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INQUIRY INTO THE INTERLOCUTION OF STUDENTS ENGAGED WITH MATHEMATICS:APPRECIATING LINKS BETWEEN RESEARCH AND PRACTICE[1]

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For either to be useful, links between research and practice are critical Just as important are connections between the practice of students engaged in mathematical activity and research that seeks to understand that practice. This research report explores lessons that researchers and practitioners can learn from an inquiry into the interlocution of students working collaboratively in small groups when engaged in talking and listening to each other. We use the term interlocution to denote discursive practices of learners in conversational exchanges. Questions that motivate this research included the following. What discursive practices do interlocutors employ as they work collaboratively to understand and resolve mathematical tasks? How do these practices influence the growth of their mathematical ideas? In what ways do their discursive practices help them move from a contextualized, situated task to generalize the task or their solution? Do students' discursive practices assist them to connect and generalize ideas from a new problem to others on which they have worked?

Background and Setting

This report focuses on the development of mathematical ideas by individual students as they work collaboratively in small groups. More particularly, we analyze how students through their discursive practice structure their own investigations. The data come from an ongoing, fourteen-year longitudinal research project of the Robert B. Davis Institute for Learning, directed by Maher (Maher, in press, and for references to relevant reports), that has been conducted in public elementary and secondary schools in a suburban, working-class, and immigrant town of New Jersey. Overall, our longitudinal study aims to contribute basic scientific understanding of cognitive behaviors as well as pedagogical conditions for which mathematics learning occurs as a process of sense making. The participants in the present study are four students—Brian, Jeff, Michael, and Romina—in their senior year of high school who, from their entry into first grade have participated in mathematical activities of our longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, and probability, both in the context of classroom investigations as well as in after school settings. For the present research report, these four students participated in an after-school session of mathematical problem solving. Data came from a culminating task, "The Taxicab Problem," in the research strand on combinatorics. The task has an underlying mathematical structure and contains concepts that resonate with those of other problems on which the students have worked during their involvement with the longitudinal study. The participants potentially revisit and deepen mathematical ideas they have already built as well build new ideas. After the students worked on the problem for about an hour and a half, we interviewed them concerning the nature of their work so that we might follow movements in their discourse toward further justification of their solution.

Theoretical Framework

Our research questions emerge from our ongoing, longitudinal investigation into the development of mathematical ideas by individual students as they work collaboratively in small groups (Davis & Maher, 1990, 1997; Maher & Martino, 1996, 2000; Maher & Speiser, 1997). The theoretical underpinnings of this investigation come from principally three sources: research on the development of representations (Davis, 1984; Davis & Maher, 1990, 1997; Speiser & Walter, 2000), models of the growth of understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994) and theories about the generation of meaning (Dörfler, 2000). This report builds on these theoretical notions and explores the applicability of an additional conceptual framework concerning a particular discursive practice with the aim of understanding how students structure their own investigations through such a practice.

Grounded in philosophical ideas of Levine (1989) concerning the social evolution of human's capacity for listening and a movement toward listening that is an antidote for "alienated meaning" (p. 8), Davis (1996) develops a theory of pedagogical listening, according to three different modes of listening, not necessarily mutually exclusive: evaluative, interpretive, and hermeneutic. Evaluative listening occurs when a teacher maintains a "detached, evaluative stance" (p. 52). In contrast, a teacher listening interpretively endeavors "to get at what learners are thinking…to open up spaces for re-presentation and revision of ideas—to *access* subjective sense rather than to merely *assess* what has been learned" (pp. 52-53, original emphasis). Finally, Davis posits hermeneutic listening as "more negotiatory, engaging, and messy, involving the hearer and the heard in a shared project…an imaginative participation in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted…the unfolding of possibilities *through collective action*" (p. 53, original emphasis).

Recently, drawing on the Pirie-Kieren theory for the growth of mathematical understanding (Pirie & Kieren, 1994) and Davis's (1996) interpretive and hermeneutic listening, Martin (2001) observes teacher interactions in an elementary classroom to analyze how mathematical understanding grows at the individual and whole-class levels. He elaborates and provides evidence for how the listening patterns of a teacher can occasion opportunities for students "to construct and modify their own images in response to her interventions" (p. 251).

Both Martin (2001) and Davis (1996) inquire into teacher listening and its consequent impact on the growth of student understanding. Our study broadens this scope of inquiry as well as applies and extends Davis's category to analyze not just listening but rather discursive practices of learners in conversational exchanges. This we call interlocution. Our category has four properties and guides the inquiry into how learners' discursive exchanges structure their investigation as well as contribute to the mathematical ideas they build and their growth in mathematical understanding. Emerging from new data within our longitudinal study, we analyze them in light of four properties of interlocution, three of which are adapted from the three listening modes that Davis (1996) proposes, which we define as follows:

- *Evaluative*: an interlocutor maintains a non-participatory and evaluative stance, judging statements of his or her conversational partner as either right or wrong, good or bad, useful or not.
- *Informative*: an interlocutor requests factual data to satisfy a doubt, a question, or a curiosity without evidence of judgment.
- *Interpretive*: an interlocutor endeavors to tease out what his or her conversational partner is thinking, wanting to say, expressing, and meaning; "to open up spaces for re-presentation and revision of ideas—to *access* subjective sense rather than to merely *assess*" it (Davis, 1996, pp. 52-53, original emphasis).
- *Hermeneutic*: an interlocutor engages and negotiates with his or her conversational partner; the interlocutors are involved in a shared project; each interlocutor participates in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted and the prejudices that frame perception and actions; the interlocutors "engage in an unfolding of possibilities *through collective action*" (Davis, 1996, p. 53, original emphasis).

Interlocution as a conceptual category in our research enables us to track the participation, growth in understanding, and autonomy of learners in their construction of mathematical ideas. Over the course of our longitudinal study, we have observed how individual learning occurs as learners participate in meaningful activities with other learners (Maher & Martino, 2000). Characteristics of these activities include not just the conceptually rich and open-ended nature of the mathematical tasks but also the research setting as well as how and when researcher-teachers intervene or not while learners engage in doing mathematics (Maher, 1998a, 1998b; Maher, Davis, & Alston, 1992). Over the course of our work, the role of researcher-teachers has evolved, increasingly becoming less interventionist and, after presenting a problem task and answering clarifying questions, being physically absent from where participants work.

Method

Our data sources consist of video images collected from the different perspective of three cameras, each of the research session video's is about two hours in length; transcript of the videotapes combined to produced a fuller, more accurate verbatim record of the research sessions; student inscriptions; and researcher field notes. Our analytic method employs a sequence of phases, informed by grounded theory (Charmaz & Mitchell, 2001; Creswell, 1998; Pirie, 1998), ethnography and microanalysis (Erickson, 1992; Goldman-Segall, 1998), and approaches for analyzing video data (Cobb & Whitenack, 1996; Pirie, 1996, 2001; Powell, Francisco, & Maher, 2001). Specifically, our method of data analysis involves the following nine non-linear, interacting phases: (1) attentively viewing the videotapes several times without intentionally imposing a specific analytic lens; (2) describing consecutive time intervals; (3) identifying critical events; (4) transcribing the video record; (5) inductive and deductive synchronous coding of transcript, videotape, and inscriptions; (6) writing analytical memoranda; (7) categorizing codes, identifying properties, and dimensionalizing properties within categories; (8) constructing a storyline; and (9) composing a narrative. (For an elaboration and examples of these phases, see Powell et al., 2001).

Results

The following is the combinatorial investigation on which participants were invited to work collaboratively, "The Taxicab Problem":

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.
>
> What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

With this problem statement, we gave the participants a map, actually, a 6 x 6 rectangular grid on which the left, uppermost intersection point represents the taxi stand. The three passengers are identified at three different intersections as blue, red, and green dots, respectively, while their respective distances from the taxi stand are four units south and one unit east, three units south and four units east, and four units south and five units east. (To more fully appreciate the episodes that we present below, you may wish first to work on the problem.)

Clockwise from the left, seated on three sides of a trapezoidal-shaped table are the four participants, Michael, Romina, Jeff, and Brian. At the start of the session, Researcher 1 pulls up a chair, sits on the right side of the table between Jeff and Brian, thanks the four students for coming, distributes the Taxicab Problem, and asks them to read and see whether they understand the task. Afterward, the researcher stands up and, backing away from the table, removes her chair. With his head bent downward, facing the problem statement, Jeff asks aloud whether one has to stay on the grid lines and whether they represent streets. The researcher responds, "Exactly." Romina, Brian, and Jeff discuss that 5 is the number of blocks it takes to reach the blue destination point and that different routes to blue are the same length as long as one does not go beyond it. Brian says that they should prove it.

Researcher 1 walks back over to the table and asks the students for their understanding of the problem. Jeff says that the task is to find the shortest route while "staying on the streets." The researcher adds that it is about finding whether there is more than one shortest route. Both Brian and Romina voice agreement. The researcher goes on to say that if there is more than one shortest route, they have to determine how many there are. Jeff inquires whether the researcher is asking how many different shortest routes? Researcher 1 says that not only do they have to find the number of shortest routes but also that they will "have to convince us" that they have found all of them. The researcher then walks away from the table.

Jeff asks for colored markers. Jeff, Romina, and Brian choose to each work on different destination points. Romina says that it is five blocks to the blue point. Brian suggests counting them and being sure. Jeff asks why the length of each route to blue is the same. Michael explains that to get the blue point one has to go "four down and right one" since one cannot move backward or diagonally. Romina asks how to devise an area for that. Jeff and Michael tell her that it's not area, it's perimeter with each segment of the grid considered as one unit.

The above descriptive account is of the first four minutes and forty-four seconds of the research session. It is worth noting that the researcher spends little time at the table with the students and responds only to student questions in a tailored yet sparse manner. Also, the students rather quickly organize themselves by requesting colored markers and assigning subtasks to each other. When Jeff inquires about why the length or routes to the blue destination point are the same, Michael explains. When Romina requests help in devising an area, Jeff and Michael respond and inform her that the applicable notion is perimeter not that of area. In general, they also carefully and respectfully listen and respond to each other's questions, statements, and ideas.

After almost fourteen minutes into the research session, there is an interesting and pivotal interchange among Romina, Brian, and Jeff:

Episode 1: Break Apart and Do Easier Ones   Time interval. 0:13:42 to 0:13:54

Romina: I think we're going to have to break it apart and draw as many as possible.

Brian: Yeah, //that's what I'm going to do.

Jeff: //And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers?

In Episode 1, An agenda for action emerges from this interlocution among the three students. Brian and Jeff accept the task implied in Romina's statement and act on her heuristic. Moreover, Jeff refines her suggestion in his interrogative: "why don't we do easier ones?" Romina's statement and Jeff's interrogative establish a new agenda for the group's actions. Importantly, this action agenda represents a milestone in their mathematical investigation. From this point onward, the students no longer work on the combinatorial problem as given and, instead, pose and work on simpler situations to glean relevant information and extract insights from those situations so as to inform their understanding and resolution of the given problem.

Following the heuristic approach that the participants articulated in the pivotal episode presented above, the participants continue their investigation with a renewed sense of purpose. They chose to work on a more general problem than the one the researcher posed but felt that it was simpler and sensed that it would, in the words of Jeff, "lead us to something." This new approach developed out of a brief but focused negotiation between Romina and Jeff. Romina presents an idea and Jeff listens, and responding to her idea, participates in an unfolding of possibilities. What emerged from their interlocution was a plan for collective action and illustrates one of four interlocutory mode or property observed within the data. Their discursive exchange is hermeneutic in the sense that both Romina and Jeff participate in a negotiation, an unfolding of possibilities, each listening to the other and co-constructing an idea.

The data suggest that particular interlocutory modes influence differently the progress of the group. We will first describe aspects of the work that the participants had accomplished by the time they were midway through the session and then quote from a long episode to illustrate further the different interlocutory modes.

In Episode 1 above, Romina, Brian, and Jeff decided to count the number of shortest routes by starting with simpler cases for intersection points nearby the taxi stand. Afterward, Romina and Jeff negotiate a method for counting. Starting with an intersection point that is 2 blocks east and 2 blocks south of the taxi stand (forming a 2 by 2 sub-grid), they count the number of shortest routes of several nearby points and record their results in the taxicab grid. Then, working with their 2 by 2 sub-grid for which they found 6 shortest routes, they work on a 3 by 3, finding 15 shortest routes. They also work on 2 by 4, 2 by 3, and 4 by 3 sub-grids. In this way, they control for variables, a heuristic that they had developed and employed in several earlier tasks in the longitudinal study.

At the start of the Episode 2 quoted below, Michael is double-checking that the number of shortest routes for the 3 by 3 sub-grid is 20. There is a brief, evaluative exchange of information between Michael and Brian, where the latter wants to know whether the former has included in his count a route whose trace has the shape of a staircase.

Episode 2: The Staircase One — Time interval: 0:55:31 to 0:56:42

Brian:	Did you figure out the five by five?
Michael:	Five by five? I'm doing three by three right now.
Brian:	Let's just agree. If we already know what it is then we have to figure out-
Michael:	I just want to make sure that's twenty. So- [Michael counts routes, moving his pen on his grid.]
Michael:	I'm missing two. That's probably right though.
Brian:	Did you get the, uh, staircase one?
Michael:	Which one? For the three by three?
Brian:	Yeah. [Inaudible]. [Romina returns.]

In the above, Brian, after Michael says that he is determining the number of shortest routes for a 3 by 3 sub-grid, inquires whether Michael found the "staircase one." The exchange involves passing on information and not evaluating responses. Brian lets Michael know the route whose shape is a staircase should not be overlooked. In this episode, the interlocutory property that characterizes the exchange between Brian and Michael is informative and leads to no externally expressed awareness or construction of mathematical ideas.

The next episode illustrates another mode of interlocution and points to its contribution the development of mathematical ideas. Romina had gone to the bathroom and rejoins the group.

Episode 3: Now It's Working — Time interval: 0:56:42 to 0.57:04

Brian:	Did Jeff tell you?
Romina:	What?
Brian:	That this one-
Romina:	For which one?
Michael:	//For-
Brian:	//Four by two.
Romina:	So you did get fifteen? So now it's working? [Meaning that the pattern of shortest routes corresponds to Pascal's triangle.] And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.

In Episode 3, Brian announces to Romina that Jeff and he have verified that 15, not 12, is the number of shortest routes in a 4 by 2 sub-grid. Romina notes that 15 must also be the number of shortest routes for a 2 by 4 sub-grid. In so doing, she voices her implicit awareness of a symmetrical property for the numerical pattern of shortest routes that Jeff and she have developed. Moreover, she observes that the pattern resembles that of Pascal's triangle. Romina listens, evaluates the information ("So now it's working?"), and interprets its meaning ("And then the two by four has to be fifteen too.") This episode establishes that a discursive exchange may involve more than one interlocutory property. In the present case, the interlocutor's words reveal that she engages both evaluative and interpretive interlocution.

At the end Episode 3, Romina suggests that if they can show that the number of shortest routes in a 3 by 3 sub-grid is 20, they can feel assured that their numerical pattern is Pascal's triangle. Michael and Jeff will assert that they will have to explain why numerical pattern is Pascal's triangle, and Romina will suggest relating their result to another problem they have worked on. However, at this point in their investigation, she, Michael, and Brian work individually to produce inscriptions of the different shortest routes in a 3 by 3 sub-grid. After about three minutes, she announces to Jeff that she is stuck.

Episode 4: How Do You Know It's Nothing Else?
Time interval: 0:59:49 to 1:00:34

Romina: I'm already stuck. [Brian draws a 3 by 3 rectangle on his paper. Romina draws in shortest routes for the "imaginary" 3 by 3 on her grid. Romina's pen stops when drawing a route.]

Jeff: You shouldn't be. Where you going?

Romina. Three by three. [She shows the paper to Jeff.]

[...]

Michael: Yeah I got twenty for that one.

Jeff. For a three by three?

Michael Yeah.

Jeff: Alright well then- I mean can't we explain why we think- well- alright. [Jeff waves his hand.]

Michael. //They're going to ask us-

Jeff: //Alright then the next question is why- //why-

Romina: //Now-

Michael: //How do you know-

Romina: //Just relate this back to the //blocks. [Jeff points to the grid on the transparency with his marker.]

Jeff: //But wait- Why is this- why does the Pascal's Triangle work for this is the question.

Romina: //Exactly. Relate it to the blocks. [The word "blocks" here refers to the Towers Problem.[2]]

Michael: //If it is how do you know it's twenty? How do you know it's nothing else?

[...]

Jeff: If we can explain why- why this is the Pascal's Triangle up to here [He points to the transparency grid.], we don't need to explain how we're positive this is twenty. //You know what I'm saying? [Jeff waves his hand again.]

A few minutes before the start of Episode 4, Romina voices concern about the group's datum for the 3 by 3 sub-grid. Brian offers to recount the routes for that one, and too creates inscriptions of the routes in a 3 by 3 sub-grid. Meanwhile, Michael has been counting the shortest routes for the 3 by 3 sub-grid and, in Episode 4, announces that he found 20. Suspecting that this confirms that their numerical pattern follows Pascal's triangle, Jeff states, "Why does Pascal's triangle work for this is the question." Michael echoes this question as well as states that the researchers are "going to ask us" this. Romina suggests relating their result to "the blocks," which is actually refers to "The Towers Problem," a class of related problems some of which are indeed isomorphic to the "Taxicab Problem."[1] Similar to Episode 3, in this one, the participants' conversational exchange has both an interpretive and hermeneutic interlocutory structure.

Discussion

The four episodes presented above point to key mathematical ideas that the four students generate as well as heuristic and content connections that they make to other problems on which they have worked. Episode 4 illustrates that the students seek to understand and explain reasons why Pascal's triangle underlies the mathematical structure of the Taxicab Problem. Moreover, in the episode, Romina suggests that they relate (find an isomorphism between) this problem to the Towers Problem, a problem they have already met and resolved. As well, at the end of Episode 4, Jeff points to the numerical pattern on his and Romina's grid for the shortest routes for sub-grids smaller than a 3 by 3. Implicitly reasoning by induction, he then asserts that if they "can explain...why Pascal's Triangle works up to here [a diagonal of numbers above the entry 20 that represents a 3 by 3 sub-grid], we don't need to explain how we're positive this is twenty."

This and the other three episodes contribute and further our understanding of how learners develop mathematical ideas through their thoughtful engagement with task situations that they mathematize and of problem solving. Under the pedagogical conditions of our research, the data suggest that learners collaborate to create agendas for action, that is, structure their method of investigation; co-construct an understanding of the task as its underlining structure; and build mathematical ideas synchronously as individuals and as collective members of a community of practice. Their co-constructions of heuristics and mathematical ideas emerge from informative, interpretive, and hermeneutic interlocution. A further outcome of the dominance of these interlocutory modes or properties is that over time learners build an *esprit de corps*.

Importantly, through specifically interpretive and hermeneutic interlocution learners do structure, monitor, and conduct their problem-solving investigation. In these interlocutory modes, the four participants of our study suggested subtasks for each other and ways of working, identified their need to verify expected outcomes and actual results, as well as prompted their need to explain and justify the underlining mathematical structures that emerged. As seen in Episodes 1 and 4, hermeneutic interlocution is a discursive practice in which learners structure their investigation of a mathematical task. As seen in Episodes 1, 3, and 4, learners engaged in interpretive and hermeneutic interlocution construct heuristics as well as mathematical ideas. Typically, evaluative and informative interlocution lead to no externally expressed awareness or construction of mathematical ideas. Tracking the properties of learners' interlocution yields insights into their mathematical ideas, growth in understanding, and autonomy as learners.

Though we have presented an analysis of episodes of a single problem-solving session, our observations of participants' mathematical engagement are informed by previous analyses in our longitudinal study (Kiczek, 2000; Kiczek, Maher, & Speiser, 2001; Maher & Martino, 2000; Martino, 1992; Muter, 1999). For instance, the evolved cultural norms of our research setting, most particularly the questioning patterns and expectations of researchers for explanations and justifications have been assimilated by research participants and infused into their ways of collaborating with each other. This is manifest in Episode 4 when Michael comments "they're going to ask us," and Jeff says "why does Pascal's Triangle work for this is the question."

Conversational interactions among learners can advance their subsequent individual and collective actions. Our data suggests that, among our four interlocutory properties, (1) interpretive and hermeneutic interlocution have the potential for advancing the mathematical understanding of individual learners working in a small group, (2) the personal or individual understanding of a learner is intermeshed with the understanding of his or her interlocutors, and (3) the mathematical ideas and understanding of an individual and his or her group emerge in a parallel fashion.

Conclusion

The results of this study show that learners can autonomously structure, monitor, and conduct their investigation of a mathematical task. They can sustain their engagement with a task for long periods of time and do so essentially without researcher-teacher intervention. We examined closely student interlocution and traced the development of their heuristic and mathematical ideas and growth of mathematical understanding. We have demonstrated how the discursive practices of learners influence how they structure their investigations, the mathematical ideas they develop, and how their understanding grows.

Notes

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[2] *n*-Tall Towers Problem: Your group has two colors of Unifix Cubes to work with. Work together and make as many different towers *n* cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers *n* cubes tall.

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