TOWARDS THE PROOF OF HOWE DUALITY. 
NOTES FOR A TALK AT RUTGERS, 30 DEC. 2011

YIANNIS SAKELLARIDIS

Abstract. I present the “soft” part of the proof of Howe duality, reducing it to a hard, explicit statement about generators of a subspace of invariants in the oscillator representation. The reference is Chapter 5 of the book of Moeglin–Vignéras–Waldspurger.

Contents

1. The lattice model 1
2. Compatible lattices 2
3. The main theorem 3
4. Deduction of Howe duality 3

1. The lattice model

We will only discuss the unramified case, i.e. we are given finite-dimensional vector spaces $W_1, W_2$ over a non-archimedean field $F$ (with ring of integers $\mathfrak{o}$), non-degenerate symmetric, resp. alternating forms on each of them (no matter which), and assume that there are self-dual sublattices $L_1, L_2$, respectively, which we fix.

Let $W = W_1 \otimes_F W_2$, $L = L_1 \otimes_{\mathfrak{o}} L_2$. Fix a character $\psi : F \to \mathbb{C}^\times$ with conductor $\mathfrak{o}$. Then the dual of a lattice $M \subset W$ can be described either as the set of elements $w \in W$ such that $\langle w, M \rangle \subset \mathfrak{o}$, or, equivalently, as the set of elements with $\psi(\langle w, M \rangle) = \{1\}$.

Consider the Heisenberg group $H = W \rtimes F$, and let $\tilde{M}$ be the inverse image of any subgroup $M \subset W$. We extend the character $\psi$ to a character $\psi_L$ of $\tilde{L}$ by setting it equal to 1 on $L$ (it is a character because of self-duality and the choice of conductor for $\psi$).

Consider the representation $(\rho, \mathcal{S}) := \text{Ind}^H_L(\psi_L)$ (smooth induction). For any $w \in W$ we will denote by $s_w$ the element of $\mathcal{S}$ which is supported on the $L$-coset of $w$, and equal to 1 on $w$.

1.0.1. Proposition. (1) Elements of $\mathcal{S}$ are compactly supported modulo $\tilde{L}$ (hence modulo the center).
(2) The representation is irreducible.
Proof. (1) Let $f \in \mathcal{S}^M$, where $M$ is a sublattice of $L$, then the character $w \psi_L$ (by which $M$ acts on $s_w$) is trivial on $M$ if and only if $w \in M^\vee$. 
(2) Let $f \in \mathcal{S}^M$ and $w \in W$. We need to show that $s_w$ is in the $H$-span of $f$. By translating, we may assume that $w = 0$ and $f(0) \neq 0$. If $1_L$ denotes the characteristic measure of $L$, then:
\[
\rho(1_L)(f)(w) = \int_L f(wl)dl = \int_L \psi(\langle w, l \rangle) f(w)dl = \int_L \psi(\langle w, l \rangle) f(w)dl = f(w) \cdot \int_L \psi(\langle w, l \rangle) dl = \begin{cases}
    f(w) & \text{if } w \in L, \\
    0 & \text{otherwise.}
\end{cases}
\]

The Weil representation $(\omega, \mathcal{S})$ (for a $\mathbb{C}^\times$-cover $\tilde{G}$ of $G = \text{Sp}(W)$, acting on $W$ on the right) is given on a section of $G$ by the formula:
\[
\omega(g)f(w) = \sum_{l \in L/L(g \cap L)} \psi\left(\frac{\langle l, w \rangle}{2}\right) f((l + w)g). \tag{1.1}
\]

In particular, for $g \in K :=$ the stabilizer of $L$ we have:
\[
\omega(g)f(w) = f(wg).
\]

2. Compatible lattices

Our goal is to define a system of open compact neighborhoods of the identity in $G_1$ using sublattices of $L_1$, and then describe, for each such lattice $M_1$, some “corresponding” lattices $M_2$ which give, similarly, compact neighborhoods of the identity in $G_2$. The idea being, roughly, that representations with $J_1(M_1)$-invariant vectors will correspond under Howe duality to representations with $J_2(M_2)$-invariant vectors (where $J_i$ denotes the subgroup corresponding to $M_i$).

More precisely, we define, for every sublattice $M_1 \subset L_1$:
\[
J_1(M_1) = \{ g \in G_1 | (g - 1)M_1^\vee \subset M_1 \},
\]
\[
H_1(M_1) = \{ g \in G_1 | (g - 1)M_1^\vee \subset L_1 \}.
\]

Whenever there is no confusion about $M_1$, we write simply $J_1, H_1$. Notice that $J_1, H_1 \subset K_1$.

We also set:
\[
B_1(M_1) = M_1^\vee \otimes L_2 \subset W.
\]

The subgroups above are convenient, because we can compute the action of $J_1$ and $H_1$ on $s_w$ when $w \in B_1(M_1)$:

2.0.2. Lemma. For $w \in B_1(M_1)$ the vector $s_w$ is an eigenvector for $H_1$, and invariant under $J_1$. 

Proof. Recall that $K_1$ acts on $S$ simply by right translations. The $L$-coset of $w$ is preserved by $H_1$:

$$w \cdot h - w = w(h - 1) \quad w \in M_1^\vee \otimes L_2 \quad w \cdot h - w \in L_1 \otimes L_2 = L.$$ 

Hence, $s_w$ is an eigenvector for $H_1$.

A similar calculation shows that $w \cdot h - w \in M_1 \otimes L_2$, and since $w \in M_1^\vee \otimes L_2 = (M_1 \otimes L_2)^\vee$, we see as in the first part of Proposition 1.0.1 that $s_w$ is invariant by $J_1$. $\square$

These vectors $s_w, w \in B_1(M_1)$, and our ability to compute the action of these subgroups on them will be the basis of the whole argument.

Facts: The subgroups $J_1(M_1)$, as $M_1$ ranges over a system of neighborhoods of the identity in $W$, form a system of neighborhoods of the identity in $G_1$. The quotient $H_1(M_1)/J_1(M_1)$ is abelian.

Now, given $M_1$ we want to define a corresponding lattice $M_2 \subset L_2$. It will not be unique. We call an element $w \in B_1(M_1)$ extreme if there is no intermediate lattice $M$ between $M_1$ and $L_1$ such that $w \in B_1(M)$. Then there is a unique lattice $M_2 \subset L_2$ such that $w$ is also “extreme” in $B_2(M_2)$ (defined analogously).

Via the dualities induced by the pairings, we can view the space $W$ as $\text{Hom}(W_1, W_2)$ or as $\text{Hom}(W_2, W_1)$. Then the above condition on $w$ is equivalent to saying that $w(L_2) + L_1 = M_1^\vee$. Now we set:

$$M_2 := (w(L_1) + L_2)^\vee. \quad (2.1)$$

Analogously, we define $J_2(M_2)$ and $H_2(M_2)$.

The indices of the lattices $M_1$, $M_2$ in $L_1$, $L_2$ are very closely related, namely:

2.0.3. Lemma. Let $w \in W$, then $|L_1/(w(L_2) + L_1)^\vee| = |L_2/(w(L_1) + L_2)^\vee|$.

3. The main theorem

Consider the subspace $S_{M_1}$ of $S$ consisting of elements which are supported on $B_1(M_1)$. The main results are:

3.0.4. Proposition. If $w, w'$ are “extreme” in $B_1(M_1)$ and the $H_1$-eigencharacters $\psi_w, \psi_{w'}$ of $s_w$ and $s_w'$ coincide then there is $k \in K_2$ such that $w \equiv w'k \mod L$.

3.0.5. Theorem. $S^{H_2}$ is generated by $S_{M_1}$ under the action of the (full) Hecke algebra $H_2$ of $M_2$.

These results require lengthy, explicit calculations in the lattice model and I will not prove them (until I find the courage to read them and understand something about them). Notice, however, that Howe presents a more conceptual proof of a slightly different result in Part I of the Piatetski-Shapiro “Festschrift”. This is probably the way to go.
4. Deduction of Howe Duality

Howe duality can be stated as follows:

4.0.6. Theorem. Let $\pi_1$ be an irreducible representation of $G_1$ and $S[\pi_1] \simeq \pi_1 \otimes V_2$ the maximal semisimple $\pi_1$-isotypic quotient of $S$. Then $V_2$ has a unique irreducible quotient.

Remark. Although we don’t really need an isomorphism of the form: $S[\pi_1] \simeq \pi_1 \otimes V_2$, let me explain where it comes from: If everything was finite dimensional, we would be able to say that $V_2 = (\text{Hom}_{G_1}(S, \pi_1))^\ast$ (linear dual), and the map $S \rightarrow V_2$ would be just the natural:

$$S \rightarrow \text{Hom}_C(\text{Hom}_{G_1}(S, \pi_1), \pi_1) \simeq \pi_1 \otimes (\text{Hom}_{G_1}(S, \pi_1))^\ast,$$

which we can easily show to be surjective.

In the infinite-dimensional case, things are not quite so: the map (*) is still defined, but it’s something like taking double dual, for instance if $G_1$ acts trivially on $S$ and $\pi_1$ is the trivial representation. Moreover, the map (**) is not an isomorphism, but rather an injection from the right to the left, with image those homomorphisms (into $\pi_1$) of finite-dimensional range.

Since $S$ is smooth and $\pi_1$ is admissible, we can see that the image of $S$ indeed lies in $\pi_1 \otimes (\text{Hom}_{G_1}(S, \pi_1))^\ast$, but this is still not an isomorphism (for instance, the elements of $(\text{Hom}_{G_1}(S, \pi_1))^\ast$ are not $G_2$-smooth, but even if we take the $G_2$-smooth subspace we don’t see anywhere the condition that elements of $S$ are actually $G$-smooth). To show that there is a subspace $V_2$ such that the image of $S$ lies in $V_2$, take a smooth linear functional $l \in \tilde{\pi_1}$ and let $V_2$ be the image of $S$ under:

$$S \rightarrow \pi_1 \otimes (\text{Hom}_{G_1}(S, \pi_1))^\ast \rightarrow (\text{Hom}_{G_1}(S, \pi_1))^\ast.$$

Since $\tilde{\pi_1}$ is irreducible and hence $C[G_1]$ acts transitively on its non-zero vectors, this image does not depend on the choice of $l$. It can then easily be seen that the image of $S$ in $\pi_1 \otimes (\text{Hom}_{G_1}(S, \pi_1))^\ast$ is equal to $\pi_1 \otimes V_1$.

Let $M_1 \subset L_1$ be maximal such that $\pi_1$ has a non-zero $M_1$-invariant vector. Let $S_1$ be the quotient of $S$ by the subrepresentation generated by all $J_1(M)$-invariants, with $M_1 \subseteq M \subset L_1$, and consider the quotient map $p : S \rightarrow S_1$.

4.0.7. Lemma. If $w \in B_1(M_1)$ is not “extreme” then $p(s_w) = 0$.

Proof. Indeed, then $w \in M^\vee \otimes L_2$, which implies that $s_w$ is invariant by $J_1(M)$.

4.0.8. Corollary. The only $H_1/J_1$-eigencharacters which appear in $S_1^{J_1}$ are those of the form $\psi_w$, with $w$ “extreme” in $B_1(M_1)$. For any “extreme” element $w \in B_1(M_1)$, the space: $S^{(H_1, \psi_w)}$ is generated by $s_w$ over $\mathcal{H}_2$.

Proof. Indeed, by the lemma the first statement is true for the subspace which is the image of $S_1^{J_1}$, but this generates $S_1^{J_1}$ over $\mathcal{H}_2$, by the Theorem. The second statement follows from Proposition 3.0.4. □
Now we fix such an “extreme” \( w \), the corresponding character \( \psi_1 = \psi_w \) of \( H_1 \), and consider the analogous data for \( G_2 \): \( M_2, J_2, H_2, \) and \( \psi_2 \) a character of \( H_2 \). Let \( e_i \) \( (i = 1, 2) \) be the idempotents in \( \mathcal{H}_1, \mathcal{H}_2 \) which project to \((H_i, \psi_i)\)-equivariant vectors, and let \( \overline{\mathcal{H}_i} = e_i \mathcal{H}_i e_i \). Notice that, \( s_w \) is an \( H_1 \times H_2 \)-eigenvector with eigencharacter \( \psi_1 \times \psi_2 \).

We have almost proven duality: for any irreducible representation \( \pi_1 \) of \( G_1 \) we can find such data \( M_i, J_i, H_i, w, \psi_i \) so that \( \pi_1^{(H_1, \psi_1)} \neq 0 \), and then it follows from the above that \( S[\pi_1]^{(H_1, \psi_1)} \) will be generated over \( \mathcal{H}_2 \) by the image of \( s_w \). In particular, every (non-zero) \( G_2 \)-equivariant quotient of \( S[\pi_1]^{(H_1, \psi_1)} \) will have non-zero \((H_2, \psi_2)\)-equivariant vectors.

We can now prove:

4.0.9. **Proposition.**

\[
S_{1}^{(H_1 \times H_2, \psi_1 \times \psi_2)} = \overline{\mathcal{H}_1} \cdot p(s_w) = \overline{\mathcal{H}_2} \cdot p(s_w).
\]

**Proof.** The equality with \( \overline{\mathcal{H}_2} \cdot p(s_w) \) has already been proven. The equality with \( \overline{\mathcal{H}_1} \cdot p(s_w) \) follows from the same steps, once we prove:

4.0.10. **Lemma.** If \( w \in B_2(M_2) \) is not extreme, then \( s_w \) has zero image in \( S_1 \).

Recall that this is the case for \( w \in B_1(M_1) \) which are not extreme, by maximality of \( M_1 \), and this was the only place where maximality of \( M_1 \) was used.

**Proof of the lemma.** If \( M'_2 = (w(L_1) + L_2)^{\vee} \) then \( M_2 \subseteq M_2 \subset K_2 \), and if \( M'_1 = (w(L_2) + L_1)^{\vee} \) then \( M_1 \subset M'_1 \subset K_1 \). By Lemma 2.0.3, \( M_1 \neq M'_1 \), hence \( p(s_w) = 0 \) because \( s_w \) is \( M'_1 \)-invariant. \( \square \)

Howe duality now follows: for a complex vector space \( V \), commuting subalgebras \( A, B \) of \( \text{End}(V) \) and an element \( v \in V \) such that \( Av = Bv = V \), \( A \) and \( B \) should be each other’s commutators. In particular, no quotient of \( V \) can admit a non-trivial direct sum \( A \)-decomposition (because projection to one summand is in the commutator of \( A \), contradicting the fact that it’s generated over \( B \) by the image of \( v \).)