CHAPTER 1:  INTRODUCTION

1.1 Nature of the Study

Situated in the discipline of mathematics education, this is a microanalytic, epistemological, and discursive investigation into specific contents of the mathematical awarenesses of learners as they collaborate to resolve a problem task. I use the verb ‘resolve’ instead of ‘solve’ to underscore that the focus my inquiry is not with the or even a solution to the specific problem task given but rather with the mathematical contours and contents of the processes in which the learners engage to satisfy themselves and researchers that they have developed a response to the problem with which they consider both sufficiently satisfying and impervious to challenge, at least for an interval of in their lives, since like most responses to problems, theirs too may be provisional and temporal and depend on context.

The investigation is microanalytic since it focuses in detail on the work of four learners, and based on their intellectual actions and products, the study illustrates possibilities for larger groups of learners. Furthermore, the investigation concerns epistemology in that it is interested in understanding not only the resolutions the learners develop but also, and more fascinatingly and importantly, how they build mathematical ideas and forms of reasoning that in some instances they discard and in others use to formulate a resolution of the problem task. The materials for inquiry are the learners’ discourse and inscriptions. With these materials and the analytic tools described in chapter 3, this study aims to understand what the learners’ discursive and inscriptive products reveal about their mathematical ideas and forms of reasoning. In these
senses, this investigation involves microanalysis, discourse analysis, and epistemology.

1.2 The Setting and the Participants

This study is based on a problem-solving session. At the time of the session, however, the participants and the context of the session have a 12-year history. The research session is part of a longitudinal, cross-sectional study, currently in its 15th year, supported by grants MDR-9053597, directed by Robert B. Davis and Carolyn A. Maher, and REC-9814846, directed by Carolyn A. Maher, from the National Science Foundation.1 The problem-solving session for this research occurred near the end of the school year on 5 May 2000 at the David Brearley High School.2 Located in Kenilworth, New Jersey, a diverse working-class, immigrant, community, the high school belongs to a school district that during the research session for this study already had a twelve-year partnership with Rutgers University and the Robert B. Davis Institute for Learning. The Rutgers-Kenilworth partnership established itself to create classroom environments in which children would be engaged actively in building mathematical models on an experiential foundation and in which curriculum would be based on students’ construction of meaning (see, for instance, Maher.

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1 Any opinions, findings, and conclusions and recommendations expressed in this dissertation are those of the author and do not necessarily reflect the views of the National Science Foundation.
2 The longitudinal study is an outgrowth of a three-year teacher development project that began in 1984 in the Harding Elementary School in Kenilworth. See O’Brien (1995) for a ten-year analysis of this teacher-development project. At the same school, the longitudinal study itself was initiated in 1989 in a class of 18 first-graders, one of three such classes. As Maher (2002) relates, “[t]he students in each class remained together for their first three years of elementary school. In grade four, new classes were formed. The study continued with a smaller subset of the original class and several other students who joined” (p. 32). Seven of the original students have been followed for 15 years, including 3 years while they attended their respective universities. The participants of this study are a subset of the seven original students.
Of significance to this study, over the span of the Rutgers longitudinal study in Kenilworth, researchers (Maher, Martino, & Pantozzi, 1995) have observed shifts in student interactions in problem-solving sessions. Significantly, student interactions have shifted toward growing independence in mathematical thinking and deeper listening to other students. When working collaboratively on a problem, participants of the longitudinal study have developed the mind habit of reflecting on their own ideas and on those of others.

The participants in this investigation are four students—Brian, Jeff, Michael, and Romina—in their senior year of high school. From their entry into first grade, they have participated in mathematical activities of the Rutgers longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, probability, and calculus both in the context of classroom investigations as well as in after school settings (Maher, 2002; Speiser, Walter, & Maher, in press).

1.3 The Problem Task

The problem-solving session for this study was held in a classroom during the late afternoon, after school hours. During the session, which lasted about 1 hour and 40 minutes, the four students collaborate on a culminating task of the research strand on combinatorics that I suggested and helped design—The Taxicab Problem:

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the
intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

Accompanying this problem statement, the participants have a map, actually, a 6 x 6 rectangular grid (see Figure 1) on which the left, uppermost intersection point represents the taxi stand. The three passengers are positioned at different intersections as blue, red, and green dots, respectively, while their respective taxicab distances from the taxi stand are one unit east and four units south, four units east and three units south, and five units east and five units south.

![Figure 1](image)

*Figure 1.* The grid of the Taxicab Problem, one quadrant of the taxicab plane.

This task is a combinatorial problem and set in a non-Euclidean context. See Appendix A for the task as it was actually given to the participants. The problem task has an underlying mathematical structure and contains concepts
that resonate with those of other problems on which the students have worked. See Appendix B for the sequence of combinatorial problem tasks on which the participants have worked preceding the task of this study. Working on the present problem task, the participants revisit mathematical ideas they have already built as well as to deepen those ideas and build new ones. The next section details the mathematical environment of the problem task and its curricular significance.

1.4 The Taxicab Problem and Its Significance

The Taxicab Problem lies at the crossroads of combinatorial analysis and non-Euclidean geometry. The particular non-Euclidean geometry in which the problem is embedded has come to be known as taxicab geometry (Golland, 1990; Golland, McGuinness, & Sklar, 1994; Krause, 1973, 1975/1986; Menger, 1952, 1979). In one of his texts on learning and pedagogy, among other topics, Polya (1981) discusses mathematical recursion and, in this context, poses the combinatorial problem of determining “the number of different shortest paths in a network between given endpoints” (p. 68). In relation to this general enumeration problem, the Taxicab Problem is a specialized instantiation of it, situated in the two-dimensional space of Menger’s taxicab geometry.

As Menger (1979) recounts, it was Hermann Minkowski (1864-1909), in a collection of his papers (1911/1967), Gesammelte Abhandlungen, published posthumously at the beginning of the Twentieth Century, who defined a metric concept “according to which a circle is indeed a square” (p. 217). Menger himself also contributed to this geometry. According to Golland (1990), he initiated the systematic development of abstract distance geometry in 1928 and was the first
to use the appellation “taxicab geometry.” The name appears in a booklet, *You Will Like Geometry*, a publication accompanying a geometry exhibit that Menger established in 1952 at the Museum of Science and Industry in Chicago, Illinois (Menger, 1952, p. 5). Menger’s exhibit attracted the attention of mathematicians as well as non-mathematicians. The taxicab metric and its conceptual implications for the spatial quality of a circle immediately caught the attention of other mathematicians (see, for example, Curtis, 1953).

Besides pedagogical and mathematical purposes, Menger was also interested in the defining metric of taxicab geometry for philosophical and polemical reasons. It is the particular case for \( n = 1 \) of Minkowski’s generalized distance function \( \left[ |x_1 - x_2|^n + |y_1 - y_2|^n \right]^{\frac{1}{n}} \). Given a fixed point with integral coordinates in the taxicab plane, \((x_0, y_0)\), using the metric of taxicab geometry, the equation—

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\left| x_0 - x \right| + \left| y_0 - y \right| = 1
\]

defines the locus of points that lie a unit distance from \((x_0, y_0)\). That is, this taxicab equation of a circle defines a unit square and thus is an example of a square circle. With this idea, Menger (1979) presents contrary evidence to the then prevailing philosophical thought:

Round squares or square circles have been considered as the pinnacle of absurdity and the quintessence of impossibility by practically all philosophers and philosophical schools in antiquity, in the middle ages and in modern times. In fact, their inconceivability has been one of the few points of universal consensus, even after Minkowski in the early 1900’s proved their existence in a new non-Euclidean geometry, then
considered as ivory tower mathematics. But, years ago, I brought square circles literally to the man in the street. I showed that in a modern city the end points of all taxi rides starting at the same point and having the same length (as proved by fares)—in other words, the points of a circle in the taxicab geometry—lie on a square. (p. 4)

As he further states elsewhere, with the metric of taxicab geometry, mathematicians give “a perfectly meaningful interpretation to the paramount example of what philosophers call meaningless” (Golland et al., 1994, p. 28).

The taxicab metric makes meaningful seemingly contradictory categories: square circle and circular squares. However, this is not its only virtue. Krause (1973; 1975/1986) argues that taxicab geometry’s accessibility and closeness to Euclidean geometry renders it more pedagogically appropriate than other non-Euclidean geometries. He advocates its educational value for developing the mathematical faculties of secondary school students. Following Krause’s lead, other mathematicians and mathematics educators (Borasi, 1981, 1991; Laatsh, 1982; Prevost, 1998; Schattschneider, 1984; Sowell, 1989) have further developed investigations and applications of taxicab geometry. Some of these intersect with areas outside of geometry such as analysis and algebra. The problem used in this study—the Taxicab Problem—among others, involves ideas reflecting Pascal’s triangle, binomial coefficients, arithmetical progressions, combinatorial analysis, and distinction between Euclidean and taxicab geometries such as their planar structures, their metrics, and the uniqueness of a path between two distinct points. In textual materials designed to introduce learners to taxicab geometry or to combinatorial analysis, a path problem similar to the one of this study is often

1.5 Research Purpose and Guiding Questions

More than a decade ago, Davis (1992a) challenged mathematics education researchers to study the emergence among learners of what lies at the core of mathematics: mathematical ideas. He noted that “very little research in mathematics education has focused on the actual ideas in students’ minds or on how well teachers are able to identify these ideas, interact with them, and help students improve on them” (p. 732). This study takes up this challenge and posits that such ideas can manifest themselves in learners’ discursive interactions and inscriptive products. Through an analysis of these complex interactions and products, this investigation has global as well as local purposes. Globally, it aims to contribute basic scientific understanding of cognitive and discursive behaviors for which mathematical ideas, forms of reasoning, and mathematics learning emerge as by-products of sense making. Locally, by inviting the participants to engage the Taxicab Problem, this investigation endeavors to identify, examine, and illustrate mathematical ideas that the four participants build through their discursive acts.

The problem task is an investigative and cognitively demanding activity. The task admits more than one resolution strategy, each capable of being represented in multiple ways. It requires learners individually and collaboratively to impose meaning and structure on it, make decisions about what to do and how to do it, interpret the reasonableness of their actions, formulate conjectures, verify their conjectures, and justify their procedures and
resolutions. In short, the problem task requires learners to reason mathematically.

The analytic foci of this investigation are on talk and inscription. Philosophically, whether revealed in talk or in writing, this investigation views that mathematical ideas, including objects and relations, as constituted in discursive activities, not as preexisting and waiting to be noticed. That is, through communicative acts, mathematical ideas derive their existence and meaning, shape and content.

The participants have been part of the Rutgers longitudinal study and, in the course of it, have worked on problem tasks in which they have built mathematical ideas similar to the underlying mathematical structure of the Taxicab Problem. In the context of this study, this problem task was proposed with the following four central guiding research questions:

1. As the participants engage collaboratively, without assigned roles, to understand and resolve the Taxicab Problem what are the features and functions of learners’ discursive practices?
2. As they articulate their emergent understandings of the problem task, what mathematical ideas, associated meanings, and forms of reasoning do they reveal in their discourse?
3. How do participants structure their investigation?
4. How do their conversational exchanges support advances in their problem solving?

1.6 Mathematical Microcultural Norms of the Participants

Over the course of their twelve-year participation in the Rutgers longitudinal research project, engaging mathematical tasks, the four participants of this study along with other members of their cohort have structured a dynamic way of working together. Characteristics of their way of working constitute what can be called their mathematical microculture. Its consists of
evolving, situational patterns of behavior that members of the cohort expect of each other and that an observer can notice when cohort members engage problem tasks of the Rutgers research project. Similar to the problem task of this study, these tasks are investigative and cognitively demanding, requiring participants to reason mathematically. For instance, the cohort members demonstrate microcultural norms that include an insistence on sense making, free exchange of ideas, as well as collective and individual justification of ideas.

A window into how these microcultural norms have evolved is to examine characteristics of the teaching interventions of the Rutgers longitudinal study. The following are findings of behavioral characteristics of teacher-researchers when engaging the cohort group in mathematical activities (see, Maher, 1998):

1. Pose tasks, invite learners to participate in building a repertoire of mathematical representations, notations, formalisms, techniques, and lines of reasoning;
2. Assess the ideas that the students have built by observing their activity (such as model building) and listening to their explanations;
3. Require the students to support ideas with suitable justifications and arguments;
4. Encourage the students make public how their ideas emerged so that they develop mathematical tools as well as apply and build upon them;
5. Make standard mathematical tools, notations, representations, and language meaningful and accessible to students;
6. Work to build a classroom culture that encourages the exchange of ideas;
7. Notice and call to the students’ attention instances where there are differences and disagreements, and then facilitate conversations among students about these discrepancies;
8. Encourage student-to-student and student-to-teacher efforts to map representations and develop modes of inquiry that might disclose deeper understanding or discrepancies;
9. Facilitate the organization and reorganization of student groups to allow for the timely sharing of information and ideas;
10. Provide multiple opportunities for students to talk about and represent ideas;
11. Keep discussion open and revisit ideas over sustained periods of time;
12. When students have solved a particular problem, challenge them to generalize and extend their solution;
13. Assist students in reflecting on their own learning process so that they can design ways of working on future investigations;
14. Supply students with materials with which to reflect on, reconsider, and make public their ideas; and
15. Treat students and their thinking with interest, dignity, and personal respect.

Remarking on teacher-researchers’ expectations, Speiser, Walter, and Maher (in press) note a habit of the mind that evolved into a feature of the microculture:

From the first grade onward, learners were expected to invent representations that made sense to them, build solutions that they found convincing, and to communicate their findings in an atmosphere of thoughtful and responsive questioning. Over the years, these students became accustomed to extended explorations, in which first impressions often proved to be inadequate.

Extended investigative tasks that require thinking deeply, beyond initial thoughts, are features that contribute to the microculture of the Rutgers longitudinal study. In studies on proofs and representations (Kiczek, 2000; Maher & Martino, 1996c, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999), detail analyses of cohort members changing views of underlying mathematical ideas provide results in the areas of proofmaking and representational systems. Major findings include the following five:

1. Students offered convincing arguments in presenting solutions to their problems. They reasoned using proof by cases and proof by contradiction. The forms of reasoning that evolved included inductive, deductive and recursive.
2. Students expressed forms of reasoning first through informal language and their own notations. The structure of their reasoning broadened, and deepened over time and became increasingly symbolic and generalized.

3. Students’ transitions to standard forms of notation occurred naturally. In explaining and justifying solutions to problems, students frequently shifted between personal notational systems and more standard forms to express their ideas.

4. Students developed a deep interest in understanding their own ideas and making sense of them, as well as the ideas of others. The expectation that all ideas would be heard, explored and discussed became the norm.

5. Follow-up interviews revealed that students were able to talk about and reconstruct earlier reasoning about problem tasks. Reconstructed solutions were generally expressed in more formal symbolic notation.

The cohort of the Rutgers longitudinal study, to which the four participants of this study belong, have evolved a microculture, a community of learners that has been shaped by their intellectual actions and the pedagogical behaviors of the teacher-researchers. Consequently, Brian, Jeff, Michael, and Romina, the participants of this study, in the context of the Rutgers longitudinal study have a rich history of problem solving, ways of working on mathematical problems, and internal awareness of researchers expectation that they present convincing arguments for their resolutions.