

“Where’s the cheese?”: Understanding Students’ Inscriptions, Mathematical Ideas, and Reasoning in Interactive Problem Solving Online

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In this chapter, we examine collaborative problem solving mediated by a computer communication system. We trace collaborative problem solving as an interactive, layered building of meaning among learners working as a small group. Researchers in the learning sciences concerned with computer-supported collaborative learning focus primarily on understanding how such environments promote and support learners’ collaborative knowledge building (Stahl, Koschmann, & Suthers, 2006). In addition, as mathematics education researchers, we are interested in discovering what mathematics learners learn. We present a case study of four students who participate in collaborative, mathematical problem solving within the online environment of Virtual Math Teams Chat (VMT Chat). Our analytic aim is to investigate how students through their inscriptive signs collaboratively build mathematical ideas, heuristics, and lines of reasoning in a virtual environment with the functionality of VMT Chat.

Similar to other computer-mediated communication systems, VMT Chat presents communicative affordances and constraints that influence users’ discursive interactions. (We present an image of the VMT Chat environment in Figure 1). We are interested in how students use the affordances of the virtual environment, including the shared, dynamic whiteboard space; chat feature; and referencing tool, as well as what mathematical ideas, heuristics, and lines of reasoning are evidenced in their interactions. In addition, we are interested in how constraints of the system intervene in student discursive interactions.

Online communication systems present affordances and constraints to researchers, as well. VMT Chat presents methodological challenges and opportunities to researchers interested in investigating how students exchange and interactively develop emergent mathematical ideas, heuristics, and lines of reasoning. Consequently, we also explore an analytic approach for inquiring into the archived interactions of students collaborating on mathematical problem solving through an online dual-interaction space. While analyses of users’ online problem solving typically focus on their chat text and referenced whiteboard inscriptions, in the analysis that we present, for reasons that we will discuss, our analytic attention focuses almost exclusively on the evolution of participants’ whiteboard inscriptions as a means to gain insight into the interactive development of their mathematical ideas, heuristics, and reasoning as they solve an open-ended mathematics problem.

CONCEPTUAL FRAMEWORK

In this study, key conceptual terms include discourse, student-to-student or peer mathematical discussion, collaborative interaction, problem solving, heuristics, mathematical ideas, and inscriptions. Discourse here refers to language (natural or

symbolic; oral, gestic, or inscriptive) used to carry out tasks—for example, social or intellectual—of a community. In agreement with Pirie and Schwarzenberger (1988), student-to-student or peer conversations are mathematical discussions when they possess the following four features: are purposeful, focused on mathematical notions, involve genuine student contributions, and are interactive. We define collaborative interaction as individuals exchanging ideas and considering and challenging each other's ideas so as to affect one another's ideas and working together for a common purpose. In addition, in the context of the data of this study, the student-to-student, discursive collaborations involve minimal, substantive interaction with a teacher or researcher.

The term 'heuristics' applied to human beings and machines has various uses and meanings in fields as diverse as philosophy, psychology, computer science, artificial intelligence, law, and mathematics education. We construe heuristics to mean actions that human problem solvers perform that serve as means to advance their understanding and resolution of a problem task. We do not imply that when problem solvers implement a set of heuristics that they will necessarily advance toward a solution but only that their intent is to do so. Our sense of heuristics includes explicit and implicit general strategies such as categories outlined by Pólya (1945/1973, pp. xvi-xvii and 112-114) and others (Brown & Walter, 1983; Engel, 1997; J. Mason, Burton, & Stacey, 1984; J. H. Mason, 1988; Schoenfeld, 1985) and pertains to other actions such as a group of problem solvers decision to assign subtasks to each other to later pool their outcomes to influence their progress on the larger problem at hand (Powell, 2003). Furthermore, we distinguish heuristics from reasoning, which we view as a broad cognitive process of building explanations for the outcome of relations, conclusions, beliefs, actions, and feelings.

A paramount goal of mathematics education is to promote among learners effective problem solving. In our view, mathematics teaching strives to enhance students' ability to solve problems individually and collaboratively that they have not previously encountered. Nevertheless, the meaning of mathematical "problem solving" is neither unique nor universal. Its meaning depends on ontological and epistemological stances, on philosophical views of mathematics and mathematics education. For the purposes of this chapter, we subscribe to how Mayer and Wittrock (1996) define problem solving and its psychological characteristics:

Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Mayer, 1992). According to this definition, problem solving has four main characteristics. First, problem solving is *cognitive*—it occurs within the problem solver's cognitive system and can be inferred indirectly from changes in the problem solver's behavior. Second, problem solving is a *process*—it involves representing and manipulating knowledge in the problem solver's cognitive system. Third, problem solving is *directed*—the problem solver's thoughts are motivated by goals. Fourth, problem solving is *personal*—the individual knowledge and skills of the problem solver help determine the difficulty or ease with which obstacles to solutions can be overcome. (p. 47)

Coupled with these cognitive and other psychological characteristics, mathematical problem solving also has social and cultural dimensions. Some features include what a social or cultural group considers to be a mathematical problem (cf., D'Ambrosio, 2001; Powell & Frankenstein, 1997), the context in which individuals may

prefer to engage in mathematical problem solving, and how problem solvers understand a given problem as well as what they consider to be adequate responses (cf., Lakatos, 1976). In instructional settings, students' problem solving are strongly influenced by teachers' representational strategies, which are constrained by cultural and social factors (Cai & Lester, 2005; Stigler & Hiebert, 1999). Moreover, with online technologies, the affordances and constraints of virtual environments provide another dimension to the social and cultural features of problem solving since "such technologies are intertwined in the practices used by humans to represent and negotiate cultural experience" (Davis, Sumara, & Luce-Kapler, 2000, p. 170) and how problem solvers think and act. Finally, the framing of abstract combinatorial concepts in the cultural context of a "pizza" problem, which is presented in the next section, also offers conceptual affordances and constraints.

In offline as well as online environments, users express objects, relations, and other ideas graphically as text and as inscriptions. These are special instances of the more general semiotic category of signs. A sign is a human product—an utterance, gesture, or mark—by which a thought, command, or wish is expressed. As Sfard notes, "in semiotics every linguistic expression, as well as every action, thought or feeling, counts as a sign" (Sfard, 2000, p. 45). A sign expresses something and, therefore, is meaningful and as such communicative, at the very least, to its producer and, perhaps, to others. Some signs are ephemeral such as unrecorded speech and gestures, while others like drawings and monuments persist. Whether ephemeral or persistent, a sign's meaning is not static; its denotation and connotation are likely to shift over time in the course of its discursive use.

As a discursive entity, a sign is a linguistic unit that can be said to contain two, associated components. Saussure (1983) proposes that a sign is the unification of the phonic substance that we know as a "word" or *signifier* and the conceptual material that it stands for or *signified*. He conceptualizes the linguistic sign (say, the written formation) as representing both the set of noises (the pronunciation or sound image) one utters for it and the meaning (the concept or idea) one attributes to it. Examples of the written formation of a linguistic sign are "chair" and " $\cos^2 x$ ", each with associated, socially constructed meanings. Saussure observes further that a linguistic sign is arbitrary, meaning that both components are arbitrary. The signifier is arbitrary since there is no inherent link between the formation and pronunciation of a word or mathematical symbol and what it indexes. A monkey is called *o macaco* in Portuguese and *le singe* in French, and further in English the animal is denoted "monkey" and not "telephone" or anything else. The arbitrariness of the signified can be understood in the sense that not every linguistic community chooses to make salient by assigning a formation and a sound image to some aspect of the experiential world, a piece of social or perceptual reality. Consider, for example, the signifieds *cursor*, *mauve*, and *zero*. They index ideas that not all linguistic communities choose to lexicalize or represent.

Signs can be considered to represent ideas. However, Sfard (2000) argues that a sign is constitutive rather than strictly representational since meaning is not only presented in the sign but also comes into existence through it. Specifically, she states that

mathematical discourse and its objects are *mutually constitutive*: It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather

the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions. (p. 47, original emphasis)

This theoretical stance on the mutually constitutive nature of meaning and sign provides a foundation for analysis of the discursive emergence of mathematical ideas, reasoning, and heuristics. On the one hand, signs can represent encoded meanings that based on previous discursive interactions interlocutors can grasp as they decode the signs. On the other hand, through moment-to-moment discursive interactions, interlocutors can create signs and, during communicative actions, achieve shared meanings of the signs. In this sense, the sameness of meaning for interlocutors that allows for successful their communication is not something pre-existing but rather an achievement of the communicative act. This accomplishment may compel interlocutors to bring into existence signs to further their discourse.

Mathematical signs—objects, relations, symbols, and so on—are components of mathematical discourse and are intertwined in constituting mathematical meanings. Signs exist in many different forms, and inscriptions or written signs are but one. They are produced for personal or public consumption and for an admixture of purposes: to discover, construct, investigate, or communicate ideas. As mathematicians and other mathematics education researchers also emphasize (Dörfler, 2000; Lesh & Lehrer, 2000; Speiser, Walter, & Maher, 2003; Speiser, Walter, & Shull, 2002), building and discussing inscriptions are essential to building and communicating mathematical and scientific concepts. In a discussion of mathematics and science teaching, Lehrer, Schauble, Carpenter, and Penner (2000) illustrate how learners work “in a world of inscriptions, so that, over time, the natural and inscribed worlds become mutually articulated” and the importance of a “shared history of inscription” (p. 357). In mathematics, the invention, application, and modification of appropriate symbols to express and extend ideas are constitutive activities in the history of mathematics (Struik, 1948/1967). Some researchers claim that mathematical meaning only exists through symbols and that symbols constitute mathematical ideas.

For researchers in mathematics education and in computer-supported collaborative learning, the arbitrariness of signifieds is a more significant point about Saussure’s observation concerning the arbitrariness of signs. The reason is that the conceptual material that a person (or a small group of people) lexicalizes, for example, with pencil and paper, with text in a chat window, or with drawn objects on a shared, digital workspace indicates to what that user attends, her insight into material reality that is external or internal to her mind. The inscriptions of individuals working online in a small-group or team provide observers, who must interpret meanings constituted in the inscriptions, evidence of individual and collective thinking. The small group’s inscriptions present ideas it chooses to lexicalize or symbolize. By analyzing the unfolding and use of inscriptions, researchers can understand how participants constitute their mathematical ideas, reasoning, and heuristics, the meanings they attribute to their inscriptions, and how their inscriptions influence emergent meanings. As Speiser, Walter and Maher (2003) underscore, what counts as mathematical in analyzing inscriptions is not the inscription itself, which are “tools or artifacts, but rather how the students have chosen to *work*” (p. 22, original emphasis) with their inscriptions. In the specific case of this study, in an online environment that offers resources for individuals to collaborate,

what work they interactively accomplish with their inscriptions reveals their ideas, heuristics, and reasoning.

METHOD

The data come from a class of undergraduate teacher candidates for positions in urban schools who are enrolled in a semester course—“Mathematics and Instructional Technology”—whose theme is the use of digital technologies for the teaching of mathematics in elementary schools. The second author taught this course, which was developed by the first author. During a particular class session, students worked on an open-ended problem, The Pizza Problem, interacting in chat-room teams of four through the online, collaborative environment, VMT Chat. When students entered their assigned chat room, the version of the problem with which they were presented is as follows:

The Pizza Problem

A local pizza shop has asked us to help them keep track of pizza sales. Their standard “plain” pizza contains cheese with tomato sauce. A customer can then select from the following toppings to add to the whole plain pizza: peppers, sausage, mushrooms, bacon, and pepperoni.

How many different choices for pizza does a customer have?

List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

We chose this mathematical problem for three reasons: (1) it relates to the course module, which concerned number and algebra, (2) its context is familiar to students from urban and suburban communities, and (3) mathematically, it affords different solutions approaches, ranging from simple listing procedures to more advanced methods involving combinatorial analysis.

Our data sources are the mathematical problem and the persistent computer log of the chat-room interactions from the dual-interaction spaces that VMT Chat maintains.

To investigate the online, problem-solving actions of learners so as to understand how they build mathematical ideas, heuristics, and reasoning, we code for instances in the data of their discursive attention to any of four markers of mathematical elements—objects, relations among objects, dynamics linking different relations, and heuristics (Gattegno, 1988; Powell, 2003). In their chat text and whiteboard inscriptions, participants either communicate affirmations or interrogatives about these mathematical elements, and as such, we attend to eight different critical events that provide insight into learners’ general mathematical behavior. The matrix in Table 1 contains the codes we used to flag these critical events in the chat text and whiteboard inscriptions. Below in Table 2, we provide an example of how we coded a version of our data.

It is possible that an interaction receives multiple codes. We analyze the mathematical ideas and forms of reasoning that learners produce working interactively in pairs and in a team in a chat room, tracing the development of their ideas and reasoning patterns over the course of the problem-solving session.

Table 1.

Matrix of Event Types Designated as Critical

Subject and type of utterance or inscription	Objects	Relations among objects	Dynamics linking different relations	Heuristics
Affirmations	AO	AR	AD	AH
Interrogatives	IO	IR	ID	IH

Epistemologically, we view learning or knowledge creation as a process of conceptual change whereby individuals and groups of individuals construct new understandings of reality. Through social interactions, learners engaged with mathematics seek meaning and search for patterns, relationships, and dynamics linking relationships among objects and events of their experiential world.

The second author grouped students into teams as they arrived in the classroom. Each team consisted of four students and was assigned to a chat room. In one virtual chat room, students were grouped in pairs, each pair at one computer. In the other chat room, three students shared a computer and one student was alone at a computer.

For this case study, we analyze data from one of the two chat rooms, the one involving two pairs of students. After reviewing data of both chat rooms, using the ConcertChat player, we chose this corpus of data since realizing that given the paucity of text chat these data would provide an interesting analytic challenge. In what follows, we refer to the two students in each pair collectively, using an abbreviation of the screen name of the one individual of the pair who signed into the chat room. We refer to the first pair as Silvestre; the participants are Sonia and Lyndsey, and they used Sonia's screen name, SOSilvestre, in the chat room. We refer to the second pair as Suzyn; the participants are Susan and Komal, and they used Susan's screen name, suzyn17, in the chat room. In this report of our case study, although we are speaking of two pairs of students, to simplify things, we will refer to each pair as a female student from each group. That is, we will discuss Silvestre and Suzyn from now on as singular feminine nouns. Although the students were colocated, the pairs were asked to interact only through the chat room, pretending that they were located at distant sites.

RESULTS: Methodological

Our study yields methodological and cognitive results. We present cognitive results of our case study after detailing how we addressed the methodological challenge of analyzing data from a virtual environment with the functionality of VMT Chat with the intent to study collaborative mathematical behavior on problem solving. An outcome of our study is the approach we used to analyze data from VMT Chat to study how students collaboratively build their mathematical ideas, heuristics, and lines of reasoning.

In analyzing our data, we realized that the data for this particular study provided an analytic challenge that had to be overcome to make sense of the chat room interaction of the participants. Specifically, the chat-room participants hardly interacted in the chat

frame of VMT and used the whiteboard almost exclusively. This meant that we had to follow the evolution of their inscriptions on the whiteboard to understand the emergence of their mathematical ideas and reasoning as they solved the Pizza Problem. In this regard, to analyze the evolution of the whiteboard inscriptions, we adapted a videodata analytic technique used for qualitative investigations into the development of learners' mathematical ideas and reasoning (Powell, Maher, Francisco, 2003). This approach allows us to view our data replayed through the ConcertChat player, much as we would a video recording, and evolved into four recursive stages.

Our first analytic move was to view attentively the data in the ConcertChat player several times at various speeds to familiarize ourselves with the real-time sequence of whiteboard actions and chat text postings. Afterwards, we discussed our sense of the data amongst ourselves. Also, as part of a professional development program for teacher candidates of secondary mathematics, we engaged undergraduate mathematics students in viewing and discussing the data.¹

After these initial viewings of the data, our second analytic move was to step carefully through the data with the ConcertChat player to create an objective description of actions that transpired in the chat and whiteboard spaces. We created these descriptions for each five-minute interval. Following the descriptions, our third move was to code the data deductively and inductively while also writing analytic, interpretative notes of the problem solving and other interactive accomplishments occurring in the chat-room session. For the deductive codes, we used the markers of attention to mathematical elements indicated in Table 1. For the inductive coding, we inquired into the heuristics and lines of reasoning evident in the data as well as to how the chat-room participants manage affordances and constraints of the virtual environment. We present the results of our coding in the next section of this report. In Table 2 below, we present an example of a description, coding, and interpretation of three intervals of the chat-room actions, each less than a minute long, in three respective columns, in which Silvestre contributes to Suzyn's solution, and then Suzyn subsequently critiques this addition and induces Silvestre to make further changes. In the interpretation column, for each five-minute interval, we include rationale for our coding of a particular chunk of data. The letters, EC, which stand for "explanation of code," precedes these rationales.

Our third analytic move proceeded from our interpretations and EC rationales. We chunk the data by reorganizing them into specific categories based the deductive and inductive codes. This allowed us further to understand the actions the team takes to make sense of the problem and the sequence of subsequent actions the participants perform to present and refine their solutions. In this stage, we also create a story line, deciding how the data informs our research question and what other interpretive frame do the data suggest. The final stage of our analytic process was to compose a narrative, the report that you are reading.

Our trajectory of analytic moves is far more recursive than the linear description we have just provided. For instance, we refined and corrected the description as we coded and composed interpretations of chunks of data. In some instances, deductive and

¹ These students are teacher candidates for teaching high school mathematics in economically impoverished, urban school districts and recipients of Robert Noyce scholarships, sponsored by National Science Foundations, and administered through a joint project of Rutgers University, New Jersey Institute of Technology, the Newark Public Schools and the Newark Museum.

inductive coding occurred almost simultaneously. In the analytic method that we have developed, we also present a detailed description of data so that readers can have access to a reduced form of our case study data. After presenting this description, we present the results of the cognitive inquiry of our case study.

Table 2.

<p><i>Example of time-interval description, interpretation, and coding of chat room (chat text and whiteboard inscriptions) data</i></p> <p>12:50:02 – 12:50:16</p>	<p>SOSilvestre creates an ellipse filled with the color red below the ellipse containing the textbox containing “M/B/R”. Within this red ellipse, SOSilvestre creates a textbox and types “P/S/R”.</p>	<p>SOSilvestre creates an ellipse on suzyn17’s side, containing a textbox listing a pizza with pepper and two other toppings, presumably because SOSilvestre is done with her work, and wants to help out suzyn17. SOSilvestre seems to color the pizza red to have more fun with the problem. This seems to be the second attempt to collaborate since suzyn17 wrote “Plain Pizza” into the chat window.</p> <p>EC: (AO) SOSilvestre creates a pizza containing peppers, sausages, and pepperoni as toppings on Suzyn17’s side of the whiteboard</p> <p>EC: (AR) By creating a pizza for Suzyn17, SOSilvestre engages in a relation among the objects on Suzyn17’s side of the whiteboard.</p> <p>EC: (II) By creating a pizza for Suzyn17, SOSilvestre essentially initiates an interaction with Suzyn17, although the “interaction” here is not verbal.</p>	<p>AO, AR, II</p>
<p>12:56:24 – 12:56:49</p>	<p>Suzyn17 types into the chat window “WHO COLORED MY PIZZA?” SOSilvestre types “i did I did”. SOSilvestre types “pizza red right?” SOSilvestre types “lol”.</p>	<p>EC: (AO) SOSilvestre creates a pizza with peppers as the only topping on Suzyn17’s side of the whiteboard. She then deletes this pizza.</p> <p>EC: (AR) By creating an additional pizza on Suzyn17’s side of the whiteboard, SOSilvestre engages in a relation among the objects on Suzyn17’s side.</p> <p>EC: (II) By asking “WHO COLORED MY PIZZA?” in the chat window, Suzyn17 attempts to initiate an interaction with SOSilvestre in the chat window around the pizza that SOSilvestre</p>	<p>AO, AR, II</p>

		has drawn for Suzyn17.	
12:57:08 – 12:57:43	SOSilvestre adjusts the size of the ellipse containing the textbox containing “S/M/B/R”. suzyn17 types “WHERE’S THE CHEESE?” SOSilvestre colors the textbox containing “P/S/R” yellow. SOSilvestre types “there it is”.	EC: (AO) SOSilvestre colors yellow the peppers, sausages, and pepperoni pizza on suzyn17’s side.	AO

DETAILED DESCRIPTION OF DATA

In this section, we provide an uninterrupted description of what transpires in the nearly two-hour, problem-solving session of the four chat-room participants. Before reading the description of the data, we recommend that readers examine Figure 1 below to have a sense of the final solution state that both pairs represented on the whiteboard. In the next sections, we present our results, followed by a discussion.

At the beginning of the session, both Suzyn and Silvestre draw ellipses on the whiteboard. Suzyn denotes one of her ellipses as a plain pizza, using the referencing tool. Three minutes later, Silvestre labels one of her ellipses as a plain tomato and cheese pizza. Afterward, Silvestre lists in a textbox four two-topping pizzas all containing the letter “P.”

At this point, Suzyn and Silvestre occupy different sides of the whiteboard. This division of the whiteboard appears to occur tacitly. Suzyn is doing all her work on the right side of the whiteboard, while Silvestre uses the left side for hers. (It important to note that each user see what the other sees in the window of VMT Chat.)

Silvestre creates a key for the pizza toppings. P stands for pepperoni, S stands for sausage, M stands for mushrooms, B stands for bacon, and R stands for pepperoni. After creating the key, Silvestre lists in separate columns two-, three-, and four-topping pizzas with peppers as one of the toppings. Silvestre lists one pizza with only peppers, four pizzas with peppers and one other topping, four pizzas with peppers and two other toppings, two pizzas with peppers and three other toppings, and an additional pizza with peppers and two other toppings. Then Silvestre starts listing pizzas with sausage in a separate textbox. She lists one pizza with sausage, one pizza with sausage and three other toppings, two pizzas with sausage and two other toppings, and three pizzas with sausage and one other topping.

Suzyn starts listing two-topping pizzas with peppers as one of the toppings by placing each combination in a separate textbox within an ellipse, and arranging these ellipses vertically, representing each topping combination as a separate pizza. She also labels their column at the top with a textbox containing the word pepper. The last pizza that she creates is a pizza with just peppers.

Suzyn uses the same representational scheme to list two topping pizzas containing sausages. She places a textbox at the top to the left of her textbox labeled peppers and draw three circles for these two-topping pizzas with sausage as one of the toppings.

Returning to Silvestre, after she lists pizzas with sausage as one of the toppings, she lists in another textbox pizzas with mushrooms as one of the toppings. She lists one pizza with mushroom, two pizzas with mushroom and one other topping, and one pizza with mushroom and two other toppings. Then Silvestre lists in yet another textbox pizzas

with Bacon as one of the toppings. She initially lists two pizzas with bacon and one other topping, and one pizza with bacon and two other toppings, and then add to the beginning of this list one pizza with bacon. Silvestre then lists one plain pizza with pepperoni also in a separate textbox. Altogether, she has five separate textboxes, one for each of the five toppings.

After listing pizzas containing sausages, Suzyn places a textbox labeled mushrooms to the left of their textbox labeled sausage, and list under this label two pizzas with mushroom and one other topping. She lists a plain pizza with mushroom as the last pizza. Then, Suzyn creates a textbox to the left of the textbox labeled mushroom, labels it bacon, and draws two ellipses. Under this textbox, Suzyn lists one pizza with bacon and one other topping, and one pizza with just bacon. Suzyn then creates a textbox labeled pepperoni to the left of the textbox labeled mushroom. Below it, she draws one ellipse and, in it, places a textbox labeled pepperoni.

Suzyn rearranges the ellipses she has created so that they are closer together. Meanwhile, Silvestre rearranges her representation of the pizzas with peppers. Pizzas with peppers and one other topping are in one column, pizzas with peppers and two other toppings are in a second column, pizzas with peppers and three other toppings are in a third column, and pizzas with peppers and four other toppings are in a row below the latter three columns.

Suzyn now lists pizzas with peppers and two other toppings, peppers and three other toppings, peppers and four other toppings, sausages and two other toppings, and sausages and three other toppings. While she does this, Silvestre put in parentheses the number of combinations in each of her textboxes. Thus far, she finds fourteen pizzas with peppers, seven pizzas with sausages (and no peppers), four pizzas with mushrooms (and no peppers or sausages), three pizzas with bacon (and no peppers, sausages, or mushrooms), and one pizza with pepperoni (and no peppers, sausages, mushrooms, or bacon).

Silvestre creates an ellipse on Suzyn's side, containing a textbox listing a pizza with pepper and two other toppings: sausage and pepperoni. They fill the space red within the ellipse. Suzyn then lists a pizza containing peppers and two other toppings as well as a pizza with mushrooms and two other toppings.

Suzyn notes Silvestre's contribution to her listings and types into the chat window, "WHO COLORED MY PIZZA?" Silvestre responds, "i did I did." Silvestre then writes, "pizza red right?" Silvestre also types, "lol [laugh out loud]." Suzyn then types, "WHERE'S THE CHEESE?" Silvestre fills the space in the textbox containing "P/S/R" yellow, and answers, "there it is."

After a nine-minute pause, Silvestre draws more ellipses on Suzyn's side of the whiteboard. Silvestre lists three pizzas with peppers and two other toppings. The chat between Suzyn and Silvestre continue. Silvestre types, "did u say u have 33 combos." Suzyn responds, "CAN YOU KEEP UP PLEASE-IT'S 34." Silvestre type, "darn I have gotten passed 29" and immediately type "haven't." Silvestre types, "what happen to the rest of the pizza pies... huh." Suzyn sate them." Suzyn types "what problem are rob and jenna doing?" Silvestre types, "mo clue prob another ICT section." Suzyn types "it." Suzyn types, "it's back to 33-we repeated...what do you want from us?" Silvestre types, "a clue.. lol." Silvestre responds, "no."

Silvestre types, “what happen to the rest of the pizza pies...huh,” indicating that she sees on whiteboard only 17 of the 34 of pizza pies that Suzyn’s claim to have. Silvestre asks Suzyn for help. Suzyn helps Silvestre by placing the combination of psbr into Silvestre’s list of pizzas with peppers as a topping. Silvestre incorporates this combination into her listing of pizzas with peppers, and changing the number of such pizzas from fourteen to fifteen.

Suzyn now lists possible pizza toppings in a different manner. In a textbox, she lists 10 possibilities for pizzas with two toppings, 10 possibilities for pizzas with three toppings, 5 possibilities for pizzas with four toppings, and one possibility for pizzas with five toppings.

Silvestre moves Suzyn’s arrangement of ellipses upwards. Suzyn also helps to move their arrangement as well. Silvestre then adds a plain pizza into their textbox of pizzas containing peppers and change their number of pizzas combinations in that textbox from fifteen to sixteen.

In the midst of moving Suzyn’s arrangement, Silvestre changes her method of rearrangement. They do not move the ellipse containing “Plain R” directly upwards; they move it directly below the textbox labeled “PEPPORONI.” The textboxes containing “Plain R” and “P/M/B” had been switched by Suzyn earlier. Silvestre replaces them into original rearrangement.

A bit later, Silvestre further rearranges the ellipses, and places one-topping pizzas in the top row, two-topping pizzas in the next two rows, three-topping pizzas in the next two rows, four-topping pizzas in the next row, and five-topping pizzas in the last row. The ellipse labeled “Plain R” is beneath the textbox labeled “PEPPORONI.” While placing three-topping pizzas into rows, Silvestre draws some ellipses.

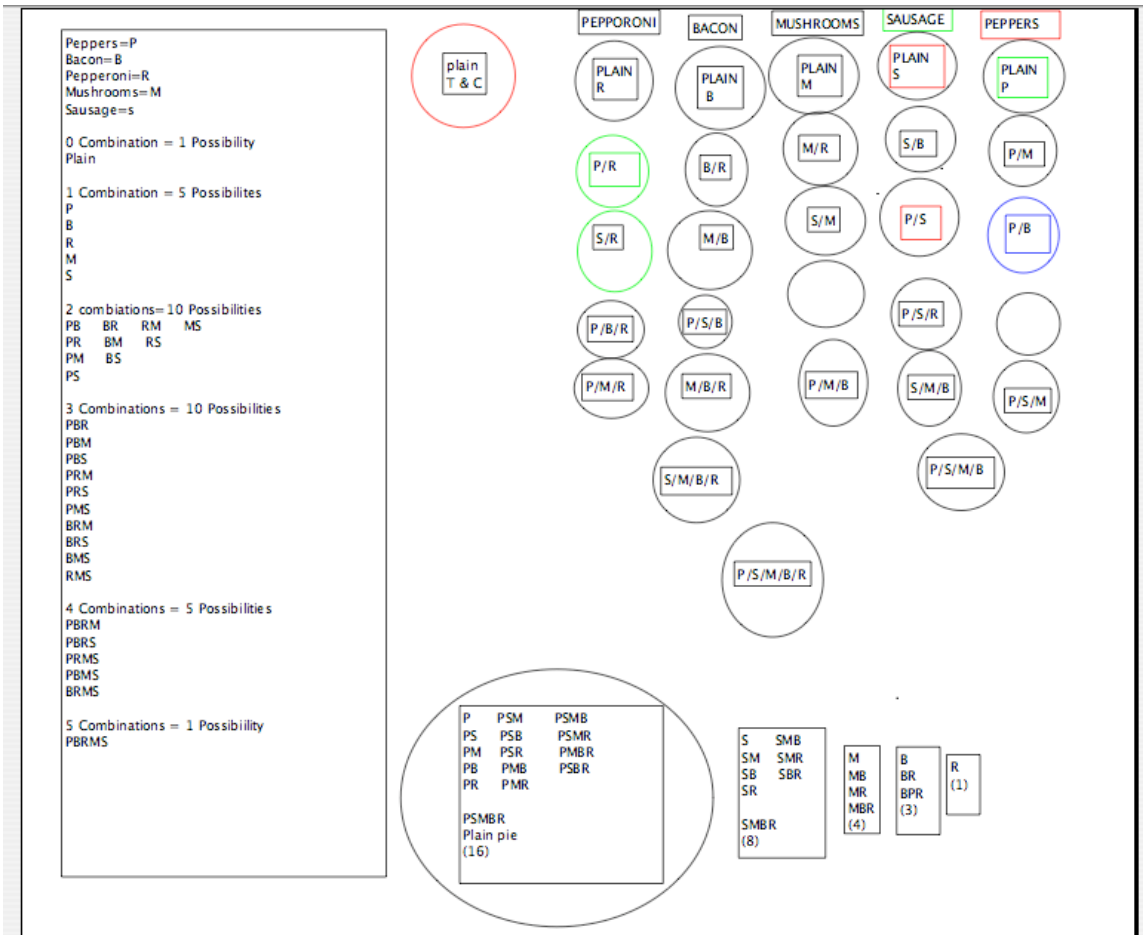


Figure 1. Screen shot of the shared, VMT Chat whiteboard containing both pairs’ final solutions.

RESULTS: Cognitive

With regards to our inquiry into the cognition of the team of participants, our investigation concerns two guiding questions: (1) How do learners interactively build (externally represented) mathematical meanings by collaborating in small groups, using a computer-mediated communication system? and (2) In the process, what mathematical ideas, heuristics, and reasoning do they develop? These are overarching questions of our research program. The data that we analyze here represents a small, preliminary case study. We present the results along several dimensions: interaction, heuristics, mathematical ideas, mathematical reasoning. Afterward, we discuss issues that emerge from our results and conclude with implications of our case study.

The data of this case surprised us in that the team communicates sparingly with chat text and mainly through whiteboard postings. From our experience, teams use the chat space to a much greater extent than this team does. Consequently, our analysis of the mathematical ideas and reasoning that the students engaged is not profoundly based on their textual communication but rather mainly on an examination of the evolution of their inscriptive whiteboard interactions.

Interaction

The student participants worked in pairs and the two pairs, as a team, interacted through the computer-mediated system, VMT Chat, using two interaction spaces, the chat and the whiteboard frames. The pairs used the chat room to work through the problem, with one student of each pair controlling the mouse and keyboard. The two pairs interacted with each other for the vast majority of the time through inscriptive postings on the whiteboard. In the nearly two hours of interaction, the students rarely used the chat frame to communicate with the other pair.

Our analyses of the data reveal how participants use the affordances of the VMT Chat environment, how they managed constraints they encountered in it, and what mathematical ideas, heuristics, and lines of reasoning are evident in their collaborative interactions. The initial work of the online group can be read as establishing its bearings. These include how to work within the affordances and constraints of the VMT environment, how to manage the shared workspace, and how to represent the object with which they will work.

Interactive Initiation of Inscriptive Phases

The two pairs of participants, collaborating in a single chat room, develop inscriptions or, more specifically, discursive objects or artifacts that serve to simultaneously represent and beget their mathematical ideas and reasoning as they build solutions to the problem. As Sfard (2000) notes, “mathematical discourse and its objects are *mutually constitutive*” (p. 47, original emphasis). While building their solutions, the development of discursive objects occurs in what we discern as phases. Phase 1 is initiated when the pair of participants, Silvestre, experiments with drawing ellipses, which seem to be analogous to pizza pies. The participant pair, Suzyn, then also experiments with drawing ellipses (see figure 1).

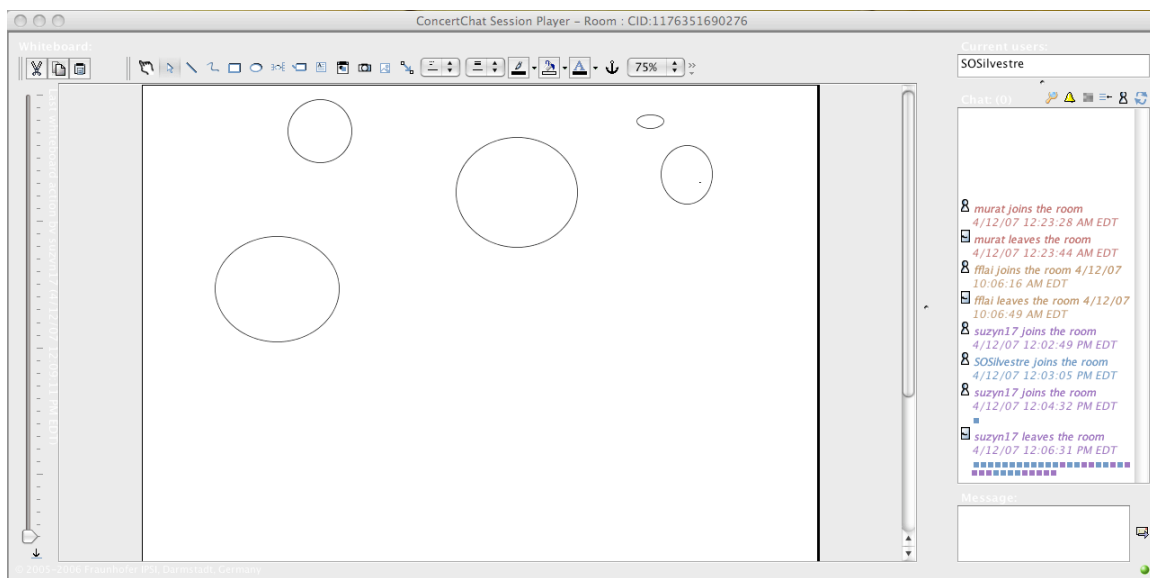


Figure 2. Screenshot of Phase 1: Suzyn and Silvestre initially experiment with drawing ellipses.

Phase 2 entails labeling ellipses. Suzyn types “Plain Pizza” in the chat frame and uses the reference tool to link this chat statement with an ellipse on the whiteboard. Afterward, Silvestre creates a textbox in an ellipse and types “plain T & C,” establishing that it perhaps is more convenient to indicate a pizza and its topping such as a plain tomato and cheese pizza with a textbox superimposed onto an ellipse rather than linking a chat statement with an element—an ellipse—drawn on the whiteboard. With this action, Silvestre appears to offer an implicit proposal (see Figures 3A and 3B).

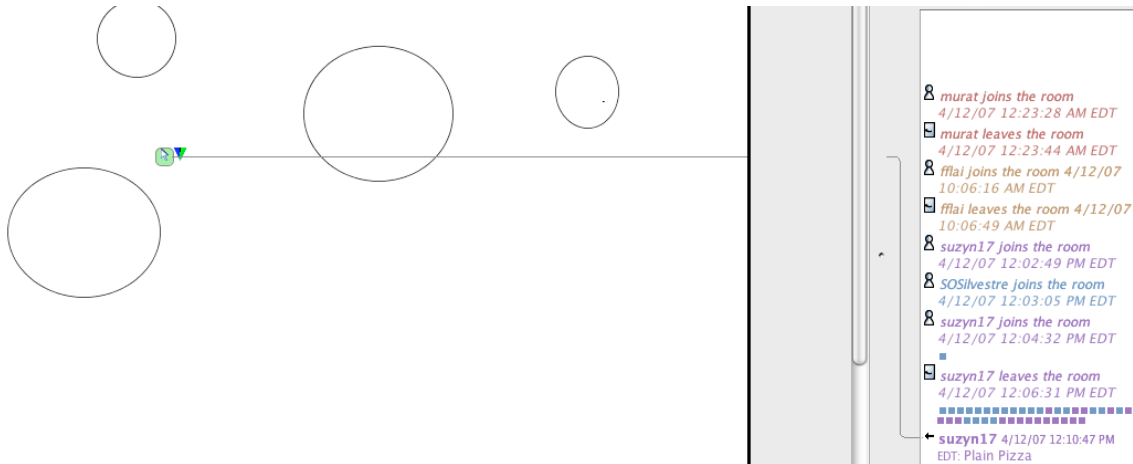


Figure 3A. Screenshot of Phase 2: Suzyn uses the reference tool to link her chat statement of “Plain Pizza” to a blank ellipse.

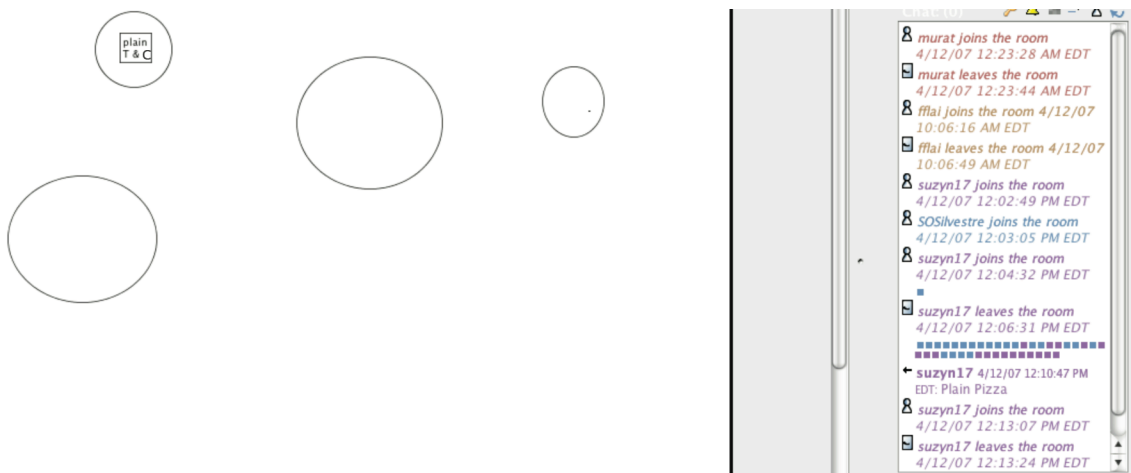


Figure 3B. Screenshot of Phase 2: Silvestre labels an ellipse with a textbox containing “plain T&C.”

Both labeling approaches seem to be cumbersome for the participants, and in the next phase, each pair modifies their approach. Now in phase 3, apparently influenced by Silvestre’s use of a textbox superimposed onto an ellipse, Suzyn incorporates this technique into her representation. Each ellipse that Suzyn creates is labeled with a textbox and represents a specific pizza with particular toppings. Suzyn employs this iconic representation for most of remaining time in which she works. By this point,

Silvestre and Suzyn type on separate parts of the whiteboard. Silvestre uses the left side while Suzyn uses the right side (see Figure 4A).

In their modified representations, each pair uses the symbol, P. However, what does P represent, peppers or pepperoni? Silvestre settles the question by creating a key in which she indicates what letter represents what topping: P for peppers, S for sausage, M for mushroom, B for Bacon, and R for pepperoni. In a different way, Suzyn also announces for what P stands. She creates a textbox, types “PEPPERS” into it, and lines up in a column under this heading her three pizzas that contain P: P/B, P/S, and P/M. Instead of an ellipse representing a class of pizzas, each ellipse represents a different pizza, differentiated from the others by its topping. These objects or pizzas are also similar to each other in that each contains two toppings, one of which is P, indicating that Suzyn is engaged with relations among objects. This pattern is indicative of thinking about grouping different, possible pizzas by cases. In this instance, the case is two-topping pizzas with each having P as a topping. Suzyn employs this iconic representation for most of remaining time in which she works (see Figure 4A).

While Suzyn modified her representation, also in phase 3, Silvestre changes her notational scheme and develops a symbolic inscription. To indicate the mathematical objects with which she is working, she types P/S, P/M, P/S, and P/R into a single textbox superimposed on an ellipse. Now, a single circular ellipse is not a single pizza but represents a class of pizzas. Her notation’s structure appears to be the following: a single pizza has two toppings and the toppings on a pizza are separated with slashes. Her inscription also indicates a relation among the objects with which she is engaged; namely, each object is a two-topping pizza with P as one of its toppings. Moreover, this pattern is suggestive of a strategy by which she may intend to list different possible pizzas. In this instance, it is grouping different, possible pizzas by cases. In this case, the case is two-topping pizzas with P as one of the two toppings (see Figure 4A).

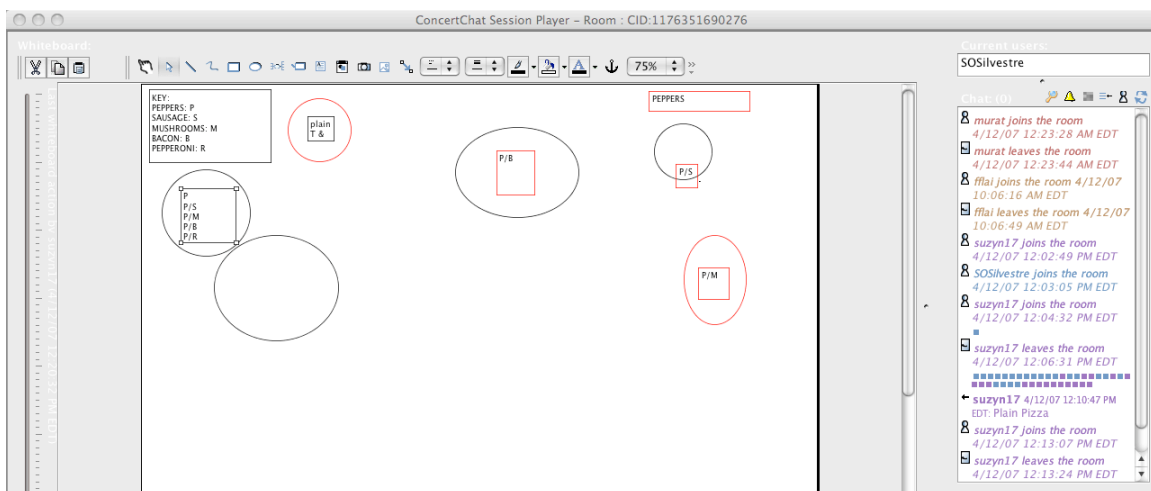


Figure 4A. Screenshot of Phase 3: Suzyn represents single pizzas by using textboxes to label ellipses; Silvestre creates a key and uses a symbolic inscription to represent her pizzas.

Silvestre further modifies her notational scheme by removing forward slashes. Instead of using P/M or P/S to represent pizzas with peppers and mushrooms or pizzas with peppers and sausages, respectively, Silvestre uses PM and PS to represent these

pizzas. In addition, she expands upon her representation and uses it to designate pizzas with more than two toppings. For instance, a pizza with sausages, bacon, and pepperoni is represented by SBR (see Figure 4B).

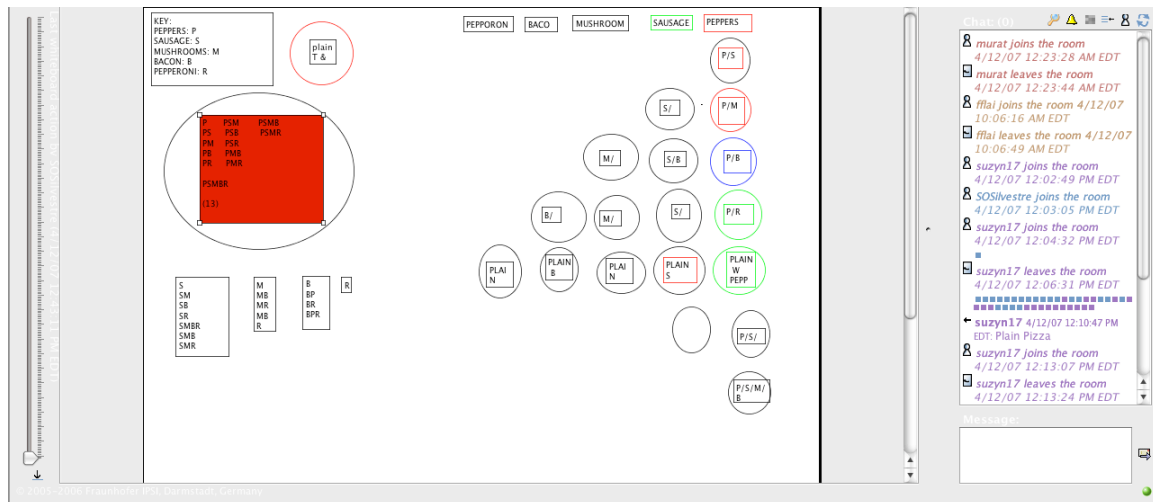


Figure 4B. Screenshot of Phase 3: Silvestre modifies and extends her notational scheme by removing forward slashes and representing pizzas with more than two toppings.

Finally, in phase 4, Suzyn seems to adopt Silvestre's notational inscription to display her way of reasoning about a solution to the problem. Suzyn develops a symbolic inscription. Placing each case within a textbox, Suzyn lists and enumerates pizzas containing certain numbers of "combinations." She lists 1 pizza with "0 Combinations" or no toppings, five pizzas with "1 Combination" or 1 topping, ten pizzas with "Two Combinations" or two toppings, ten pizzas with "Three Combinations" or three toppings, five pizzas with "Four Combinations" or four toppings, and one pizza with "Five Combinations" or five toppings. Like Silvestre, Suzyn uses combinations of letters as the objects with which she exhibits her thinking about different possible pizza pies and relationships among these possibilities, but groups her pizzas according to total number of toppings as opposed to common toppings (see Figure 5).

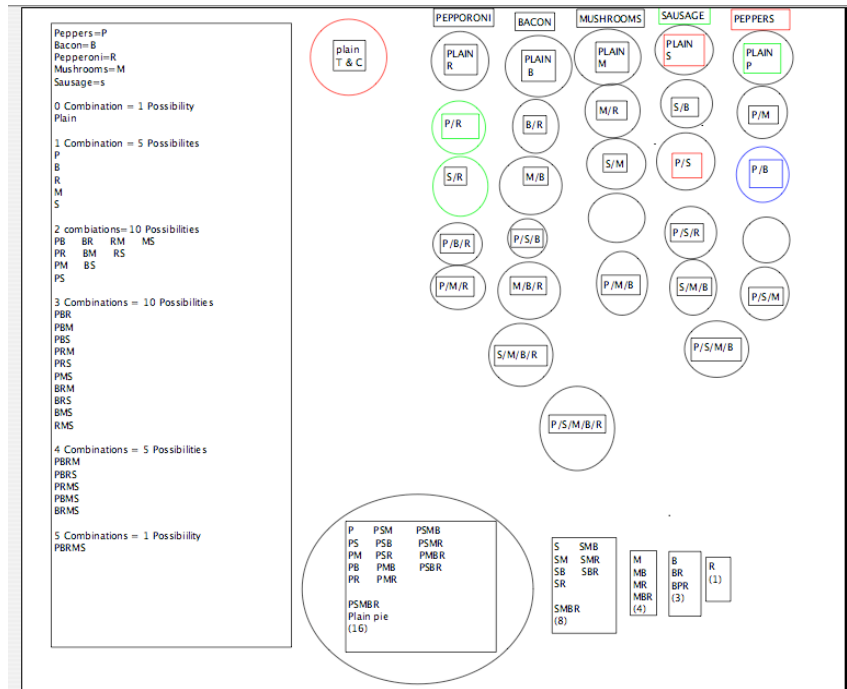


Figure 5. Screenshot of Phase IV: Suzyn adopts a symbolic notation.

Representing objects and engaging with relations among objects

As we have seen, the groups develop two different representations for the objects with which they develop mathematical ideas and reasoning. The pair designated by Suzyn uses initially an iconic inscription for each of their pizzas. It consists of an ellipse formed into a circle and, a textbox with letters. The letters P, S, M, B, and R are toppings and combinations of them of placed in a textbox atop an ellipse. Suzyn uses the two inscriptions—ellipse and a non-empty textbox—to represent a particular pizza pie.

Different from Suzyn’s iconic representation, Silvestre develops a symbolic inscription. She uses combinations of letters as the objects with which she exhibits her thinking about different possible pizza pies and relationships among these possibilities. For instance, P, S, M, B, and R stand for objects or pizza toppings and combinations of these letters such as M, PS, or SBR designate different possible pizzas. In their semiotic system, Silvestre uses a letter or combination of letters to represent both particular toppings and pizza pies with particular toppings. That is, P can stand for one of the available toppings (peppers) or a one-topping pizza (of peppers). Unlike Suzyn’s inscriptive system, where two distinct types of inscriptions represent toppings and pizzas with toppings, Silvestre’s symbols play dual roles. The economy of this semiotic system will later be appreciated by Suzyn and she will shift her notational usage.

Interestingly, although at the start of the group’s problem-solving session Silvestre initiated constructing ellipses on the whiteboard and use of textboxes to label an ellipse such as when she created a “plain T & C” pizza, the chore or cumbersomeness of drawing and labeling within the VMT-Chat system may have contributed to her development of another, more convenient representation. Drawing elliptical shapes and creating textboxes on the shared workspace are affordances of the system, which at same

time represent a constraint because of mechanical or motor difficulties involved in creating and coordinating these objects. This constraint may have impelled Silvestre to find a less representational, more symbolic, and therefore, computationally more powerful inscription.

With her inscriptive objects, Silvestre engages with relations among their objects. Just as Silvestre and Suzyn developed different representations, they also engage with different relations among the objects or pizzas. Suzyn indicate relations among her objects spatially by locating pizzas that contain a particular, common topping, like peppers, under a column head by the name of *the* common topping. The column headed by “Peppers” has four pizzas each containing peppers with one different other topping and one pizza with just peppers as its topping; the column headed by “Sausage” has three pizzas each containing sausage with one different other topping and one pizza with just sausages as the topping; the column headed by “Mushroom” has two pizzas each containing mushroom with one different other topping and one pizza with just mushrooms as the topping; the column headed by “Bacon” has one pizza containing bacon with one different other topping and one pizza with just bacon as the topping; the column headed by “Pepperoni” has one pizza with just peppers as the topping. Each successive column had one less pizza than the one before it because it does not include the topping used in the previous column. Suzyn seems to realize this before labeling her pizzas since, as she went along, she drew just the right number of ellipses under each column heading.

Silvestre presents her perception of relationships among objects, the different possible pizzas. In turn, she considers each available topping and, in separate textboxes, lists all possible pizzas that contain it as a topping (see the five textboxes at the bottom of Figure 1). That is, first, she lists all possible different pizzas containing P or peppers; second, all possible, different pizzas containing S or sausage, except for those that contain P since they were already accounted for; third, all possible, different pizzas containing M or mushroom, except for those that contain P or S since they have already been accounted for; fourth all possible, different pizzas containing B, except for those that contain P, S, or M since they have already been represented, and finally, all possible, different pizzas containing R, except for those that contain P, S, M, or B since they have already been indicated.

Engaging with dynamics linking different relations

The work of Suzyn and Silvestre evidence their engagement with dynamics linking different relations or, to say in other words, relations among relations. Silvestre listed, for example, pizzas containing peppers, P, in a textbox. This listing by itself is a relation. Ultimately, she arranged the possible pizzas containing in turn each of the five available toppings into separate textboxes. The textbox containing pepper pizzas is to the left of the textbox containing sausage pizzas, which is to the left of the textbox containing mushroom pizzas, which is to the left of the textbox containing bacon pizzas, which is to the left of the textbox containing the pepperoni pizza. Her inscriptive and spatial work indicates that Silvestre view each listing as distinct from the others. In this sense, she is also engaged with dynamics linking—by distinction—different relations.

In an analogous manner, Suzyn signals her engagement with relations among relations. She lists different possible pizzas by considering cases. In the long, rectangular textbox on the left in Figure 1, Suzyn lists in turn all possible pizzas with 0

toppings, 1 toppings, 2 toppings, 3 toppings, 4 toppings, and 5 toppings. Each case indexes a relation and is distinct from the others. Her listing indicates Suzyn's engagement with dynamics linking different relations.

Each of the student participants— Suzyn and Silvestre—considers different dynamics linking different relations. The structure of their thinking in this regard reveals different perceptions of the underlying mathematical structure of the problem. We elaborate on this in the discussion section below.

Inventing heuristics

Both Suzyn and Silvestre seemed to invent heuristics based on the resources within the VMT chat environment. For example, both started off by drawing ellipses using the ellipse tool. It seems that Suzyn then realized, after using the referencing tool to label an ellipse as a plain pizza, that the textbox could be better used for this purpose. Thus, for the early part of her work session, Suzyn used ellipses labeled by textboxes to represent their pizzas. Her solution representation is iconic.

For Silvestre, the drawing of ellipses may have seemed too cumbersome. Silvestre used a symbolic method of representation. Specifically, she used the textbox tool to list pizza possibilities. Within separate textboxes, Silvestre listed pizzas containing peppers as a topping, pizzas containing sausage as a topping that have not already been listed, pizzas containing mushroom as a topping that have not already been listed, pizzas containing bacon as a topping that have not already been listed, and pizzas containing pepperoni as a topping that have not already been listed.

Suzyn's evolution of heuristic from iconic representation to symbolic representation may have been influenced by Silvestre's use of symbolic representation in her solution method. That Silvestre was able to list more pizzas with her method than Suzyn was able to list with her iconic representation may have influenced Suzyn to use a symbolic representation to complete her solution. Interestingly, although both Suzyn and Silvestre end up with symbolic representations, their solutions are quite different.

Reasoning about possibilities

The work of the team and of each of the two pairs in the team exemplifies particular types of mathematical analysis: reasoning by cases and reasoning by controlling variables. The team of Silvestre and Suzyn begin their work by indicating possible pizzas with two toppings in which one is P, peppers. On the one hand, this line of reasoning continues to dominate the work of Suzyn throughout the session. Suzyn reasons by cases by counting and listing pizzas with one topping, pizzas with two toppings, pizzas with three toppings, and pizzas with four toppings. Suzyn continues reasoning by cases by listing additional combinations in her textbox. She lists the combination of no toppings, a plain pizza, and the combination of all toppings, a pizza containing peppers, sausages, mushrooms, bacon, and pepperoni.

On the other hand, Silvestre shifts from reasoning by cases to reasoning by controlling for variables. When Silvestre creates a textbox and type in four pizzas containing peppers as a topping with the combinations of peppers and sausage, peppers, and mushroom, peppers and sausage, and peppers and pepperoni, this is the first instance of reasoning by cases. Later on, that Silvestre controls for the variable P as they list one-, two-, three-, and four-topping pizzas containing peppers. Silvestre then creates a textbox

and lists pizzas containing sausages, sausages and two other toppings, and sausages and three other toppings.

Within each textbox, Silvestre also engages reasoning by cases. She adjusts her list of pizzas with peppers so that pizzas with one-, two-, three-, and four- topping pizzas all appear in separate columns. Before the adjustment, pizzas with three and four toppings appeared in the same column. In a similar fashion, Silvestre arranges her listing of pizzas containing sausage, not containing peppers, by grouping the possibilities according to the number of toppings.

DISCUSSION

Our aims were to investigate, based on data gathered from chat-room participants' mathematical problem solving within the VMT Chat environment, how to study chat-room participants' development of mathematical ideas and lines of reasoning and, in the interactive spaces of VMT Chat, what ideas and reasoning are evident.

In the following sections, we discuss the significance of the results in the light of our theoretical perspective and cognitive perspectives.

Discourse creating objects and objects shaping discourse

To explore, develop, and communicate their mathematical ideas, the four students—as Suzyn and Silvestre—interactively unfold an inscriptive system composed of objects as well as implicit relations among the objects and relations among the relations. After entering their assigned chat room in the VMT-Chat environment and after reading the statement of the Pizza Problem, the students experiment drawing circular ellipses and initially default to pictorial or iconic representations of pizzas. Suzyn types “Plain Pizza” and use the reference tool to link this chat statement to an ellipse on the whiteboard. Immediately afterward, Silvestre creates a textbox superimposed on an ellipse and type “plain T & C,” which we understand to mean a plain pizza of tomato and cheese. In subsequent actions of creating objects on the whiteboard, Suzyn incorporates Silvestre’s technique of drawing an ellipse and labeling it by typing into a textbox superimposed on the ellipse. Each ellipse that Suzyn creates is labeled with a textbox and represents a specific pizza with particular toppings, with each of the toppings separated by a slash. Each ellipse also represents a different pizza, differentiated from the others by its indicated toppings. As can be seen below in Figure 6, both Suzyn and Silvestre use ellipses in their representations of pizzas perhaps because representing pizzas in a pictorial manner makes the problem more personal, less abstract, and easier to work with in early stages of their thinking. Their initial discourse in the chat and whiteboard spaces concerns experiments with designs for the objects on which they will work.

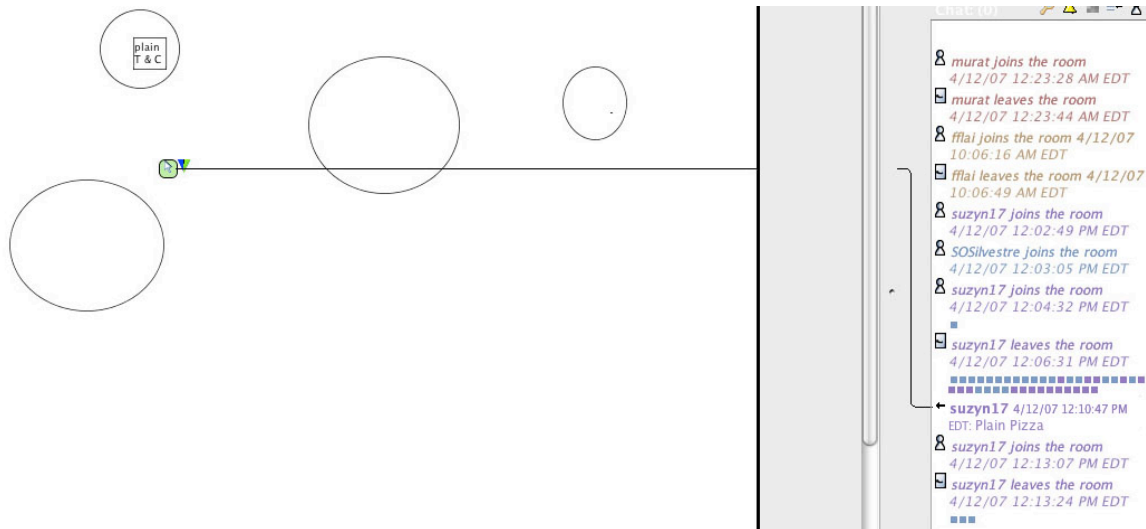


Figure 6. Screen shot of the VMT Chat whiteboard and chat spaces, showing how both dyads initially use ellipses to represent pizzas.

After Silvestre experiments with using an ellipse labeled with a textbox as a way of indicating pizzas, they change from an iconic to a symbolic representational scheme. The chore of drawing and labeling within the VMT-Chat system—a constraint of the environment—may have contributed to their development of another, more convenient representation. Drawing elliptical shapes and creating textboxes on the shared workspace are affordances of the system, but also represent a constraint because of mechanical or motor difficulties involved in creating and coordinating these objects. This constraint may have impelled Silvestre to conceive of a less pictorial, more symbolic, and therefore, computationally more powerful inscription.

To indicate the mathematical objects with which they are engaged, Silvestre's initial symbolic inscription involved a list of letters and slashes—P/S, P/M, P/S, and P/R—typed into a single textbox superimposed on an ellipse, indicating pizzas and their toppings (Figure 7). The structure of the notation appears to be the following: each group of two letters with a slash between them is a single pizza with two toppings with each topping indicated by a letter. The ellipse is not a single pizza but indexes a class of pizzas and a relation among them. The relation that they engage seems to be all two-topping pizzas containing peppers, P. This pattern may suggest how they may intend to list different possible pizzas, distinguishing classes of pizzas by means of ellipses.

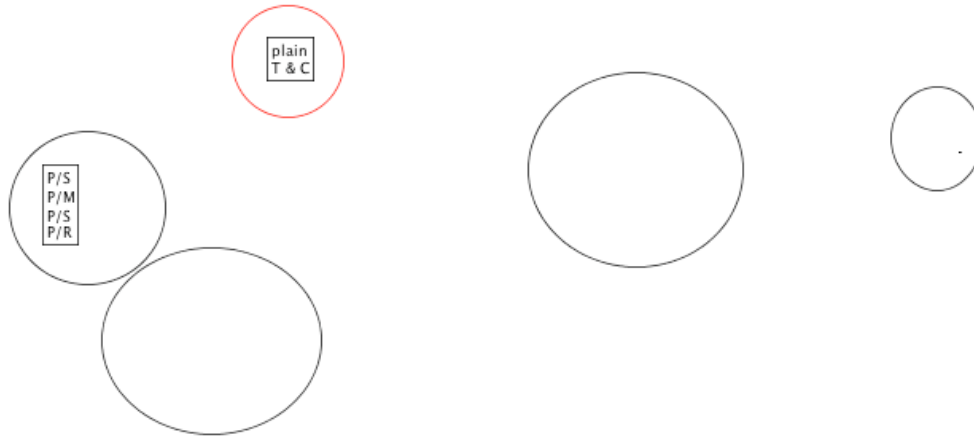


Figure 7. Screen shot of the VMT Chat whiteboard space, showing Silvestre’s initial symbolic inscription.

There is an interaction between Silvestre’s objects and her problem-solving strategy. The objects push her discourse in new directions. Silvestre modifies and extends her inscriptive and problem-solving strategy. She uses combinations of letters without slashes as objects to represent different possible pizza pies and relationships among these possibilities. For instance, P, S, M, B, and R stand for the five different pizza toppings and combinations of these letters such as M, PS, or SBR designate different possible pizzas. Silvestre uses a letter or combination of letters to represent both particular toppings and pizza pies with particular toppings. That is, P can stand for one of the available toppings (peppers) or a one-topping pizza (of peppers).

Silvestre’s development of a more cogent and computationally powerful inscription parallels shifts in her discourse. That is, this notational scheme allows them to not only illustrate pizzas with different combinations of topping but also to engage with patterns and relationships of these combinations and to use these patterns and relationships to engage with and illustrate relations among the relations. In her final solution, she presents in five different textboxes different classes of pizzas; first, all different possible pizzas containing peppers, P; second, all pizzas containing sausage, S, but not containing peppers; and so on. Examining these textboxes makes Silvestre’s strategy for listing pizzas evident. In the textbox containing pizzas containing peppers, a one-topping pizza containing peppers is listed first. Then, for each two-topping combination containing peppers, peppers is listed first, followed by single-topping combinations of sausages, mushrooms, bacon, or pepperoni, listed in this order. For each three-topping combination containing peppers, peppers is listed first, followed by two-topping combinations of sausages, mushrooms, bacon, or pepperoni. For each four-topping combination containing peppers, peppers is listed first, followed by three-topping combinations of sausages, mushrooms, bacon, or pepperoni. Finally, for each five-topping combination containing peppers, peppers is listed first, followed by four-topping combinations of sausages, mushrooms, bacon, and pepperoni. A similar strategy is followed for pizzas containing sausages, mushrooms, bacon, and pepperoni. Note that the order in which pizzas containing certain toppings are presented (pizzas containing peppers, pizzas containing sausages, pizzas containing mushrooms, pizzas containing

bacon, and finally pizzas containing pepperoni) is the same as the order of the toppings presented in each textbox (Figure 8).

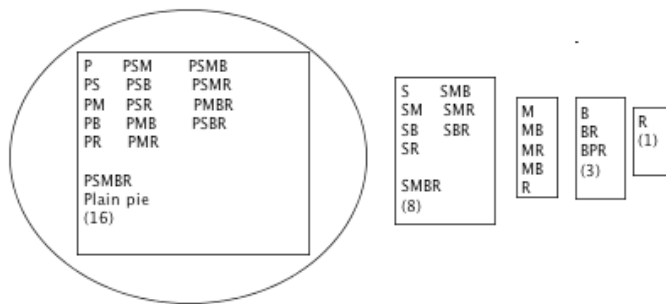


Figure 8. Screenshot of the VMT Chat whiteboard space showing Silvestre's final solution.

The content of the textboxes displays particular relations among the pizzas and the different textboxes distinguishes relations among these relations. This inscriptive system that Silvestre develops illustrates the theoretical point about how the signs learners choose and how their signs provide an analytic window into the signified field of conceptual material or ideas with which they engage (Powell, 2003).

The team's initial and later work to create and use their mathematical objects exemplifies another theoretical point. Sfard (2000) theorizes, "mathematical discourse and its objects are *mutually constitutive*" (p. 47, original emphasis). Through their discourse, they interactively develop approaches to represent the objects on which they work in their solution space. They interactively consider and modify an initial proposal for how to represent their objects—pizzas with particular toppings. Each pair elects to work with a different representation, one iconic and the other symbolic. The emergence to these inscriptive systems usher into the discourse two directions of work toward a solution of the problem. Indeed, the process of designing objects shapes their respective solution space. Silvestre's symbolic representation supports her reasoning about the different possible pizzas as collections in which they control variables, holding P (peppers) fixed and listing first all possible pizzas containing P. Whereas, though Suzyn's iconic representation supports her reasoning—cases defined by the number of toppings—it proves cumbersome and inefficient. Toward the end of the problem solving session, she abandons it in favor of Silvestre's symbolic representation. The iconic representation communicates the physicality of a pizza—an ellipse—and in a textbox displays its toppings. The symbolic representation—concatenated letters—simultaneously lists the toppings of a pizza and stands for the pizza itself. The meaning of the objects and the meaning presented through the objects are constituted through their use. From their discursive interactions in the two interactive spaces of VMT Chat, the team implicitly agree that what distinguish pizzas from one another are their toppings. Therefore, it is sufficient to list their toppings without having to draw pictures of pizzas. Through their discursive interaction the team constitutes their objects and, in turn, their objects shape and advance their discourse. This point is further evidenced in the next section.

Dyads influencing dyads

When Suzyn and Silvestre enter the VMT space, they both begin by drawing ellipses. Suzyn uses a chat posting of “Plain pizza” to link to one of her ellipses as a way of labeling it as a plain pizza. Silvestre takes one of her ellipses and places a textbox inside, and labels it “plain T&C.” Suzyn seems to be influenced by this and subsequently use this notational scheme to develop her solution.

While both groups use letters to represent the pizza toppings, because two toppings share the same first letter, the letter P can represent either peppers or pepperoni. Silvestre settles this problem by creating a key in which she indicates what letter represents what topping: P for peppers, S for sausage, M for mushroom, B for Bacon, and R for pepperoni. In a different way, Suzyn also announce for what P stands. She creates a textbox, types “PEPPERS” into it, and lines up in a column under this heading her three pizzas that contain P. After Silvestre creates this key, Suzyn appears to adopt Silvestre’s notation.

Unlike Suzyn’s inscriptive system, where two distinct types of inscriptions represent toppings and pizzas with toppings, Silvestre’s symbols play dual roles. The economy of this semiotic system will later be appreciated by Suzyn and she will shift her notational usage to a symbolic notational scheme.

Throughout the session, both Suzyn and Silvestre influenced each other in various ways. In the beginning of the session, Silvestre was influenced by Suzyn to use ellipses to represent pizzas, but used textboxes instead of linked chat statements to label the ellipse. Later on, Silvestre switched to a symbolic representation of pizzas, perhaps seeing that the iconic representation was not suitable for generating large numbers of possibilities. Near the end, that Silvestre was able to list more pizzas with her method than Suzyn was able to list with her iconic representation may have influenced Suzyn to use a symbolic representation to complete her solution.

Dyads reasoning differently about pizzas

Interestingly, although both Suzyn and Silvestre end up with symbolic representations, their solutions are quite different. Silvestre groups her pizzas by placing all pizzas involving a certain topping into separate textboxes. One textbox contains all pizzas with peppers as a topping, a second textbox contains all pizzas with sausages but not peppers as topping, a third textbox contains all pizzas with mushrooms but not sausages or peppers as a topping, a fourth textbox contains all pizzas with bacon but not mushrooms, sausages, or peppers as a topping, and a fifth textbox contains all pizzas with pepperoni but not bacon, mushrooms, sausages, or peppers as a topping. Silvestre’s solution indicates that she is reasoning by cases, with each case being all pizzas containing a certain topping. Each successive case following the first case does not include pizzas containing the particular topping used in a previous case.

Suzyn’s symbolic inscriptive system is different from the one used by Silvestre. While Suzyn is also reasoning by cases, the cases being used are different. Placing each case within a textbox, Suzyn lists and enumerates pizzas containing certain numbers of “combinations.” She lists 1 pizza with “0 Combinations” or no toppings, five pizzas with “1 Combination” or 1 topping, ten pizzas with “Two Combinations” or two toppings, ten pizzas with “Three Combinations” or three toppings, five pizzas with “Four

Combinations” or four toppings, and one pizza with “Five Combinations” or five toppings.

It seems that by the end, Suzyn’s thinking has progressed to the point where Silvestre was earlier in their thinking. Both are now thinking beyond the idea of just generating different pizzas, but of producing combinations of toppings to generate patterns, and to use these patterns to ensure that they have accounted for all combinations.

Cognitive significance

As a team, Suzyn and Silvestre built sophisticated cognitive structures that can provide insight into Pascal’s triangle and combinatorial analyses. From an analytical viewpoint, we find these structures to be significant since they establish cognitive foundations upon which students can build and extend their understanding of binomial structures. In Figure 1, Silvestre’s representation of her solution nearly mimics successive rows of Pascal’s Triangle. Her listing of pizzas with peppers almost represents the fourth row of Pascal’s Triangle (see Figure 9). First, she lists a pizza with peppers, pizzas with peppers and one other topping, pizzas with peppers and two other toppings, pizzas with peppers and three other toppings, and a pizza with peppers and four other toppings. The number of pizzas in each of these sub-categories is the same as one of the numbers in the fourth row of Pascal’s Triangle: 1 4 6 4 1. That is, there is one pizza with peppers only, four pizzas with peppers and one other topping, six pizzas with peppers and two other toppings, four pizzas with peppers and three other toppings, and one pizza with peppers and four other toppings.

Zeroth row	1
First row	1 1
Second row	1 2 1
Third row	1 3 3 1
Fourth row	1 4 6 4 1
Fifth row	1 5 10 10 5 1

Figure 9. The zeroth to fifth rows of Pascal’s triangle.

Combinatorially speaking, for the case of all possible pizzas containing peppers, since each pizza must have peppers as a topping, there are four remaining toppings from which to choose. Using combinatorial notation, $\binom{n}{r}$, which means the number of ways to select r items from a collection of n of them, the following are the possible pizzas containing peppers:

- pizzas with peppers only = $\binom{4}{0} = 1$, since out of four choices of toppings none are chosen,

- pizzas with peppers and one other topping = $\binom{4}{1} = 4$, since out of four choices of topping one is chosen,
- pizzas with peppers and two other toppings = $\binom{4}{2} = 6$, since out of four choices of topping two are chosen,
- pizzas with peppers and three other toppings = $\binom{4}{3} = 4$, since out of four choices of topping three are chosen, and
- pizzas with peppers and four other toppings = $\binom{4}{4} = 1$, since out of four choices of topping four are chosen.

In Silvestre’s representation of all possible pizzas with peppers as a topping, she misses the pizza with peppers, bacon, and pepperoni, and instead lists it as a pizza with bacon, peppers and pepperoni under their listing with pizzas with bacon as a topping (see Figure 1). They also place within their listing of pizzas with peppers as a topping a plain pizza. Aside from these two inconsistencies, Silvestre’s listing of pizzas with sausages as a topping, mushrooms as a topping, bacon as a topping, and pepperoni as a topping represent the third, second, first, and zeroth rows of Pascal’s Triangle, respectively, and can also be described in a combinatorial fashion as above. We illustrate this in the Table 3 below. Finally, Silvestre’s solution method represents the sum of all rows of Pascal’s triangle previous to the fifth row plus one.

Table 3.

Silvestre’s solution method represented as the fourth to zeroth rows of Pascal’s triangle

Peppers	Sausage	Mushroom	Bacon	Pepperoni
$P = \binom{4}{0}$	$S = \binom{3}{0}$	$M = \binom{2}{0}$	$B = \binom{1}{0}$	$R = \binom{0}{0}$
$\left. \begin{matrix} PS \\ PM \\ PB \\ PR \end{matrix} \right\} = \binom{4}{1}$	$\left. \begin{matrix} SM \\ SB \\ SR \end{matrix} \right\} = \binom{3}{1}$	$\left. \begin{matrix} MB \\ MR \end{matrix} \right\} = \binom{2}{1}$	$BR = \binom{1}{1}$	$1 = 1$
	$\left. \begin{matrix} SMB \\ SMR \\ SBR \end{matrix} \right\} = \binom{3}{2}$	$MBR = \binom{2}{2}$	$2 = 1 + 1$	
	$SMBR = \binom{3}{3}$	$4 = 1 + 2 + 1$		

$\left. \begin{array}{l} \text{PSM} \\ \text{PSB} \\ \text{PSR} \\ \text{PMB} \\ \text{PMR} \\ \text{PBR} \end{array} \right\} = \binom{4}{2}$	$8 = 1 + 3 + 3 + 1$			
$\left. \begin{array}{l} \text{PSMBR} \\ \text{PSMR} \\ \text{PSBR} \\ \text{PMBR} \end{array} \right\} = \binom{4}{3}$				
$\text{PSMBR} = \binom{4}{4}$				
$16 = 1 + 4 + 6 + 4 + 1$				

In contrast to Silvestre’s solution method, which represents the sums of each of the first five rows of Pascal’s Triangle, Suzyn’s solution method mimics the sixth row of Pascal’s Triangle: 1 5 10 10 5 1. Within a textbox, she first lists at the top a key containing abbreviations for each of the toppings (Peppers = P, Bacon = B, Pepperoni = R, Mushrooms = M, and Sausage = S), then underneath in successive rows they list pizzas under the following headings:

- “0 Combination = 1 Possibility” (One possible pizza with no toppings)
- “1 Combination = 5 Possibilities” (Five possible pizzas with one topping)
- “2 Combinations = 10 Possibilities” (Ten possible pizzas with two toppings)
- “3 Combinations = 10 Possibilities” (Ten possible pizzas with three toppings)
- “4 Combinations = 5 Possibilities (Five possible pizzas with four toppings), and
- “5 Combinations = 1 Possibility (One possible pizzas with five toppings).

As seen above, the number of pizzas described by each heading is the same as an entry in the fifth row of Pascal’s Triangle. This solution also has a combinatorial explanation. The following are all possible pizzas with no toppings, one, two, three, four, and five toppings when choosing from five different toppings:

- pizzas with no toppings = $\binom{5}{0} = 1$, since out of five choices of toppings, none are chosen,
- pizzas with one topping = $\binom{5}{1} = 5$, since out of five choices of topping, one is being chosen,

- pizzas with two toppings = $\binom{5}{2} = 10$, since out of five choices of topping, two are being chosen,
- pizzas with three toppings = $\binom{5}{3} = 10$, since out of five choices of topping, three are being chosen,
- pizzas with four toppings = $\binom{5}{4} = 5$, since out of five choices of topping, four are being chosen, and
- pizzas with five toppings = $\binom{5}{5} = 1$, since out of five choices of topping, five are being chosen.

In our analysis of the data, we did not find evidence in the students' discourse that they were aware of Pascal's triangle or of the mathematics of binomial structures. Nevertheless, if the student sessions were to be continued over a longer period of time, students could be engaged with other problems that would provide them with opportunities to construct mathematical ideas and frameworks that underlie the rich concepts and structures of Pascal's arithmetic triangle (Edwards, 1987). In a mathematics classroom or during virtual mathematics problem-solving sessions, to promote the construction of ideas and framework, a curriculum unit could be built around a sequence of open-ended, well-designed mathematics tasks that engage teams of students with binomial structures in varied contexts. To make information on Pascal's triangle available to students as they are solving these tasks, we may do one of several things: Either an online moderator can direct students while they are in VMT Chat to websites featuring discussions of Pascal's triangle, or a time-sensitive Wiki on Pascal's triangle may be made available to students only after they have completed a certain problem within a sequence of related problems.

In our study, the sophisticated cognitive structure that the two pairs build and made available on the shared whiteboard for each other emerged from interactional work. After the first ten minutes of nearly 120 minutes of work, the two pairs of students started to work as two separate units. In this sense, they were like two entities of a single dyad. In the psychological literature on problem solving, it is accepted that when a dyad is engaged in solving a problem that typically one entity begins to solve the problem while the other listens to the ensuing solution attempt (cf., Shirouzu, Miyake, & Masukawa, 2002). The speaker may be talking out loud while solving a problem while her partner listens. Analogously, one pair presenting their solution on the whiteboard is like a speaker talking aloud their problem-solving process. However, the data of this case study shows that instead both entities of the dyad simultaneously "talked" aloud their ensuing solution and that the non-ephemeral nature of their communication medium allowed each entity to "hear" the other while "talking" aloud their problem-solving attempt. An affordance of the virtual environment may have allowed for this simultaneous solving of the problem by both entities of the dyad. In a traditional dyad, it would be difficult for both members to solve a problem out loud while paying attention to each other as well as to her own work, because two people cannot speak at once. Moreover, it is difficult to

think in one way when a different way of thinking is being described aloud. In this virtual environment, perhaps because the workspace is shared, relatively large, equally visible to both pairs, and communication is non-aural, it is easier for both pairs to go about problem solving individually while still paying attention to what the other group was doing.

As we have attempted to demonstrate, while solving the Pizza Problem, the interactional and collaborative work of Silvestre and Suzyn establishes important mathematical bases for their future action. These include ways of reasoning mathematically as well as particular combinatorial structures. Stahl (2006) suggests that “[t]he being-there-together in a chat is temporally structured as a world of future possible activities with shared meaningful objects” (2006, p. 115). The interactive work of the four students in the chat room that we have analyzed leaves them with tools for future collaboration. Interactively, they have built a discursive world of mathematical entities with which to engage particular combinatorial ideas and lines of reasoning. Silvestre and Suzyn experience together interacting in the chat room, and this leaves them prepared for further collaborative, mathematical actions with sets of shared, meaningful mathematical objects, relations among the objects, and dynamics linking relations.

References

- Brown, S. I., & Walter, M. I. (1983). *The art of problem posing*. Philadelphia: The Franklin Institute.
- Cai, J., & Lester, Jr., F. K. (2005). Solution representations and pedagogical representations in Chinese and U.S. classrooms. *Journal of Mathematical Behavior*, 24(3/4), 221-237.
- D'Ambrosio, U. (2001). *Etnomatemática: Elo entre as tradições e a modernidade [Ethnomathematics: Link between tradition and modernity]*. Belo Horizonte, MG: Autêntica.
- Davis, B., Sumara, D., & Luce-Kapler, R. (2000). *Engaging minds: Learning and teaching in a complex world*. Mahwah, NJ: Lawrence Erlbaum.
- Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 99-131). Mahwah, NJ: Lawrence Erlbaum Associates.
- Edwards, A. W. F. (1987). *Pascal's arithmetical triangle*. London: Griffin & Oxford.
- Engel, A. (1997). *Problem-solving strategies*. New York: Springer.
- Gattegno, C. (1988). *The science of education: Part 2B: The awareness of mathematization*. New York: Educational Solution.
- Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. New York: Cambridge.
- Lesh, R., & Lehrer, R. (Eds.). (2000). *Handbook of research data design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum.
- Mason, J., Burton, L., & Stacey, K. (1984). *Thinking mathematically*. London: Addison Wesley.
- Mason, J. H. (1988). *Learning and doing mathematics*. London: Macmillan.
- Mayer, R. E. (1992). *Thinking, problem solving, cognition* (2nd ed.). New York: Freeman.
- Mayer, R. E., & Wittrock, M. C. (1996). Problem-solving transfer. In D. C. Berlin & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 47-62). New York: Macmillan.
- Pirie, S., & Schwarzenberger, R. (1988). Mathematical discussion and mathematical understanding. *Educational Studies in Mathematics*, 19(4), 459-470.
- Polya, G. (1945/1973). *How to solve it: A new aspect of mathematical method*. Princeton: Princeton University.
- Powell, A. B. (2003). *"So let's prove it!": Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task*. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick.
- Powell, A. B., & Frankenstein, M. (Eds.). (1997). *Ethnomathematics: Challenging eurocentrism in mathematics education*. Albany, New York: State University of New York.
- Saussure, F. d. (1983). *Course in general linguistics* (R. Harris, Trans.). London: Duckworth.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

- Sfard, A. (2000). Symbolizing mathematical reality into being—Or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Lawrence Erlbaum Associates.
- Shirouzu, H., Miyake, N., & Masukawa, H. (2002). Cognitively active externalization for situated reflection. *Cognitive Science*, 26(4), 469-501.
- Speiser, B., Walter, C., & Maher, C. A. (2003). Representing motion: An experiment in learning. *The Journal of Mathematical Behavior*, 22(1), 1-35.
- Speiser, B., Walter, C., & Shull, B. (2002). Preservice teachers undertake division in base five: How inscriptions support thinking and communication. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Athens, Georgia)* (Vol. 3, pp. 1153-1162). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
- Stahl, G. (2006). Supporting group cognition in an online chat community: A cognitive tool for small-group referencing in text chat. *Journal of Educational Computing Research*, 35(2), 103-122.
- Stahl, G., Koschmann, T., & Suthers, D. (2006). Computer-supported collaborative learning: An historical perspective. In R. K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 409-426). Cambridge: Cambridge.
- Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
- Struik, D. J. (1948/1967). *A concise history of mathematics* (3rd Revised ed.). New York: Dover.