of his journal, though he computed the LCM correctly, it appears that his available language did not permit him to describe accurately his perception and action. On the other hand, part of his verbal confusion relates to what the adjective *common* qualifies and also to what can be visualized in the prime factorizations of a group of integers. To find the GCF, the word *common* relates to what one sees directly in the prime factorizations, the *common* elements. However, given the prime factorizations of a group of integers, the LCM does not refer to common *visible* elements. That is, the multiples of the integers are not displayed; one cannot see the LCM of a group of integers by inspecting their prime factorizations in the same, direct way one can see their GCF, if it is other than one.

Eventually, because he could mechanically find the LCM, this student had to mine language that corresponded to his perceptions and actions. When his instructor had asked him to reflect critically—to review and comment—on a group of journal entries that contained the one above, this student ultimately found appropriate language to describe correctly the process he engaged in to find the LCM. The task required him the think deeply about how the common and non-common visible elements of the prime factorization of a group of positive integers relate to their GCF and their LCM. The following is excerpted from the journal entry he wrote after reflecting critically on his previous journal entries on LCMs:

To find the LCM, least common multiple, of a group of numbers one must take the distinct prime factors of the group that one expressed to the highest power.

E g , LCM($2^5 \times 5^9 \times 3^3$, $2^1 \times 5^3 \times 3^1$, $7^1 \times 19^1 \times 13^1$) $= 2^5 \times 3^3 \times 5^9 \times 7^1 \times 13^1 \times 19^1$

This excerpt contains three particularly fascinating aspects. First, this students represents integers not in standard form such as 750 but rather in their prime factored form, $2^1 \times 5^3 \times 3^1$. That is, his writing evidences a certain ease in handling this more abstract representation of integers. Second, he describes how to calculate the LCM of a group of integers generally and concisely. He achieves this level of generality and conciseness by reflecting and critically reflecting in writing and then revising the written representations of his thinking. Third, in the description above, if the word *take* were replaced by *multiply* and the phrase *that one* replaced by *each*, which also express his actions, then his description would appear to come from an edition of James & James (1963: 262)!

The manner in which this one student grappled in writing with his thinking about GCF and LCM, illustrates how journal writing can be used powerfully to capture, examine, and respond to student's mathematical thinking. We see that writing forces students to reflect on mathematical experiences and that as students examine their written reflections, writing leads them to reflect on their ideas critically. Moreover, reflecting and then reflecting critically on one's mathematical experiences in writing presuppose an active, not a passive, learner. This presupposition coupled with the revelatory character of their reflective writing suggest that writing can significantly influence students' cognition and metacognition. Writing, because the writer and others can see it, allows one to explore relationships, make meaning, and manipulate thoughts; to extend, expand, or drop ideas; and to review, comment upon and monitor reflections. Expressive writing supports those cognitive and metacognitive acts.[5] After establishing a degree of confidence in one's ideas, it seems almost natural to move from expressive to transactional prose. Such a movement occurred with this student as he wrestled with his ideas on how to determine the least common multiple of a group of integers. He constructed and reconstructed meaning. He wrote and revised his reflections, a process mediated by external comments. As he began to express his ideas with greater clarity and confidence and selected language that more accurately described his perceptions and actions, his writing shifted from expressive to transactional.[6]

We have also seen that writing helps students acquire a rich, functional vocabulary and use it in the context of their understanding of mathematics. Mayher, Lester, & Pradl (1983) make this point in regard to learning in general.

> Writing's capacity to place the learner at the center of her own learning can and should make writing an important facilitator of learning anything that involves language. Writing that involves language choice requires each writer to find her own words to express whatever is being learned. Such a process may initially serve to reveal more gaps than mastery of a particular subject, but even that can be of immense diagnostic value for teacher and learner alike. And as the process is repeated, real and lasting mastery of the subject and its technical vocabulary is achieved (p. 79)

By providing students with opportunities to work with mathematical ideas in their own language and on their own terms, writing helps students develop confidence in their understanding of mathematics and become more thoroughly engaged with mathematics.

The above excerpts from a student's journal writings illustrate that, as teachers, we accomplish important pedagogical objectives when students write about the mathematics in which they are engaged. Students capture important evidence of their mathematical thinking whatever the writing activity—as long as it requires students to probe their ideas and understanding. Unlike speech, which is ephemeral, writing is a stable medium that allows for both student and teachers to examine and respond to the student's mathematical thinking. When a teacher responds to students' journals, for example, a powerful medium for dialogue between teacher and students is established. Teachers can provide precise feedback on students' statements, interpretations, questions, discoveries, and misconceptions. Important opportunities often present themselves for teachers to encourage students to reconsider, deepen, and extend their ideas. Such personal dialogue can reassure students that their concerns and ideas matter. Furthermore, the revelatory character of students' expressive writings provide teachers with feedback on their own instruction.

Reflecting on the mathematics they are learning leaves students with crucial cognitive and affective insights. They acquire greater control over their learning and develop criteria for monitoring their progress. Such acquisition of control and monitoring capabilities engenders in students feelings of accomplishment; those feelings, in turn, produce a positive effect on students' affective responses to mathematics. Furthermore, students develop confidence in themselves as learners

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[5] In composition theory, researchers distinguish between different modes of writing: expressive and transactional. Transactional writing uses language "to get things done: to inform people (telling them what they need or want to know or what we think they ought to know), to advise or persuade or instruct people." It is used whenever an "accurate and specific reference to what is known about reality" is needed. Expressive writing is "'thinking aloud on paper.' It has the function of revealing the speaker, verbalizing his consciousness ... [It] submits itself to the free flow of ideas and feelings.'" (Britton *et al* 1975: 88-90)

[6] For an example of a writing activity that prompts learning and requires students to write transactionally, see Powell (1993).

capable of doing *and* understanding mathematics.

Using multiple-entry logs in mathematics classes

When we incorporate writing activities into mathematics instruction, we support an important pedagogical percept: Learning is enhanced when students reflect critically on their mathematical experiences and respond to mathematical situations and questions that are personal and of their own choosing. Another example of such a writing activity is the multiple-entry log.[7] A multiple-entry log is, first, a vehicle to prompt students to reflect on and form images of a piece of mathematics and, second, a medium in which students record, in prose, multiple and layered versions of their reflections and images. Students create this personal, reflective vehicle by creasing a sheet of loose-leaf paper width-wise into three equal sections. In the left-hand column, they write down a text of their own choosing that particularly interests or strikes them. (We interpret the word *text* broadly to mean some combination of mathematical prose or notational expressions selected from a textbook, lecture, problem set, computer or calculator screen, or any other course material. Students may also extract text from a mathematical discussion in which they participate or one that they, as it were, witness.) In the middle column, students reflect on the text by writing a commentary, an interpretation, an evaluation, a summary, or any other type of elaboration of their thoughts. Finally, in the most crucial part of maintaining multiple-entry logs, students, some time later, reflect again ("meta-reflect") on previous text-reflection entries and, in the right-hand column, revise, reconsider, refine, or otherwise comment on their previous reflections.

The excerpt shown in Table 1 is from a multiple-entry log that a student wrote during the second week of an algebra course. It illustrates several interesting characteristics of writing students produce using this tool. First, the entries in the middle and right-hand columns are examples of personal or expressive writing that are both reflective and analytical. Second, in the middle column, this student states that she is "not sure how to set up the problem," as if by declaring this she permits herself to become unstuck and to continue working on the problem. Third, she uses the opportunity to write to explore her understanding of the problem. Specifically, she appears to wrestle with the significance of Doug having worked six hours overtime and the impact that that has on the pay he received for the week. Fourth, as the student writes, her understanding of the problem seems to deepen, and she discovers a way to express one of the unknown quantities. She determines that the variable 'x' can represent Doug's normal rate of pay and that '$2x$' would therefore stand for his overtime rate. Then she establishes an equation: Doug's "typical weekly earning" plus his overtime earning equals his total earnings for the week.

In the right-hand column, which contains the student's second or critical reflection, she discusses two important insights. The second of these is a generalization of the particular instance in which she became aware that if one unknown quantity could be denoted by a variable, then the same variable can be used to assist in representing other unknown quantities in the problem. Later, of course, she will need to specialize this awareness to determine the domain for which her insight holds. The point here, however, is that generalizing is an important aspect of mathematical thinking and that responding reflectively in writing afforded this student an opportunity to engage in meta-cognition and to examine deeply her understanding of a mathematical idea: In this case, the use of a single variable to construct a notational expression. for another, related unknown. Naturally, one could continue to reflect on multiple-entry log reflections and generate new questions, issues, and understandings.[8]

Conclusion

Writing about mathematical ideas is an inexpensive and non-intrusive technology that allows students and teachers to capture, examine, and respond to mathematical thinking. The two writing activities that I presented—journals and multiple-entry logs—are effective tools for implementing writing in mathematics classrooms since they prompt students to write in particularly useful ways. Generally, different writing-to-learn activities prompt students to produce different kinds of writings. As Hoffman & Powell (1989) theorize, those writings exist within a matrix of categories: non-personal and non-reflective; non-personal and reflective; personal and non-reflective; and personal and reflective. The category that best supports mathematical thinking

[7] For further discussions of multiple-entry logs see. Hoffman and Powell (1989) as well as Powell and Ramnauth (1992a and 1992b;

[8] For an example of this recursive use of multiple-entry logs, see Powell and Ramnauth (1992a)

according, to Hoffman & Powell, is "personal, reflective writing in which the content is mathematics and students' affective responses to it" (1989: 132). The use of journal and multiple-entry logs that I have suggested entails teachers inviting students to reflect on their own learning by examining and responding to previous entries. This use prompts learners to enter into communication not only with their instructor but also with the text they select and, rather than mere summary, encourage interpretation and analysis of the text.

TEXT	REFLECTION #1	REFLECTION #2
Doug is paid double time for each hour worked over 40 hours in a week Last week he worked 46 hours and earned \$468 What is his normal hourly rate?	He only worked 6 hours overtime That means only 6 of those hours were double time I'm not sure how to set up the problem The \$468 represents the amount he was paid for the regular 40 hrs plus the 6 hrs overtime which were double time What ever his hourly rate is, for the 6 hrs he worked overtime it will be doubled. Let x represent his normal hourly rate He normally works 40 hrs /wk, so $40x$ represents a typical weekly earning $6(2x)$ represents the six hrs worked overtime at double his normal hourly pay $40x + 6(2x) = 468$ $40x + 12x = 468$ $52x = 468$ $x = 9$ His normal hourly rate is \$9/hr For the six hours he earned \$18/hr	Although the problem doesn't ask how much he earned per wk. without overtime, I can now answer that question Also I can answer the question of how much he earned for the 6 hrs he worked overtime 40(\$9) = \$360/wk 6(\$18) = \$108 for O.T After reflecting on this problem, I have come to the conclusion that if I can represent an unknown quantity with a variable I could find the other unknown quantities of a problem using that variable

Table 1

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