

ON SPLITTING INVARIANTS AND SIGN CONVENTIONS IN ENDOSCOPIC TRANSFER

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ABSTRACT. The transfer factors for standard endoscopy involve, among other things, the Langlands-Shelstad splitting invariant. This note introduces a twisted version of that splitting invariant. The twisted splitting invariant is then used to define a better twisted factor Δ_I . In addition we correct a sign error in the definition of twisted transfers. There are two ways to correct the sign error. One way yields twisted transfer factors Δ' that are compatible with the classical Langlands correspondence. The other way yields twisted transfer factors Δ_D that are compatible with a renormalized version of the Langlands correspondence.

1. INTRODUCTION

Waldspurger has observed that in order to have smooth matching [W1] of functions for twisted endoscopy, the definition of twisted transfer factors given in [KS] must be modified if the attached restricted root system is non-reduced. This happens only when the root system itself has an irreducible component of type A_{2n} such that

- (1) some power θ^i of θ preserves that irreducible component, and
- (2) θ^i acts nontrivially on the Dynkin diagram of that component.

The modification is of course needed when θ is, up to inner automorphism, transpose-inverse on $GL(2n+1)$. On the other hand, it never arises in the context of cyclic base change.

Waldspurger has made a specific proposal [W2] for modifying the definition of twisted transfer factors and has shown that his modified factors do yield smooth matching for twisted endoscopy over p -adic fields. Waldspurger proposes to modify the term Δ_{II} by replacing the expression (4.3.4) in [KS], namely

$$(1.0.1) \quad \chi_{\alpha_{\text{res}}}(N\alpha(\delta^*) + 1),$$

by the slightly different expression

$$(1.0.2) \quad \chi_{\alpha_{\text{res}}}((N\alpha(\delta^*) + 1)/2).$$

When 2 is a nonzero square in the ground field k (for example, when k is \mathbb{R} or \mathbb{C}) the terms (1.0.1) and (1.0.2) are equal, but in general they can certainly be different.

It might be thought that nothing further need be said, since Waldspurger's modification yields a satisfactory theory for all local fields of characteristic 0. The trouble is that (1.0.2) is undefined for local fields of characteristic 2. For such local fields it seems unlikely that there is any way to fix the definition of twisted transfer factors by modifying Δ_{II} . However, twisted transfer factors are the product of four

terms, which leaves open the possibility of modifying one of the three other terms in such a way as to obtain the same overall result, i.e., to multiply the twisted transfer factor of [KS] by the sign

$$\prod_{\beta} \operatorname{sgn}_{F_{\beta}/F_{\pm\beta}}(2),$$

where the product is taken over a set of representatives β for the symmetric Γ -orbits in the set of restricted roots β that are of type R_3 and come from H .

In this note we will show how this can be done. We will leave Δ_{II} unchanged and instead modify Δ_I . This approach seems quite natural. The term Δ_I of [KS] was defined in terms of the Langlands-Shelstad splitting invariant $\lambda(T^x) \in H^1(k, T^x)$. The torus T^x is defined after passage to the simply-connected cover of the derived group of G . It is harmless to assume that G itself is semisimple and simply-connected. Then $T^x = T^{\theta}$ and so we have $\lambda(T^{\theta}) \in H^1(k, T^{\theta})$. We will define a new version Δ_I^{new} of Δ_I by introducing a *twisted* splitting invariant $\lambda(T, \theta) \in H^1(k, T^{\theta})$. The basic idea is quite simple: $\lambda(T, \theta)$ is just a *refinement* of the untwisted splitting invariant $\lambda(T) \in H^1(k, T)$ of [LS] obtained from θ -invariant a -data, in the sense that the image of $\lambda(T, \theta)$ under $H^1(k, T^{\theta}) \rightarrow H^1(k, T)$ is equal to the Langlands-Shelstad splitting invariant $\lambda(T)$.

Waldspurger also found a sign error in the definition of twisted transfer factors. Contrary to what is stated in [KS], the factor $\Delta = \Delta_I \Delta_{II} \Delta_{III} \Delta_{IV}$ proposed there is not independent of the choice of χ -data, because changing the choice of χ -data multiplies Δ_{II} and Δ_{III} by the same factor. In order that twisted transfer factors be independent of χ -data, either Δ_{II} or Δ_{III} needs to occur with exponent -1 . Since Δ_I and Δ_{III} are linked together for other reasons, one cannot invert Δ_{III} without inverting Δ_I at the same time. These considerations suggest two possible corrected versions Δ_D and Δ' of twisted transfer factors, namely

$$(1.0.3) \quad \Delta_D := \Delta_I^{\text{new}} \Delta_{II}^{-1} \Delta_{III} \Delta_{IV},$$

$$(1.0.4) \quad \Delta' := (\Delta_I^{\text{new}} \Delta_{III})^{-1} \Delta_{II} \Delta_{IV}.$$

The factors Δ' are compatible with the classical Langlands correspondence. The factors Δ_D are compatible with the renormalized Langlands correspondence discussed in section 4.

This note is organized as follows. In section 2 we define the twisted splitting invariant. In section 3 we define and study the improved version Δ_I^{new} of Δ_I . In section 4 we discuss the renormalized version of the local Langlands correspondence. In section 5 we define corrected versions Δ_D and Δ' of twisted transfer factors. Subsection 5.5 treats the Whittaker normalization of twisted transfer factors. Subsection 5.6 relates the twisted transfer factors Δ' for cyclic base change to the transfer factors Δ' for standard endoscopy; this corrects the slightly flawed treatment given in [K2].

It remains to thank Waldspurger for pointing out the described errors in [KS] and for observing that the factor Δ' works.

2. DEFINITION OF THE TWISTED SPLITTING INVARIANT

2.1. Review of the Langlands-Shelstad splitting invariant. Since our twisted splitting invariant will be a refinement of the one in [LS], our first task is to review the relevant constructions from that article, whose notation we adopt almost without change.

We work over an arbitrary ground field k . We do not assume that k has characteristic 0. We choose a separable closure \bar{k} of k and put $\Gamma = \text{Gal}(\bar{k}/k)$. We consider a connected reductive group G over k . We assume that G is quasi-split over k . It is convenient, and harmless for our purposes, to assume further that G is semisimple and simply-connected. We fix a k -splitting $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ of G . We denote by $\sigma_{\mathbf{T}}$ the action of $\sigma \in \Gamma$ on \mathbf{T} and set $\mathbf{\Gamma} = \{\sigma_{\mathbf{T}} : \sigma \in \Gamma\}$. We write $\mathbf{\Omega}$ for the Weyl group $\mathbf{\Omega}(G, \mathbf{T})$.

For each simple root α of \mathbf{T} we denote by M_α the Levi subgroup of G containing \mathbf{T} and having root system $\{\pm\alpha\}$; the group M_α and its derived group G_α are defined over \bar{k} . Now G_α is isomorphic to SL_2 by virtue of our assumption that G is semisimple and simply-connected. In fact there exists a unique \bar{k} -isomorphism $\xi_\alpha : SL_2 \rightarrow G_\alpha$ such that

- (1) ξ_α maps the diagonal subgroup of SL_2 isomorphically to the maximal torus $\mathbf{T} \cap G_\alpha$ of G_α ,
- (2) ξ_α maps the upper triangular Borel subgroup of SL_2 isomorphically to the Borel subgroup $\mathbf{B} \cap G_\alpha$ of G_α ,
- (3) ξ_α maps $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ to the root vector X_α occurring in our chosen k -splitting.

The element $n(\alpha) \in \text{Norm}(\mathbf{T}, G)(\bar{k})$ obtained as the image under ξ_α of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ lifts the simple reflection $\omega(\alpha) \in \mathbf{\Omega}$. For any $\omega \in \mathbf{\Omega}$ one obtains a lifting $n(\omega) \in \text{Norm}(\mathbf{T}, G)(\bar{k})$ of ω by putting

$$n(\omega) = n(\alpha_1) \cdots n(\alpha_r)$$

for any reduced expression $\omega = \omega(\alpha_1) \cdots \omega(\alpha_r)$.

With this notation in place we are ready to review the construction in section 2.3 of [LS]. We consider a maximal k -torus T in G . We fix a -data $\{a_\alpha\}_{\alpha \in R(G, T)}$ for the root system $R(G, T)$ of T . Thus the elements $a_\alpha \in \bar{k}$ satisfy the conditions

- (1) $a_{\sigma\alpha} = \sigma(a_\alpha)$ for all $\sigma \in \Gamma$, and
- (2) $a_{-\alpha} = -a_\alpha$.

In order to define the splitting invariant $\lambda_{\{a_\alpha\}}(T) \in H^1(k, T)$, we begin by choosing a Borel subgroup B of G (over \bar{k}) that contains T , as well as an element $h \in G(\bar{k})$ such that $(B, T)^h = (\mathbf{B}, \mathbf{T})$. Denote by σ_T both the action of $\sigma \in \Gamma$ on T and its transport to \mathbf{T} by $\text{Int}(h^{-1})$. We then have

$$(2.1.1) \quad \sigma_T = \omega_T(\sigma) \rtimes \sigma_{\mathbf{T}} \in \mathbf{\Omega} \rtimes \mathbf{\Gamma}$$

where $\omega_T(\sigma)$ is the image in $\mathbf{\Omega}$ of the element $h^{-1}\sigma(h) \in \text{Norm}(\mathbf{T}, G)(\bar{k})$. We use $\text{Int}(h^{-1})$ to transport our a -data from T to \mathbf{T} .

Now $\mathbf{\Omega} \rtimes \mathbf{\Gamma}$ is a group of automorphisms of $R(G, \mathbf{T})$. For any automorphism ζ of $R(G, \mathbf{T})$ we consider the element $x(\zeta) \in \mathbf{T}(\bar{k})$ defined by

$$(2.1.2) \quad x(\zeta) = \prod_{\alpha \in R(\zeta)} a_\alpha^{\alpha^\vee},$$

where $R(\zeta) := \{\alpha \in R(G, \mathbf{T}) : \alpha > 0, \zeta^{-1}\alpha < 0\}$. Here $\alpha > 0$ means that α is a root of \mathbf{T} in \mathbf{B} . In [LS] it is shown that

$$(2.1.3) \quad m(\sigma) := x(\sigma_T)n(\omega_T(\sigma))$$

is a 1-cocycle of Γ in $\text{Norm}(\mathbf{T}, G)$ and that

$$(2.1.4) \quad t(\sigma) := hm(\sigma)\sigma(h^{-1})$$

is a 1-cocycle of Γ in T . The class in $H^1(k, T)$ of the 1-cocycle $t(\sigma)$ is by definition the splitting invariant $\lambda_{\{a_\alpha\}}(T)$; it is independent of the choice of h . It depends on the chosen k -splitting of G , even though this dependence is not reflected in the notation.

However, $\lambda_{\{a_\alpha\}}(T)$ does not depend on the choice of Borel subgroup B containing T [LS, 2.3.3]. We need to recall why this is so. Suppose that B is replaced by $B' = vBv^{-1}$ with $v \in \text{Norm}(T, G)$. Set $u = h^{-1}vh \in \text{Norm}(\mathbf{T}, G)$ and let μ be the image of u in Ω . We may as well choose v so that $u = n(\mu)$. Obviously the element $h' = vh$ satisfies $(B', T)^{h'} = (\mathbf{B}, \mathbf{T})$. Using B' , h' in place of B , h , we obtain a 1-cocycle $t'(\sigma)$ of Γ in T , and in [LS] it is shown that $t'(\sigma)$ is cohomologous to $t(\sigma)$. In fact, from the proofs of Lemmas 2.3.A and 2.3.B of [LS] it is clear, with our choice of v , that $t'(\sigma)$ is the product of $t(\sigma)$ and the coboundary of the element

$$(2.1.5) \quad hx(\mu)h^{-1} \in T(\bar{k}).$$

This completes our review of the untwisted splitting invariant.

2.2. Definition of twisted splitting invariants. We retain all the previous notation and assumptions, but now we further consider a k -automorphism θ of G that preserves the k -splitting $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. We need to understand how θ interacts with the constructions made in the untwisted case.

By assumption θ preserves our chosen k -splitting of G . It follows easily that

$$(2.2.1) \quad n(\theta(\omega)) = \theta(n(\omega))$$

for all $\omega \in \Omega$.

In this twisted situation we are interested exclusively in θ -admissible maximal k -tori T , by which we mean that $\theta(T) = T$ and that there exists a Borel subgroup B (over \bar{k}) containing T and satisfying $\theta(B) = B$. For such T the automorphism θ acts on $R(T, G)$, and we are only interested in *twisted a -data* for T , by which we mean a -data for the root system $R(T, G)$ that satisfy the additional condition

$$(2.2.2) \quad a_{\theta(\alpha)} = a_\alpha$$

for all $\alpha \in R(T, G)$. In other words, twisted a -data is nothing but standard a -data that happens to be invariant under the obvious action of θ on the set of all standard a -data.

The first step in defining the untwisted splitting invariant was to choose B and h . In the twisted situation we begin by choosing a Borel subgroup B containing T such that $\theta(B) = B$.

Lemma 2.2.1. *There exists $h \in G^\theta(\bar{k})$ such that $(B, T)^h = (\mathbf{B}, \mathbf{T})$.*

Proof. Steinberg [St] proved this for algebraically closed fields. Fortunately his proof carries over to the case of separably closed fields. \square

We now choose h as in the lemma. We use $\text{Int}(h^{-1})$ to transport σ_T to \mathbf{T} , and we define $\omega_T(\sigma)$ as before. We use our chosen twisted a -data to form the elements $x(\zeta)$ in equation (2.1.2).

Lemma 2.2.2. *Let ζ be an automorphism of $R(G, \mathbf{T})$ that commutes with the natural action of θ on \mathbf{T} . Then the element $x(\zeta) \in \mathbf{T}(\bar{k})$ lies in \mathbf{T}^θ . In particular this is so for the automorphisms σ_T of \mathbf{T} , and also for the automorphisms induced by elements in Ω^θ .*

Proof. We claim that the automorphism θ preserves $R(\zeta)$. This, together with the invariance property $a_{\theta(\alpha)} = a_\alpha$ of our twisted a -data, proves that $\theta(x(\zeta)) = x(\zeta)$. The claim holds because (i) ζ commutes with θ , and (ii) θ preserves the set of positive roots of \mathbf{T} . \square

We use h and our twisted a -data to form the 1-cocycle $t(\sigma)$ of Γ in T (see (2.1.4)).

Proposition 2.2.3. *The 1-cocycle $t(\sigma)$ takes values in the subtorus T^θ . Its class in $H^1(k, T^\theta)$ is independent of the choice of B and h .*

Proof. That T^θ is connected and hence a subtorus of T follows from our assumption that G is semisimple and simply-connected.

Next we show that $t(\sigma)$ is fixed by θ . It follows from the previous lemma that $x(\sigma_T)$ is fixed by θ . Since σ_T and $\sigma_{\mathbf{T}}$ both commute with θ , so too does $\omega_T(\sigma)$, and it then follows from equation (2.2.1) that $n(\omega_T(\sigma))$ is fixed by θ . We conclude that $m(\sigma) = x(\sigma_T)n(\omega_T(\sigma))$ is also fixed by θ . Finally, it follows from (2.1.4) that $t(\sigma)$ is fixed by θ .

It is obvious that the cohomology class of $t(\sigma)$ is independent of the choice of h . That it is also independent of the choice of θ -stable B containing T follows from the fact the element $hx(\mu)h^{-1}$ occurring in (2.1.5) is fixed by θ when $\mu \in \Omega^\theta$, and this too is a consequence of the previous lemma. \square

The class of $t(\sigma)$ in $H^1(k, T^\theta)$ is the desired twisted splitting invariant and will be denoted by $\lambda_{\{a_\alpha\}}(T, \theta)$.

3. COMPARISON OF Δ_I^{new} WITH Δ_I

As in the previous section G is semisimple and simply-connected, and T is a θ -admissible maximal k -torus in G . In this situation T^θ is connected and hence a subtorus of T . We fix a θ -stable Borel subgroup (over \bar{k}) containing T . As positive system $R^+(G, T)$ we take the roots of T in B . For $\alpha \in R(G, T)$ we denote by α_{res} the restriction of α to T^θ , and we put $R_{\text{res}}(G, T) = \{\alpha_{\text{res}} : \alpha \in R(G, T)\}$. We denote by $\pi : R(G, T) \rightarrow R_{\text{res}}(G, T)$ the map $\alpha \mapsto \alpha_{\text{res}}$.

3.1. Review of some results of Steinberg. We remind the reader of the following results of Steinberg concerning the relation between $R_{\text{res}}(G, T)$ and $R(G, T)$ for θ -admissible T . The results are purely root-theoretic and so the characteristic of the ground field plays no role here.

Theorem 3.1.1 (Steinberg).

- (1) $R_{\text{res}}(G, T)$ is a root system in $X^*(T^\theta)$, possibly non-reduced, whose Weyl group will be denoted by $\Omega_{\text{res}}(G, T)$.
- (2) $R_{\text{res}}^+(G, T) := \{\alpha_{\text{res}} : \alpha \in R^+(G, T)\}$ is a positive system in $R_{\text{res}}(G, T)$.

- (3) The set of simple roots in $R_{\text{res}}(G, T)$ is the image under π of the set of simple roots in $R(G, T)$. The set of simple roots in $R(G, T)$ is the preimage under π of the set of simple roots in $R_{\text{res}}(G, T)$.
- (4) The map π induces a bijection from the set of orbits of θ in $R(G, T)$ to the set $R_{\text{res}}(G, T)$.
- (5) There is a unique homomorphism $\Omega_{\text{res}}(G, T) \rightarrow \Omega(G, T)$ for which the restriction map $X^*(T) \rightarrow X^*(T^\theta)$ is $\Omega_{\text{res}}(G, T)$ -equivariant. This homomorphism identifies $\Omega_{\text{res}}(G, T)$ with $\Omega(G, T)^\theta$.
- (6) Let β be a simple root in $R_{\text{res}}(G, T)$. We consider the subset $\mathbb{Z}\beta \cap R_{\text{res}}(G, T)$, the intersection being taken in $X^*(T^\theta)$, and we denote by M_β the unique Levi subgroup of G over \bar{k} that contains T and has root system $\pi^{-1}(\mathbb{Z}\beta \cap R_{\text{res}}(G, T))$. Since this preimage is stable under θ , so too is M_β . Under the isomorphism $\Omega_{\text{res}}(G, T) \simeq \Omega(G, T)^\theta$ the simple reflection $\omega(\beta) \in \Omega_{\text{res}}(G, T)$ corresponds to the longest element in the Weyl group $\Omega(M_\beta, T)$. When 2β is not a restricted root, the Dynkin diagram of M_β is a disjoint union of copies of A_1 , these being permuted transitively by θ . When $2\beta \in R_{\text{res}}(G, T)$, the Dynkin diagram of M_β is a disjoint union of copies of A_2 , these being permuted transitively by θ ; moreover, if there are r copies of A_2 , then θ^r preserves each copy and acts nontrivially on it.

Proof. See [St]. □

3.2. Comparison of twisted a -data with the a -data used in [KS]. Part (4) of Steinberg's theorem allows us to view twisted a -data $\{a_\alpha\}$ for T as a -data $\{a_\beta\}$ for the restricted root system, the two viewpoints being related by the equalities $a_{\alpha_{\text{res}}} = a_\alpha$. We say that a -data for $R_{\text{res}}(G, T)$ are *special* if

$$a_{2\beta} = a_\beta$$

whenever both $\beta, 2\beta$ are restricted roots. Only special a -data were considered in [KS] (though the word "special" was not used there). Now that we have introduced twisted splitting invariants, we may as well allow arbitrary a -data. Indeed, it would be awkward to compare Δ_I^{new} with Δ_I without doing so, as the first part of Proposition 3.5.2 can only be formulated using the non-special twisted a -data $\{\tilde{a}_\alpha\}$ appearing there.

3.3. Review of Δ_I . In this subsection we assume that k is a local field F of characteristic 0, so that Δ_I is defined. We now review the relevant definitions. Because G is quasi-split and θ preserves our given F -splitting, we should use the Δ_I specified in section 5.3 of [KS]. In other words, when we use the definition in [KS, 4.2] to form Δ_I , we should use the F -splitting of G^θ obtained from our given F -splitting of G .

We now recall the definition of this F -splitting of G^θ . The group G^θ is quasi-split, connected, reductive, with maximal torus \mathbf{T}^θ and Borel subgroup \mathbf{B}^θ , and the root system $R(G^\theta, \mathbf{T}^\theta)$ is the set of indivisible roots in $R_{\text{res}}(G, \mathbf{T})$. The set of simple roots in $R(G^\theta, \mathbf{T}^\theta)$ coincides with the set of simple roots in $R_{\text{res}}(G, \mathbf{T})$. We complete the pair $(\mathbf{B}^\theta, \mathbf{T}^\theta)$ to an F -splitting $(\mathbf{B}^\theta, \mathbf{T}^\theta, \{X_\beta\})$ of G^θ by putting

$$X_\beta := \sum_{\alpha \in \pi^{-1}(\beta)} X_\alpha$$

for every simple root β of $R_{\text{res}}(G, T)$.

We now choose special a -data $\{a_\beta\}$ on $R_{\text{res}}(G, T)$. In [KS] the term Δ_I was defined by

$$\Delta_I = \langle \lambda_{\{a_\beta\}}(T^\theta), s_{T,\theta} \rangle$$

for a certain element $s_{T,\theta} \in (\hat{T})_\theta^\Gamma$ that will be discussed further in the next subsection. The pairing $\langle \cdot, \cdot \rangle$ is the Tate-Nakayama pairing between $H^1(F, T^\theta)$ and $(\hat{T})_\theta^\Gamma$. Note that $(\hat{T})_\theta$ is Langlands dual to T^θ . The splitting invariant occurring in this definition is the untwisted one from [LS] for the group G^θ , and is formed using the given a -data on $R(G^\theta, T^\theta) \subset R_{\text{res}}(G, T)$ and the F -splitting of G^θ specified above. It is worth noting that no information is lost when one restricts special a -data to the subset $R(G^\theta, T^\theta)$ of indivisible roots in $R_{\text{res}}(G, T)$.

3.4. Definition of Δ_I^{new} . In this section k is a local field F of arbitrary characteristic. We consider twisted a -data $\{a_\alpha\}$ for $R(G, T)$ (equivalently, a -data $\{a_\beta\}$ for $R_{\text{res}}(G, T)$) and use it to form the twisted splitting invariant $\lambda_{\{a_\alpha\}}(T, \theta)$. We then put

$$\Delta_I^{\text{new}} = \langle \lambda_{\{a_\alpha\}}(T, \theta), s_{T,\theta} \rangle$$

The element $s_{T,\theta}$ is the same as the one used to define Δ_I . The term Δ_I^{new} depends on the choice of θ -invariant F -splitting $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ as well as the choice of twisted a -data.

Our next goal is to understand the dependence of Δ_I^{new} on the choice of twisted a -data. Before doing so we must review some material from [LS] and [KS]. As in [KS] there are three types of restricted roots:

- (1) type R_1 , for which neither 2β nor $\frac{1}{2}\beta$ is a root,
- (2) type R_2 , for which 2β is a root,
- (3) type R_3 , for which $\frac{1}{2}\beta$ is a root.

The indivisible roots are the ones of types R_1 and R_2 .

As in [LS] and [KS] there are two kinds of orbits \mathcal{O} of Γ in $R_{\text{res}}(G, T)$:

- (1) symmetric, for which $\beta \in \mathcal{O} \implies -\beta \in \mathcal{O}$,
- (2) asymmetric, for which $\beta \in \mathcal{O} \implies -\beta \notin \mathcal{O}$.

For β lying in a symmetric orbit the field of definition F_β of β is a separable quadratic extension of the field of definition $F_{\pm\beta}$ of $\pm\beta$. We denote by $\text{sgn}_{F_\beta/F_{\pm\beta}}$ the sign character on $F_{\pm\beta}^\times$ associated by local class field theory to the quadratic extension $F_\beta/F_{\pm\beta}$.

We also need to review $s_{T,\theta}$. It comes from twisted endoscopic data (H, \dots) , but for the present purposes we only need to know what it means to say that a restricted root *comes from* H . To understand this point one must remember that the coroot system for a twisted endoscopic group H is a subsystem of the restricted root system of the Langlands dual group \hat{G} . The element $s_{T,\theta}$ tells us what this subsystem is, in a way that we will now recall. For this the reader may find it helpful to consult the discussion of twisted centralizers on page 16 of [KS].

We need one more piece of notation. For $\alpha^\vee \in R^\vee(G, T)$ we denote by $N(\alpha^\vee) \in X^*(\hat{T})$ the sum of the elements in the θ -orbit of α^\vee . Then $N(\alpha^\vee) \in X^*(\hat{T})^\theta = X^*((\hat{T})_\theta)$, which is to say that $N(\alpha^\vee)$ may be viewed as a character on $(\hat{T})_\theta$.

The coroot system of (H, T_θ) can be identified with a certain subset of the set of θ -orbits in the coroot system $R^\vee(G, T)$. The θ -orbit of $\alpha^\vee \in R^\vee(G, T)$ lies in this subset when one of the following two conditions holds

- (1) α is of type R_1 or R_2 , and $(N(\alpha^\vee))(s_{T,\theta}) = 1$, or

(2) α is of type R_3 and $(N(\alpha^\vee))(s_{T,\theta}) = -1$.

As in [KS] we say that a restricted root $\beta = \alpha_{\text{res}} \in R_{\text{res}}(G, T)$ comes from H when α^\vee satisfies either (1) or (2); this condition is obviously independent of the choice of α such that $\alpha_{\text{res}} = \beta$.

Now we are almost ready to explain how Δ_I^{new} depends on the choice of twisted a -data. We continue to view twisted a -data as being a -data $\{a_\beta\}$ for the restricted root system $R_{\text{res}}(G, T)$, and we do not assume that it is special. We want to see how Δ_I^{new} changes when $\{a_\beta\}$ is replaced by another choice $\{a'_\beta\}$ of a -data on $R_{\text{res}}(G, T)$.

Write $a'_\beta = a_\beta b_\beta$ and note that $b_\beta \in F_{\pm\beta}^\times$. Thus the sign $\text{sgn}_{F_\beta/F_{\pm\beta}}(b_\beta)$ is defined whenever β lies in a symmetric Γ -orbit.

Lemma 3.4.1. *When $\{a_\beta\}$ is replaced by $\{a'_\beta\}$ the term Δ_I^{new} is multiplied by the sign*

$$(3.4.1) \quad \prod_{\beta} \text{sgn}_{F_\beta/F_{\pm\beta}}(b_\beta)$$

where the product is taken over a set of representatives for the symmetric Γ -orbits in the set of elements $\beta \in R_{\text{res}}(G, T)$ that satisfy one of the following two conditions:

- (1) β is of type R_3 and comes from H , or
- (2) β is not of type R_3 and does not come from H .

Note that the set of elements satisfying one of these two conditions is Γ -stable and hence a union of Γ -orbits.

Proof. We claim that, when α_{res} lies in a symmetric Γ -orbit in $R_{\text{res}}(G, T)$, the value of $N(\alpha^\vee)$ on $s_{T,\theta}$ is ± 1 . Indeed, there exists $\sigma \in \Gamma$ such that $\sigma\alpha_{\text{res}} = -\alpha_{\text{res}}$, and so from the Γ -invariance of $s_{T,\theta}$ it follows that

$$(N(\alpha^\vee))(s_{T,\theta}) = (-N(\alpha^\vee))(s_{T,\theta}).$$

This just says that the square of $(N(\alpha^\vee))(s_{T,\theta})$ is 1, as claimed.

The method of proof of Lemmas 3.2.C and 3.2.D in [LS] applies without change to show that Δ_I^{new} is multiplied by

$$\prod_{\beta} \text{sgn}_{F_\beta/F_{\pm\beta}}(b_\beta)$$

where the product is taken over a set of representatives for the symmetric Γ -orbits in the set

$$\{\alpha_{\text{res}} \in R_{\text{res}}(G, T) : (N(\alpha^\vee))(s_{T,\theta}) = -1\}.$$

Glancing back at what it means for β to come from H , we see that the lemma has been proved. \square

3.5. Comparison of Δ_I^{new} and Δ_I . In this subsection we work over a local field F of characteristic 0. Let us fix special a -data $\{a_\beta\}$ on $R_{\text{res}}(G, T)$. For such a -data both Δ_I and Δ_I^{new} are defined, and our next goal is to compute their ratio.

Now Δ_I involves a splitting invariant for T^θ , while Δ_I^{new} involves a twisted splitting invariant for (T, θ) . The former involves the liftings $n(\omega)$ for the Weyl group of G^θ , the latter the liftings $n(\omega)$ for the θ -fixed points in the Weyl group of G . Our first task is to compare these two liftings, and to do so we need notation that keeps track of which group we are using.

For $\omega \in \Omega$ we write $n(\omega) \in \text{Norm}(\mathbf{T}, G)$ for the lifting of ω provided by our θ -stable F -splitting $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. For $\omega \in \Omega^\theta$, which we also view as the Weyl group of \mathbf{T}^θ in G^θ , we write $n'(\omega) \in \text{Norm}(\mathbf{T}^\theta, G^\theta)$ for the lifting of ω provided by the F -splitting $(\mathbf{B}^\theta, \mathbf{T}^\theta, \{X_\beta\})$ specified in subsection 3.3. The next lemma is valid for any field of characteristic 0. To formulate the lemma we need a definition: for $\alpha \in R(G, \mathbf{T})$ we put

$$b_\alpha := \begin{cases} \frac{1}{2} & \text{if } \alpha_{\text{res}} \text{ has type } R_3, \\ 1 & \text{otherwise} \end{cases}$$

Lemma 3.5.1. *Let $\omega \in \Omega^\theta$. Then both liftings of ω are defined, and they are related by*

$$n'(\omega) = \left(\prod_{\alpha \in R(\omega)} b_\alpha^{\alpha^\vee} \right) n(\omega),$$

where $R(\omega) = \{\alpha \in R(G, \mathbf{T}) : \alpha > 0, \omega^{-1}\alpha < 0\}$.

Proof. An easy argument using reduced expressions shows that it is enough to prove this for simple reflections in Ω^θ . Then, by part (6) of Steinberg's theorem, the lemma reduces to a calculation in root systems of type A_1 and A_2 . The case of A_1 is trivial. The case of A_2 is easy but interesting. We have included the calculation in an appendix. \square

Now we can formulate the main result of this section.

Proposition 3.5.2.

- (1) *Using our given special a -data $\{a_\beta\}$, we define another set $\{\tilde{a}_\beta\}$ of a -data on $R_{\text{res}}(G, T)$ by the rule*

$$\tilde{a}_\beta = \begin{cases} \frac{1}{2}a_\beta & \text{if } \beta \text{ has type } R_3, \\ a_\beta & \text{otherwise.} \end{cases}$$

We then have the equality

$$\lambda_{\{a_\beta\}}(T^\theta) = \lambda_{\{\tilde{a}_\beta\}}(T, \theta).$$

Notice that $\{\tilde{a}_\beta\}$ is non-special when restricted roots of type R_3 exist.

- (2) *There is an equality*

$$\frac{\Delta_I^{\text{new}}}{\Delta_I} = \prod_{\beta} \text{sgn}_{F_\beta/F_{\pm\beta}}(2)$$

where the product is taken over a set of representatives β for the symmetric Γ -orbits in the set of restricted roots having type R_3 and coming from H .

Consequently, replacing Δ_I by Δ_I^{new} has the same effect as modifying Δ_{II} in the way proposed by Waldspurger [W2].

Proof. (2) follows from (1) and Lemma 3.4.1, so it suffices to prove (1).

In order to define the twisted splitting invariant we need to choose $h \in G^\theta(\bar{F})$ such that $(B, T)^h = (\mathbf{B}, \mathbf{T})$. We then have $(B^\theta, T^\theta)^h = (\mathbf{B}^\theta, \mathbf{T}^\theta)$, so h also serves to define the untwisted splitting invariant for T^θ .

Our task is then to compare the 1-cocycle $t(\sigma)$ (see section 2) obtained from $\{\tilde{a}_\beta\}$ and (G, T, θ) with the 1-cocycle $t'(\sigma)$ obtained from $\{a_\beta\}$ and (G^θ, T^θ) . We just need to show that the two cocycles are cohomologous. Because we use the same element h to define both, they will turn out to be equal.

From (2.1.2), (2.1.3), (2.1.4) we see that it suffices to prove that $m(\sigma) = m'(\sigma)$, where

$$m(\sigma) = \left(\prod_{\alpha \in R(\sigma_T)} \tilde{a}_{\alpha_{\text{res}}}^{\alpha^\vee} \right) n(\omega_T(\sigma))$$

and

$$m'(\sigma) = \left(\prod_{\beta \in R'(\sigma_{T\theta})} a_{\beta}^{\beta^\vee} \right) n'(\omega_T(\sigma));$$

here R' is the root system $R(G^\theta, \mathbf{T}^\theta)$, which we identify with the set of indivisible roots in $R_{\text{res}}(G, T)$, and β^\vee is the coroot for the group G^θ associated to $\beta \in R'$. As in section 2 we transport a -data from T to \mathbf{T} without change of notation.

For $\beta \in R(G^\theta, \mathbf{T}^\theta)$ a computation in root systems of types A_1 and A_2 shows that $\beta^\vee \in X_*(T^\theta) = X_*(T)^\theta$ is as follows. Choose $\alpha \in R(G, T)$ such that $\alpha_{\text{res}} = \beta$; then

$$\beta^\vee = \begin{cases} N(\alpha^\vee) & \text{if } \beta \text{ is of type } R_1, \\ 2N(\alpha^\vee) & \text{if } \beta \text{ is of type } R_2. \end{cases}$$

From this it follows easily that

$$\prod_{\alpha \in R(\sigma_T)} a_{\alpha_{\text{res}}}^{\alpha^\vee} = \prod_{\beta \in R'(\sigma_{T\theta})} a_{\beta}^{\beta^\vee}.$$

Therefore it suffices to show that

$$(3.5.1) \quad n'(\omega_T(\sigma)) = \left(\prod_{\alpha \in R(\sigma_T)} b_{\alpha}^{\alpha^\vee} \right) n(\omega_T(\sigma))$$

where b_{α} is defined by

$$b_{\alpha} = \begin{cases} \frac{1}{2} & \text{if } \alpha_{\text{res}} \text{ has type } R_3, \\ 1 & \text{otherwise.} \end{cases}$$

Now $R(\sigma_T) = R(\omega_T(\sigma))$; this is an immediate consequence of the fact that σ_T preserves the set of positive roots. Therefore the equality (3.5.1) follows from Lemma 3.5.1. \square

4. TWO NORMALIZATIONS OF THE LOCAL LANGLANDS CORRESPONDENCE

4.1. Two normalizations of the reciprocity law isomorphism. Consider a nonarchimedean local field F . There are two ways to normalize the reciprocity law isomorphism $F^\times \rightarrow W_F^{\text{ab}}$. In the classical version uniformizers correspond to the Frobenius automorphism, and in Deligne's version uniformizers correspond to the inverse of the Frobenius automorphism. Tate's article [T] makes use of Deligne's normalization, while Borel's article [B] makes use of the classical one.

4.2. Two normalizations of the local Langlands correspondence for tori. More generally, there are two ways to normalize the Langlands correspondence for tori T over F . Borel, following Langlands's conventions [L], uses the version which is compatible with the classical reciprocity law when $T = \mathbb{G}_m$. In order to obtain a version of the Langlands correspondence for tori that is compatible with Deligne's normalization of the reciprocity law, one has simply to build in an inverse. In other words, if a quasicharacter χ on $T(F)$ corresponds to a Langlands parameter $\varphi : W_F \rightarrow {}^L T$ under the classical Langlands correspondence for tori, then it is χ^{-1} that corresponds to φ in the version compatible with Deligne's conventions.

If one wants to follow Deligne's conventions for nonarchimedean local fields, and one wants to have local-global compatibility, one is forced to build in an inverse when one consider the Langlands correspondence for tori over global fields, and then one is forced to build it in for \mathbb{R} and \mathbb{C} as well.

4.3. Two normalizations of the Langlands pairing. Let F be a local field. The two versions of the Langlands correspondence for tori T over F give rise to two versions of the Langlands pairing between $T(F)$ and $H^1(W_F, \hat{T})$. We need a system of notation that distinguishes between them. Let $t \in T(F)$ and $\mathbf{a} \in H^1(W_F, \hat{T})$. The classical Langlands pairing is defined by $\langle t, \mathbf{a} \rangle := \chi(t)$, where χ is the quasi-character on $T(F)$ corresponding to \mathbf{a} under the classical Langlands correspondence for tori.

We define a renormalized Langlands pairing by

$$\langle t, \mathbf{a} \rangle_D := \langle t, \mathbf{a} \rangle^{-1}.$$

The subscript is meant to remind us that the version $\langle t, \mathbf{a} \rangle_D$ of the Langlands pairing is the one compatible with Deligne's normalization of the reciprocity law.

Now let G be a connected reductive group over F . In this setting too there is a Langlands pairing as well as a renormalized version of it. For $g \in G(F)$ and $\mathbf{a} \in H^1(W_F, Z(\hat{G}))$ we denote by $\langle g, \mathbf{a} \rangle$ the classical Langlands pairing between g and \mathbf{a} . We then define a renormalized version by putting

$$\langle g, \mathbf{a} \rangle_D := \langle g, \mathbf{a} \rangle^{-1}.$$

In the case where G is a torus, these two pairings coincide with the ones we have just discussed.

In this system of notation (A.3.13) on p. 137 of [KS] becomes

$$\langle j(u), \hat{z} \rangle = \langle u, \hat{i}(\hat{z}) \rangle_D^{-1} = \langle u, \hat{i}(\hat{z}) \rangle.$$

4.4. Two normalizations of the local Langlands correspondence for connected reductive groups. Still more generally there should be two ways to normalize the conjectural Langlands correspondence for arbitrary connected reductive groups over a local field F . There should be one version that is compatible with the classical Langlands correspondence for tori. It should also be compatible with the classical version of the local Langlands correspondence for unramified representations [B]. There should be another version that is compatible with the inverse normalization of the Langlands correspondence for tori, and also with the local Langlands correspondence for unramified representations defined using the geometric Frobenius in place of the (arithmetic) Frobenius automorphism. Number theorists, when considering the local Langlands correspondence for GL_n , tend to use this second version.

These two versions of the local Langlands correspondence should be related in the following way. The L -group of a connected reductive group G is defined as a semidirect product $\hat{G} \rtimes W_F$. The action of W_F on \hat{G} preserves some splitting $(\mathcal{B}, \mathcal{T}, \{\chi\})$ (in the notation of [KS]). There is a unique automorphism θ_0 of \hat{G} that preserves $(\mathcal{B}, \mathcal{T}, \{\chi\})$ and on \mathcal{T} induces the automorphism $t \mapsto \omega_0(t)^{-1}$, where ω_0 is the longest element in the Weyl group of \mathcal{T} . The automorphism θ_0 commutes with the action of W_F , and therefore we obtain an automorphism ${}^L\theta_0$ of ${}^L G$ defined by ${}^L\theta_0(g\sigma) = \theta_0(g)\sigma$ for $g \in \hat{G}$, $\sigma \in W_F$. Let $\varphi : W_F \rightarrow {}^L G$ be a Langlands parameter, and let Π be the (conjectural) L -packet attached to φ by the classical

normalization of the local Langlands correspondence. Then in the other version of the Langlands correspondence that same packet Π should be attached to the Langlands parameter ${}^L\theta_0 \circ \varphi$ obtained by composing ${}^L\theta_0$ with φ . Notice that for tori this procedure does agree with the one described earlier, because θ_0 is simply the inversion map on \mathcal{T} . For GL_n a Langlands parameter φ amounts to an n -dimensional representation of W_F , and replacing φ by ${}^L\theta_0 \circ \varphi$ amounts to replacing that n -dimensional representation by its contragredient. The situation in general is entirely analogous, because, for any finite dimensional complex representation r of ${}^L G$, the representation $r \circ {}^L\theta_0$ is isomorphic to the contragredient of r . Thus, in a global context, replacing φ by ${}^L\theta_0 \circ \varphi$ amounts to replacing the arithmetic Frobenius by the geometric Frobenius when defining automorphic L -functions.

Our discussion may be summarized as follows. There should be two versions of the local Langlands correspondence. The first is the *classical* one, based on the arithmetic Frobenius. The second, based on the geometric Frobenius, and compatible with Deligne's normalization of the reciprocity law, will be referred to here as the *renormalized* Langlands correspondence. We will write $\Pi(\varphi)$ for the conjectural L -packet attached to a Langlands parameter φ by means of the classical Langlands correspondence, and we will write $\Pi_D(\varphi)$ for the one obtained using the renormalized Langlands correspondence. Thus $\Pi_D(\varphi) := \Pi({}^L\theta_0 \circ \varphi)$ (in situations in which the classical Langlands correspondence is known).

5. TWO VERSIONS Δ_D AND Δ' OF CORRECTED TWISTED TRANSFER FACTORS

5.1. Standard endoscopy. One goal of standard endoscopy is to provide character identities associated to endoscopic data (H, s, ξ) . Recall that ξ is an L -homomorphism $\xi : {}^L H \rightarrow {}^L G$. Given a tempered Langlands parameter $\varphi_H : W_F \rightarrow {}^L H$ for H , one forms the Langlands parameter $\varphi := \xi \circ \varphi_H$ for G , and then one expects to have character identities involving the members of $\Pi(\varphi_H)$ on one side, and the members of $\Pi(\varphi)$ on the other. The transfer factors in [LS] are expected to produce a notion of endoscopic transfer yielding such character identities.

If, however, one prefers to use the renormalized local Langlands correspondence, then one needs to renormalize the transfer factors $\Delta = \Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_2} \Delta_{IV}$ of [LS]. The renormalized factors, that we will denote by Δ_D , are easy to define. Only the term Δ_{III_2} is affected, which is to say that Δ_D takes the form $\Delta_I \Delta_{II} \Delta_{III_1} \Delta_{III_2, D} \Delta_{IV}$. Moreover the renormalized term $\Delta_{III_2, D}$ is formed as follows. Recall that $\Delta_{III_2} = \langle \gamma, \mathbf{a} \rangle$ (see [LS, p. 247]). We now define $\Delta_{III_2, D}$ to be $\langle \gamma, \mathbf{a}_D \rangle_D$, but we need to explain the notation used in this expression.

The renormalized Langlands pairing $\langle \cdot, \cdot \rangle_D$ was defined in subsection 4.3; recall that $\langle \gamma, \mathbf{a}_D \rangle_D = \langle \gamma, \mathbf{a}_D \rangle^{-1}$. The relationship between \mathbf{a} and \mathbf{a}_D is a bit more subtle. Recall from [LS, §3.5] that $\mathbf{a} \in H^1(W_F, \hat{T})$ depends on the choice of χ -data $\chi = \{\chi_\alpha\}$, and so we should write $\mathbf{a}(\chi)$ when we need to keep track of this dependence. Now $\{\chi_\alpha^{-1}\}$ are also χ -data and we put $\mathbf{a}_D(\chi) = \mathbf{a}(\chi^{-1})$, where χ^{-1} is an abbreviation for $\{\chi_\alpha^{-1}\}$. In this system of notation we have

$$(5.1.1) \quad \Delta_{III_2} = \langle \gamma, \mathbf{a}(\chi) \rangle,$$

$$(5.1.2) \quad \Delta_{III_2, D} = \langle \gamma, \mathbf{a}_D(\chi) \rangle_D = \langle \gamma, \mathbf{a}(\chi^{-1}) \rangle^{-1}.$$

There is yet another way in which transfer factors can be modified. For this one needs to notice that if (H, s, ξ) is endoscopic data, so too is (H, s^{-1}, ξ) . We obtain transfer factors Δ and Δ_D from (H, s, ξ) , and we also obtain transfer factors Δ'

and Δ'_D from (H, s^{-1}, ξ) . However, as in [K1], we can take a different point of view by regarding Δ' as an alternative version of transfer factors for the original endoscopic data (H, s, ξ) . The same applies to Δ'_D , and altogether there are four useful variants of transfer factors for (H, s, ξ) , namely Δ , Δ' , Δ_D and Δ'_D . It will be necessary to bear this in mind as we turn now to twisted transfer factors.

5.2. Twisted endoscopy. One goal of twisted endoscopy is to provide twisted character identities, and therefore one should expect to have twisted transfer factors adapted to the renormalized Langlands correspondence as well as ones adapted to the classical Langlands correspondence. We are now going to elaborate on this point, and at the same time correct an error in [KS] that was found by Waldspurger.

The error in [KS] arose from being inconsistent about the normalization of the Langlands correspondence for tori. The appendices of [KS] make use of the renormalized version of the Langlands correspondence for tori, as one sees from the presence of a^{-1} rather than a in the expression

$$\prod_{a \in K^\times} x_a(a^{-1})$$

displayed in the middle of page 131 of [KS]. However, in the course of defining twisted Δ_{III} (see p. 40 of [KS]) we made use of the admissible embeddings ${}^L T_H \hookrightarrow {}^L H$ and ${}^L(T_{\theta^*}) \hookrightarrow {}^L G^1$ obtained by applying [LS, §2.6] to our chosen χ -data. This leads to nonsense because the construction in [LS] is adapted to the classical Langlands correspondence. Indeed, as Waldspurger pointed out to us, the dependence on χ -data of the terms Δ_{II} and Δ_{III} defined in [KS] does not cancel in the way that it should. (We of course want twisted transfer factors to be independent of the choice of χ -data.)

There are two ways to fix this error. One leads to twisted transfer factors Δ' adapted to the classical Langlands correspondence, and one leads to twisted transfer factors Δ_D adapted to the renormalized Langlands correspondence. We begin with twisted Δ_D , which requires only a small modification of [KS].

5.3. Twisted Δ_D . We will define twisted Δ_D to be the product $\Delta_I^{\text{new}} \Delta_{II} \Delta_{III}^{\text{new}} \Delta_{IV}$. In this product Δ_{II} and Δ_{IV} are the terms defined in [KS], and Δ_I^{new} is the modified version of Δ_I that was defined earlier in this note. The only new term is $\Delta_{III}^{\text{new}}$, which we will now explain.

The term Δ_{III} in [KS] was defined in section 4.4 of [KS]. The definition was given first in the special case in which $H_1 = H$ and was then given in the general case. Moreover, in section 5.3 of [KS] a simpler version of Δ_{III} was defined in the special case when G is quasisplit and θ preserves an F -splitting. All three definitions follow the same pattern, and the modification needed to obtain $\Delta_{III}^{\text{new}}$ is the same in all three cases. The modification is easy to explain. The term Δ_{III} of [KS] depends on a choice of χ -data. We now define $\Delta_{III}^{\text{new}}$ to be the term Δ_{III} for the *inverse* set of χ -data. (As in 5.1 the inverse set of χ -data is obtained by replacing each $\chi_{\alpha_{\text{res}}}$ by $\chi_{\alpha_{\text{res}}}^{-1}$.)

There is an equivalent way to define twisted Δ_D , due to the fact that Δ_D is independent of the choice of χ -data. Replacing our given set of χ -data by its inverse, we obtain the equality

$$(5.3.1) \quad \Delta_D = \Delta_I^{\text{new}} \Delta_{II}^{-1} \Delta_{III} \Delta_{IV}.$$

We used the obvious fact that replacing χ -data by its inverse replaces Δ_{II} by its inverse.

The twisted transfer factor Δ_D has all the properties stated in [KS]. No further modifications need be made to [KS] (as far as we know), though one must always remember to interpret the phrases “the Langlands correspondence” and “the Langlands pairing” as referring to the renormalized versions. For example the quasicharacter ω on p. 17 of [KS] needs to be defined by $\omega(g) = \langle g, \mathbf{a} \rangle_D$. Similarly, the quasicharacter λ_{H_1} of [KS] needs to be defined by $\lambda_{H_1}(z_1) = \langle z_1, \mathbf{b} \rangle_D$, where \mathbf{b} is the Langlands parameter

$$W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\xi_{H_1}} {}^L H_1 \rightarrow {}^L Z_1$$

considered on p. 23 of [KS].

Remark 5.3.1. In the case of standard endoscopy, the Langlands parameter \mathbf{b} above is *inverse*, in the group $H^1(W_F, \hat{Z}_1)$, to the Langlands parameter $W_F \rightarrow {}^L G_1 \rightarrow {}^L Z_1$ defined on p. 253 of [LS]. This remark clarifies the meaning of the comment made on lines -4,-5 on p. 23 of [KS].

In the case of standard endoscopy the twisted factor Δ_D reduces to the factor Δ_D discussed in the previous subsection, which explains our choice of notation. It is also desirable to have twisted transfer factors adapted to the classical Langlands correspondence. In the next subsection we will see how to define twisted transfer factors Δ' that reduce to the factor Δ' discussed in the previous subsection in the case of standard endoscopy.

5.4. Twisted Δ' . Following a suggestion made to us by Waldspurger (and incorporating the improved version Δ_I^{new} of twisted Δ_I discussed earlier in this note), we now put

$$(5.4.1) \quad \Delta' := (\Delta_I^{\text{new}} \Delta_{III})^{-1} \Delta_{II} \Delta_{IV}.$$

In this product Δ_{II} , Δ_{III} and Δ_{IV} are the terms defined in [KS].

If one uses the twisted transfer factors Δ' , then one needs to insert many minus signs in [KS], especially in the part concerning the stabilization of the twisted trace formula. A complete list of the necessary changes would be rather long. We will discuss only the most significant ones.

The first point to make is that now the quasicharacters ω and λ_{H_1} need to be defined using the classical Langlands pairing. In other words, the quasicharacter ω on p. 17 of [KS] now needs to be defined by $\omega(g) = \langle g, \mathbf{a} \rangle$. Similarly, the quasicharacter λ_{H_1} of [KS] now needs to be defined by $\lambda_{H_1}(z_1) = \langle z_1, \mathbf{b} \rangle$, with \mathbf{b} as in the previous subsection. With this understanding Lemma 5.1.C of [KS] is correct for Δ' , as is part (2) of Theorem 5.1.D. However part (1) of that theorem needs to be replaced by

$$\Delta'(\gamma_1, \delta') = \langle \text{inv}(\delta, \delta'), \kappa_\delta \rangle^{-1} \Delta'(\gamma_1, \delta).$$

When one uses Δ' , one must insert many minus signs in the stabilization of the elliptic θ -regular terms on the geometric side of the twisted trace formula. We will now discuss all the basic definitions and key statements that need to be modified. These modifications entail changes in some of the formulas in the proofs, but only a few of these are listed here. The rest will be obvious to anyone who systematically works through the proof of the stabilization. All the page references in the rest of this subsection are to [KS].

The equality in the fourth line on p. 81 must be replaced by

$$\langle \text{inv}'(\delta, \delta'), \beta(\mathbf{a}) \rangle^{-1} = \omega(h),$$

and the equality on the ninth and tenth lines of that page must be replaced by

$$O_{\delta'\theta}(f) = \langle \text{inv}'(\delta, \delta'), \beta(\mathbf{a}) \rangle O_{\delta\theta}(f).$$

There is no change in the displayed formula (6.2.2) on that page.

On p. 89 the definition of Φ must be replaced by

$$\Phi(x) = \langle \text{obs}(\delta), \kappa_0 \rangle O_{\delta\theta}(f).$$

The displayed formula in line -7 of that page must be replaced by

$$\langle \text{inv}(\delta, \delta'), \kappa_0 \rangle^{-1} = \omega(h).$$

On p. 93 the expression (6.4.8) must be replaced by

$$\langle \text{inv}(\delta_0, \delta(v)), \kappa_0 \rangle^{-1} O_{\delta(v)\theta}(f_v).$$

On p. 94 the first factor in the expression displayed on line 12 should be replaced by $\langle \text{inv}(\delta_0, \delta), \kappa_0 \rangle^{-1}$.

On p. 96 the definition of the twisted κ -orbital integral should be replaced by

$$O_{\delta_0\theta}^\kappa(f) = \int_{\mathcal{D}(T, \theta, \mathbb{A})} \langle e, \kappa \rangle^{-1} O_{\delta_e\theta}(f) de_{\text{Tam}}.$$

The factor $\langle e, \kappa \rangle$ again needs to be replaced by $\langle e, \kappa \rangle^{-1}$ in the expression on line 13 of p. 102.

The last changes are especially significant. The right side of the equality in Lemma 7.3.A on p. 109 should be replaced by its inverse (when Δ' is used on the left side). The same is true for the equality in Corollary 7.3.B, which must be replaced by

$$\Delta'_\mathbb{A}(\gamma_1, \delta) = \langle \text{obs}(\delta), \kappa \rangle^{-1}.$$

5.5. Whittaker normalization of transfer factors. Assume that G is quasi-split, and choose an F -splitting $(B, T, \{X_\alpha\})$. Assume further that θ preserves the chosen F -splitting. In this situation one can consider the Whittaker normalization Δ_λ of twisted transfer factors introduced in section 5.3 of [KS]. We remind the reader that Δ_λ was defined by

$$\Delta_\lambda = \varepsilon_L(V, \psi) \Delta_I \Delta_{II} \Delta_{III} \Delta_{IV}.$$

(see pages 63 and 65 of [KS]).

Of course this definition too needs to be modified. Again there are two variants, these being Whittaker normalized versions $\Delta_D^\lambda, \Delta'_\lambda$ of Δ_D, Δ' respectively. The two variants are defined by

$$(5.5.1) \quad \begin{aligned} \Delta_D^\lambda &= \varepsilon_L(V, \psi) \Delta_I^{\text{new}} \Delta_{II} \Delta_{III}^{\text{new}} \Delta_{IV}, \\ &= \varepsilon_L(V, \psi) \Delta_I^{\text{new}} \Delta_{II}^{-1} \Delta_{III} \Delta_{IV}, \end{aligned}$$

$$(5.5.2) \quad \Delta'_\lambda = \varepsilon_L(V, \psi) (\Delta_I^{\text{new}} \Delta_{III})^{-1} \Delta_{II} \Delta_{IV}.$$

The square of $\varepsilon_L(V, \psi)$ is ± 1 (see p. 65 of [KS]). Therefore it may well happen that $\varepsilon_L(V, \psi)$ is not equal to its inverse, and it should be emphasized that we use $\varepsilon_L(V, \psi)$, as opposed to its inverse, in the definitions of both Δ_D^λ and Δ'_λ .

In the next subsection we will make use of the following definition:

$$\Delta'_0 := (\Delta_I^{\text{new}} \Delta_{III})^{-1} \Delta_{II} \Delta_{IV}.$$

Of course we then have the equality $\Delta'_\lambda = \varepsilon_L(V, \psi)\Delta'_0$. In the case of standard endoscopy Δ'_0 coincides with the factor Δ_0 defined on p. 248 of [LS], but with (H, s, ξ) replaced by (H, s^{-1}, ξ) . In other words, Δ'_0 bears the same relation to Δ_0 as Δ' does to Δ (see subsection 5.1).

5.6. Twisted transfer factors for cyclic base change. The twisted transfer factor Δ'_0 considered in the last subsection differs in two ways from the incorrectly defined twisted transfer factor Δ_0 on p. 63 of [KS]. It uses Δ_I^{new} rather than Δ_I , and both Δ_I^{new} and Δ_{III} occur with an inverse. Only the second of these two differences is relevant in the special case of cyclic base change, since $\Delta_I^{\text{new}} = \Delta_I$ in that case.

Since twisted Δ_0 was incorrectly defined in [KS], it is necessary to reformulate Proposition A.1.10 and Corollary A.2.10 in [K2] by using Δ'_0 in place of Δ_0 . Proposition A.1.10 in [K2] should be replaced by the following result.

Proposition 5.6.1. *There is an equality*

$$(5.6.1) \quad \Delta'_0(\gamma_H, \delta) = \Delta'_0(\gamma_H, \gamma) \langle \text{inv}(\gamma, \delta), (a^{-1}, \tilde{s}) \rangle^{-1}.$$

The notation used in the second factor in the righthand side of (5.6.1) is the same as in [K2], but the exponent -1 occurring in this factor is new.

Proof. One just needs to work through the proof of Proposition A.1.10, making sure to use the Langlands pairing more carefully than was done there. In the displayed formula $\Delta_{III}(\gamma_H, \gamma) = \langle \gamma, b \rangle$ in the middle of p. 194 of [K2], the Langlands pairing occurring on the right side is the one used in [LS], namely the classical one. In the formula $\langle (\gamma, \delta), (b, 1) \rangle^{-1} = \langle \gamma, b \rangle$ on the third line from the bottom on p. 194 of [K2], the Langlands pairing occurring on the right side is the renormalized one $\langle \gamma, b \rangle_D$ discussed in subsection 4.3. \square

Corollary A.2.10 in [K2] should be replaced by the following result.

Theorem 5.6.2. *There is an equality*

$$(5.6.2) \quad \Delta'_0(\gamma_H, \delta) = \Delta'_0(\gamma_H, \gamma) \langle \alpha(\gamma, \delta), s \rangle.$$

The notation used in the second factor in the righthand side of (5.6.2) is the same as in [K2], but the exponent -1 occurring in [K2] is no longer present.

Proof. Combine Proposition 5.6.1 above with Theorem A.2.9 in [K2]. \square

Fortunately Theorem 5.6.2 is exactly the result needed to justify the main contention made in [K2], namely that it was legitimate to use $\langle \alpha(\gamma_0; \delta), s \rangle \Delta_p(\gamma_H, \gamma_0)$ as twisted transfer factors in [K1]. The point is that the factor denoted by $\Delta_p(\gamma_H, \gamma_0)$ in [K1, p. 178] coincides with the factor denoted by $\Delta'_0(\gamma_H, \gamma_0)$ in this note.

5.7. Another correction. We also take this opportunity to point out that the definition of hypercohomology groups given in [KS, A.1] is correct only for complexes of G -modules that are bounded below. (For Tate hypercohomology the complexes even need to be bounded above and below.) This does not affect the main results in that appendix, which only involve bounded complexes.

APPENDIX A. COMPUTATIONS IN SL_3

A.1. Standard splitting and automorphism for $SL(3)$. Let k be a field. We consider the group $G = SL(3)$ and the automorphism θ of G given by

$$\theta(g) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} {}^t g^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}.$$

The automorphism θ has order 2 and preserves the following splitting $(\mathbf{B}, \mathbf{T}, \xi_1, \xi_2)$ of G . As maximal torus \mathbf{T} , we take the diagonal matrices in G . As Borel subgroup \mathbf{B} , we take the upper triangular matrices in G . The simple roots α_1, α_2 take the values $\alpha_1 = a/b$ and $\alpha_2 = b/c$ on the diagonal matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

and the remaining positive root is $\alpha_3 = \alpha_1 + \alpha_2$. The last ingredient in the standard splitting consists of the following two homomorphisms ξ_1, ξ_2 from $SL(2)$ to G :

$$\begin{aligned} \xi_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \xi_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \end{aligned}$$

It is obvious that $\theta\mathbf{T} = \mathbf{T}$, $\theta\mathbf{B} = \mathbf{B}$ and $\theta \circ \xi_1 = \xi_2$.

We write N for the normalizer of \mathbf{T} in G . Recall that untwisted splitting invariants for G are constructed using the elements $n_1, n_2 \in N$ defined by

$$n_i = \xi_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The group of fixed points of θ in the Weyl group $\Omega = N/\mathbf{T}$ is $\{1, \omega_0\}$, where ω_0 is the longest element of Ω . The standard lift of ω_0 is $n_3 := n_1 n_2 n_1 = n_2 n_1 n_2$. Note that $\theta(n_3) = n_3$, simply because θ exchanges n_1 and n_2 . Explicitly, we have

$$n_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A.2. The homomorphism $SL(2) \rightarrow SL(3)$ determined by the adjoint representation of $SL(2)$. The adjoint representation of $SL(2)$ will give us a particular homomorphism $\text{Ad} : SL(2) \rightarrow SL(3)$ once we choose a basis for the Lie algebra of $SL(2)$. We take a slightly non-traditional basis, namely

$$X = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The non-traditional minus sign occurring in the matrix defining X makes the explicit formula for $\text{Ad} : SL(2) \rightarrow SL(3)$ free of minus signs. This formula is as

follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

The number 2 appearing in this last 3×3 matrix is the source of the difficulty in finding a good definition of Δ_I . The point is that the adjoint representation of $SL(2)$ is a bit pathological in characteristic 2.

Now let us assume that the characteristic of k is not 2. We are then free to conjugate the homomorphism Ad by the diagonal matrix $(1, 2, 2)$, thus obtaining the homomorphism $\text{Ad}' : SL(2) \rightarrow SL(3)$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & ab & \frac{1}{2}b^2 \\ 2ac & ad+bc & bd \\ 2c^2 & 2cd & d^2 \end{bmatrix}$$

In particular

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} =: n'_3$$

Note that Ad' induces an isomorphism from $PGL(2)$ to G^θ , the group of fixed points of θ on G . Thus n'_3 is used to form untwisted splitting invariants for G^θ , and it is interesting to compare n'_3 with n_3 . Inspecting the two matrices, one finds that $n'_3 = (\frac{1}{2})^{\alpha_3^\vee} n_3$.

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