TOWARDS THE PROOF OF HOWE DUALITY. NOTES FOR A TALK AT RUTGERS, 30 DEC. 2011

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Abstract. I present the “soft” part of the proof of Howe duality, reducing it to a hard, explicit statement about generators of a subspace of invariants in the oscilator representation. The reference is Chapter 5 of the book of Moeglin–Vignéras–Waldspurger.

Contents

1. The lattice model

We will only discuss the *unramified case*, i.e. we are given finite-dimensional vector spaces W_1, W_2 over a non-archimedean field F (with ring of integers $\mathfrak{o}$), non-degenerate symmetric, resp. alternating forms on each of them (no matter which), and assume that there are *self-dual* sublattices L_1, L_2, respectively, which we fix.

Let $W = W_1 \otimes_F W_2$, $L = L_1 \otimes_{\mathfrak{o}} L_2$. Fix a character $\psi : F \to \mathbb{C}^\times$ with conductor $\mathfrak{o}$. Then the dual of a lattice $M \subset W$ can be described either as the set of elements $w \in W$ such that $\langle w, M\rangle \subset \mathfrak{o}$, or, equivalently, as the set of elements with $\psi(\langle w, M\rangle) = \{1\}$.

Consider the Heisenberg group $H = W \ltimes F$, and let $\tilde{M}$ be the inverse image of any subgroup $M \subset W$. We extend the character ψ to a character ψ_L of $\tilde{L}$ by setting it equal to 1 on L (it is a character because of self-duality and the choice of conductor for ψ).

Consider the representation $(\rho, \mathcal{S}) := \mathrm{Ind}_{\tilde{L}}^{H}(\psi_L)$ (smooth induction). For any $w \in W$ we will denote by s_w the element of $\mathcal{S}$ which is supported on the $\tilde{L}$-coset of w, and equal to 1 on w.

1.0.1. **Proposition.** (1) *Elements of $\mathcal{S}$ are compactly supported modulo $\tilde{L}$ (hence modulo the center).*
(2) *The representation is irreducible.*

Proof.

1. Let $f \in \mathcal{S}^M$, where M is a sublattice of L, then the character ${}^w\psi_L$ (by which M acts on s_w) is trivial on M if and only if $w \in M^\vee$.
2. Let $f \in \mathcal{S}^M$ and $w \in W$. We need to show that s_w is in the H-span of f. By translating, we may assume that $w = 0$ and $f(0) \neq 0$. If 1_L denotes the characteristic measure of L, then:

$$\rho(1_L)(f)(w) = \int_L f(wl)dl = \int_L \psi(\langle w, l\rangle) f(w) dl =$$

$$= f(w) \cdot \int_L \psi(\langle w, l\rangle) dl = \begin{cases} f(w) & \text{if } w \in L, \\ 0 & \text{otherwise.} \end{cases}$$

□

The Weil representation $(\omega, \mathcal{S})$ (for a $\mathbb{C}^\times$-cover $\tilde{G}$ of $G = \mathrm{Sp}(W)$, acting on W on the right) is given on a section of G by the formula:

$$\omega(g) f(w) = \sum_{l \in L/Lg \cap L} \psi\left(\frac{\langle l, w\rangle}{2}\right) f((l+w)g). \tag{1.1}$$

In particular, for $g \in K$:=the stabilizer of L we have:

$$\omega(g) f(w) = f(wg).$$

2. Compatible lattices

Our goal is to define a system of open compact neighborhoods of the identity in G_1 using sublattices of L_1, and then describe, for each such lattice M_1, some "corresponding" lattices M_2 which give, similarly, compact neighborhoods of the identity in G_2. The idea being, roughly, that representations with $J_1(M_1)$-invariant vectors will correspond under Howe duality to representations with $J_2(M_2)$-invariant vectors (where J_i denotes the subgroup corresponding to M_i).

More precisely, we define, for every sublattice $M_1 \subset L_1$:

$$J_1(M_1) = \{g \in G_1 | (g-1) M_1^\vee \subset M_1\},$$

$$H_1(M_1) = \{g \in G_1 | (g-1) M_1^\vee \subset L_1\}.$$

Whenever there is no confusion about M_1, we write simply J_1, H_1. Notice that $J_1, H_1 \subset K_1$.

We also set:

$$B_1(M_1) = M_1^\vee \otimes L_2 \subset W.$$

The subgroups above are convenient, because we can compute the action of J_1 and H_1 on s_w when $w \in B_1(M_1)$:

2.0.2. **Lemma.** *For $w \in B_1(M_1)$ the vector s_w is an eigenvector for H_1, and invariant under J_1.*

Proof. Recall that K_1 acts on $\mathcal{S}$ simply by right translations. The L-coset of w is preserved by H_1:

$$w \cdot h - w = w(h-1) \underset{M_1^\vee(h-1)\subset L_1}{\overset{w\in M_1^\vee\otimes L_2}{\Longrightarrow}} w\cdot h - w \in L_1 \otimes L_2 = L.$$

Hence, s_w is an eigenvector for H_1.

A similar calculation shows that $w \cdot h - w \in M_1 \otimes L_2$, and since $w \in M_1^\vee \otimes L_2 = (M_1 \otimes L_2)^\vee$, we see as in the first part of Proposition 1.0.1 that s_w is invariant by J_1. ☐

These vectors s_w, $w \in B_1(M_1)$, and our ability to compute the action of these subgroups on them will be the basis of the whole argument.

Facts: The subgroups $J_1(M_1)$, as M_1 ranges over a system of neighborhoods of the identity in W, form a system of neighborhoods of the identity in G_1. The quotient $H_1(M_1)/J_1(M_1)$ is abelian.

Now, given M_1 we want to define a corresponding lattice $M_2 \subset L_2$. It will not be unique. We call an element $w \in B_1(M_1)$ *extreme* if there is no intermediate lattice M between M_1 and L_1 such that $w \in B_1(M)$. Then there is a unique lattice $M_2 \subset L_2$ such that w is also "extreme" in $B_2(M_2)$ (defined analogously).

Via the dualities induced by the pairings, we can view the space W as $\mathrm{Hom}(W_1, W_2)$ or as $\mathrm{Hom}(W_2, W_1)$. Then the above condition on w is equivalent to saying that $w(L_2) + L_1 = M_1^\vee$. Now we set:

$$M_2 := (w(L_1) + L_2)^\vee. \tag{2.1}$$

Analogously, we define $J_2(M_2)$ and $H_2(M_2)$.

The indices of the lattices M_1, M_2 in L_1, L_2 are very closely related, namely:

2.0.3. **Lemma.** *Let $w \in W$, then $|L_1/(w(L_2)+L_1)^\vee| = |L_2/(w(L_1)+L_2)^\vee|$.*

3. THE MAIN THEOREM

Consider the subspace $\mathcal{S}_{M_1}$ of $\mathcal{S}$ consisting of elements which are supported on $B_1(M_1)$. The main results are:

3.0.4. **Proposition.** *If w, w' are "extreme" in $B_1(M_1)$ and the H_1-eigencharacters ψ_w, $\psi_{w'}$ of s_w and s'_w coincide then there is $k \in K_2$ such that $w \equiv w'k \mod L$.*

3.0.5. **Theorem.** *$\mathcal{S}^{J_1}$ is generated by $\mathcal{S}_{M_1}$ under the action of the (full) Hecke algebra $\mathcal{H}_2$ of M_2.*

These results require legthy, explicit calculations in the lattice model and I will not prove them (until I find the courage to read them and understand something about them). Notice, however, that Howe presents a more conceptual proof of a slightly different result in Part I of the Piatetski-Shapiro "Festschrift". This is probably the way to go.

4. Deduction of Howe duality

Howe duality can be stated as follows:

4.0.6. **Theorem.** *Let π_1 be an irreducible representation of G_1 and $\mathcal{S}[\pi_1] \simeq \pi_1 \otimes V_2$ the maximal semisimple π_1-isotypic quotient of $\mathcal{S}$. Then V_2 has a unique irreducible quotient.*

Remark. Although we don't really need an isomorphism of the form: $\mathcal{S}[\pi_1] \simeq \pi_1 \otimes V_2$, let me explain where it comes from: If everything was finite dimensional, we would be able to say that $V_2 = (\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1))^*$ (linear dual), and the map $\mathcal{S} \to V_2$ would be just the natural:

$$\mathcal{S} \xrightarrow{*} \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1), \pi_1\right) \overset{**}{\simeq} \pi_1 \otimes \left(\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1)\right)^*,$$

which we can easily show to be surjective.

In the infinite-dimensional case, things are not quite so: the map (*) is still defined, but it's something like taking double dual, for instance if G_1 acts trivially on $\mathcal{S}$ and π_1 is the trivial representation. Moreover, the map (**) is not an isomorphism, but rather an injection from the right to the left, with image those homomorphisms (into π_1) of finite-dimensional range. Since $\mathcal{S}$ is smooth and π_1 is admissible, we can see that the image of $\mathcal{S}$ indeed lies in $\pi_1 \otimes (\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1))^*$, but this is still not an isomorphism (for instance, the elements of $(\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1))^*$ are not G_2-smooth, but even if we take the G_2-smooth subspace we don't see anywhere the condition that elements of $\mathcal{S}$ are actually $\widetilde{G}$-smooth). To show that there is a subspace V_2 such that the image of $\mathcal{S}$ lies in V_2, take a smooth linear functional $l \in \widetilde{\pi_1}$ and let V_2 be the image of $\mathcal{S}$ under:

$$\mathcal{S} \to \pi_1 \otimes \left(\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1)\right)^* \xrightarrow{l} \left(\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1)\right)^*.$$

Since $\widetilde{\pi_1}$ is irreducible and hence $\mathbb{C}[G_1]$ acts transitively on its non-zero vectors, this image does not depend on the choice of l. It can then easily be seen that the image of $\mathcal{S}$ in $\pi_1 \otimes (\mathrm{Hom}_{G_1}(\mathcal{S}, \pi_1))^*$ is equal to $\pi_1 \otimes V_1$.

Let $M_1 \subset L_1$ be maximal such that π_1 has a non-zero M_1-invariant vector. Let $\mathcal{S}_1$ be the quotient of $\mathcal{S}$ by the subrepresentation generated by all $J_1(M)$-invariants, with $M_1 \subsetneq M \subset L_1$, and consider the quotient map $p : \mathcal{S} \to \mathcal{S}_1$.

4.0.7. **Lemma.** *If $w \in B_1(M_1)$ is not "extreme" then $p(s_w) = 0$.*

Proof. Indeed, then $w \in M^\vee \otimes L_2$, which implies that s_w is invariant by $J_1(M)$. □

4.0.8. **Corollary.** *The only H_1/J_1-eigencharacters which appear in $\mathcal{S}_1^{J_1}$ are those of the form ψ_w, with w "extreme" in $B_1(M_1)$. For any "extreme" element $w \in B_1(M_1)$, the space: $\mathcal{S}^{(H_1,\psi_w)}$ is generated by s_w over $\mathcal{H}_2$.*

Proof. Indeed, by the lemma the first statement is true for the subspace which is the image of $\mathcal{S}_{M_1}^{J_1}$, but this generates $\mathcal{S}_1^{J_1}$ over $\mathcal{H}_2$, by the Theorem. The second statement follows from Proposition 3.0.4. □

Now we fix such an "extreme" w, the corresponding character $\psi_1 = \psi_w$ of H_1, and consider the analogous data for G_2: M_2, J_2, H_2, and ψ_2 a character of H_2. Let e_i ($i = 1, 2$) be the idempotents in $\mathcal{H}_1$, $\mathcal{H}_2$ which project to (H_i, ψ_i)-equivariant vectors, and let $\overline{\mathcal{H}_i} = e_i \mathcal{H}_i e_i$. Notice that, s_w is an $H_1 \times H_2$-eigenvector with eigencharacter $\psi_1 \times \psi_2$.

We have almost proven duality: for any irreducible representation π_1 of G_1 we can find such data M_i, J_i, H_i, w, ψ_i so that $\pi_1^{(H_1,\psi_1)} \neq 0$, and then it follows from the above that $\mathcal{S}[\pi_1]^{(H_1,\psi_1)}$ will be generated over $\mathcal{H}_2$ by the image of s_w. In particular, every (non-zero) G_2-equivariant quotient of $\mathcal{S}[\pi_1]^{(H_1,\psi_1)}$ will have non-zero (H_2, ψ_2)-equivariant vectors.

We can now prove:

4.0.9. **Proposition.**

$$\mathcal{S}_1^{(H_1 \times H_2, \psi_1 \times \psi_2)} = \overline{\mathcal{H}_1} \cdot p(s_w) = \overline{\mathcal{H}_2} \cdot p(s_w). \tag{4.1}$$

Proof. The equality with $\overline{\mathcal{H}_2} \cdot p(s_w)$ has already been proven. The equality with $\overline{\mathcal{H}_1} \cdot p(s_w)$ follows from the same steps, once we prove:

4.0.10. **Lemma.** *If $w \in B_2(M_2)$ is not extreme, then s_w has zero image in $\mathcal{S}_1$.*

Recall that this is the case for $w \in B_1(M_1)$ which are not extreme, by maximality of M_1, and this was the only place where maximality of M_1 was used.

Proof of the lemma. If $M_2' = (w(L_1) + L_2)^\vee$ then $M_2 \subsetneq M_2 \subset K_2$, and if $M_1' = (w(L_2) + L_1)^\vee$ then $M_1 \subset M_1' \subset K_1$. By Lemma 2.0.3, $M_1 \neq M_1'$, hence $p(s_w) = 0$ because s_w is M_1'-invariant. $\square$

$\square$

Howe duality now follows: for a complex vector space V, commuting subalgebras A, B of $\mathrm{End}(V)$ and an element $v \in V$ such that $Av = Bv = V$, A and B should be each other's commutators. In particular, no quotient of V can admit a non-trivial direct sum A-decomposition (because projection to one summand is in the commutator of A, contradicting the fact that it's generated over B by the image of v.)