A NOTE ON REAL ENDOSCOPIC TRANSFER AND PSEUDO-COEFFICIENTS

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In memory of Roo

1. Introduction

We gather results about transfer using canonical factors in order to establish some formulas for evaluating stable tempered traces on the transfer of a pseudo-coefficient for a discrete series representation, or of a genuine coefficient if Schwartz functions are allowed. With a good choice of normalization for the absolute transfer factors, these formulas are simple and easy to compute. We finish with a general remark on the definition of spectral factors.

2. Endoscopic transfer

Let (G, ψ) be an inner form of a connected reductive algebraic group G^* quasi-split over $\mathbb{R}$, and let $\mathfrak{e}$ be a set of ordinary endoscopic data for G. Let (H_1, ξ_1) be a z-pair for $\mathfrak{e}$. We consider transfer between $G(\mathbb{R})$ and the endoscopic group $H_1(\mathbb{R})$. For dependence solely on the choice of normalization of transfer factors, we write the transfer statement in terms of measures of the form fdg, where $f \in C_c^\infty(G(\mathbb{R}))$ and dg is a Haar measure; on $H_1(\mathbb{R})$ the measures will be of the form $f_1 dh_1$, where $f_1 \in C_c^\infty(H_1(\mathbb{R}), \lambda_1)$ and dh_1 is a Haar measure (see [S4] where the more general twisted setting is described). Here λ_1 is a character on a certain central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$; the pair (Z_1, λ_1) is prescribed by the choice of z-pair. We could also work with a fixed central character for each of $G(\mathbb{R}), H_1(\mathbb{R})$. Regard the two characters as a character on the product $Z_{H_1}(\mathbb{R}) \times Z_G(\mathbb{R})$ of the centers. Let C be the fiber product of Z_{H_1} and Z_G over Z_H, where $H = H_1/Z_1$, with the obvious maps (the constructions in Section 5.1 of [KS] of course simplify when there is no twisting). Then the restriction of the character to the subgroup $C(\mathbb{R})$ must coincide with the character λ_C described in [KS] at the generalization of Lemma 5.1.C. See also Lemma 7.3 of [S5] for the role of λ_C (denoted ϖ_C there) on the spectral side. Finally we follow familiar conventions that introduce no dependence on the choice of invariant measures on the conjugacy classes we consider (see, for example, [S4]).

Theorem 2.1. *(Transfer theorem, [S3]) Assume that $\Delta_{geom}, \Delta_{spec}$ are transfer factors with compatible normalization. Then for each measure fdg on $G(\mathbb{R})$ there exists a measure $f_1 dh_1$ on $H_1(\mathbb{R})$ such that*

$$SO(\gamma_1, f_1 dh_1) = \sum\nolimits_{\delta,\ conj} \Delta_{geom}(\gamma_1, \delta)\ O(\delta, fdg)$$

for all strongly G-regular $\gamma_1 \in H_1(\mathbb{R})$, and then

$$St\text{-}Trace\ \pi_1(f_1 dh_1) = \sum\nolimits_{\pi, temp} \Delta_{spec}(\pi_1, \pi)\ Trace\ \pi(fdg)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1.

The geometric transfer factors Δ_{geom} are defined in [LS] for any local field of characteristic zero, while the spectral factors Δ_{spec} are defined with similar structure, but only for the archimedean case, in [S2]. Compatibility is defined in terms of a canonical compatibility factor ([S2], Section 12) which exploits common features of the geometric and spectral constructions. This last factor appears to be the hardest to calculate concretely, but in practice there appears to be little reason to do so. Direct calculation can be avoided entirely in the case of quasi-split groups: because the relative term Δ_{III} is a quotient there, transitivity properties imply quickly that geometric and spectral Whittaker (or standard Δ_0) normalizations are compatible; see Lemma 12.3 of [S2]. In general, once we decide on a normalization for Δ_{geom}, the normalization of compatible Δ_{spec} is of course determined uniquely, and vice versa.

Theorem 2.1 is a condensed version of Theorem 6.2 in [S3], where the results are stated for K-groups, *i.e.* uniformly for certain families of inner twists. One purpose of that formulation is to extend Whittaker normalization for transfer factors to a broader collection of groups. This will be useful below, although we prefer to work with the less cumbersome notation for a single group.

We now apply Theorem 2.1 to pseudo-coefficients. Assume then G is cuspidal, making the discrete series for $G(\mathbb{R})$ nonempty, and that $\mathfrak{e}$ is elliptic, making H_1 also cuspidal. Given a discrete series representation π of $G(\mathbb{R})$, we define c to be a normalized pseudo-coefficient fdg for π, so that

$$Trace\ \pi'(fdg) = \delta_{\pi,\pi'},$$

for all tempered irreducible representations π' of $G(\mathbb{R})$. Here $\delta_{\pi,\pi'} = 1$ if $\pi = \pi'$ and $\delta_{\pi,\pi'} = 0$ otherwise. Existence of such a measure c is proved in [CD] where c is also required to be K-finite, with K a maximal compact subgroup of $G(\mathbb{R})$.

We use the transfer theorem to attach measure c_1 to c. If c' is chosen in place of c then $c - c'$ is annihilated by all tempered irreducible traces and therefore by the orbital integrals for all strongly regular elements in $G(\mathbb{R})$. Thus if c_1' is attached to c' by geometric transfer then c_1' is stably equivalent to c_1 in the sense that $c_1 - c_1'$ is annihilated by the stable orbital integrals for all strongly G-regular elements in $H_1(\mathbb{R})$ and hence by all stable tempered traces, by which we mean all *St-Trace* π_1, for π_1 tempered, irreducible, with $Z_1(\mathbb{R})$ acting by λ_1. It is clear then that *St-Trace* $\pi_1(c_1)$ depends only on π_1, π and the normalization of transfer factors Δ_{geom}. Moreover, since we also have transfer for Schwartz functions we could just as well have used a normalized genuine coefficient in place of c. Finally, compatible factors $\Delta_{geom}, \Delta_{spec}$ may be replaced only by $z\Delta_{geom}$, $z\Delta_{spec}$,where $z \in \mathbb{C}^\times$, and then c_1 is replaced by zc_1.

3. Stable traces, Whittaker normalization

Continue with the transfer c_1 of the pseudo-coefficient c for the discrete series representation π, and fix compatible transfer factors. Now write $\Delta(\pi_1, \pi)$ in place of $\Delta_{spec}(\pi_1, \pi)$. The transfer theorem implies immediately the following.

Lemma 3.1. *For all tempered irreducible representations π_1 of the endoscopic group $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1, we have*

$$St\text{-}Trace\ \pi_1(c_1) = \Delta(\pi_1, \pi).$$

Let φ_1 be the Langlands parameter for the packet Π_1 containing π_1. By construction, $\Delta(\pi_1, \pi) = \Delta(\pi'_1, \pi)$ for all $\pi'_1 \in \Pi_1$, and so we could write instead $\Delta(\varphi_1, \pi)$.

Lemma 3.2. *There are only finitely many parameters φ_1 for which $\Delta(\pi_1, \pi) \neq 0$. Such parameters are regular elliptic, so that Π_1 consists of discrete series representations. Moreover, if $\Delta(\pi_1, \pi) \neq 0$ then $\Delta(\pi_1, \pi') \neq 0$ for all π' in the packet of π.*

Proof. This is a familiar result which follows quickly from an explicit description of transfer for discrete series representations: see Section 11 of [S1] together with [S2] for this description. □

Corollary 3.3. *The measure c_1 is stably equivalent to a finite linear combination of pseudo-coefficients of discrete series representations of $H_1(\mathbb{R})$.*

An analysis of compatibility for $\Delta_{geom}, \Delta_{spec}$ [S2] shows that for suitable normalizations of Δ_{geom}, this linear combination is just a signed sum of normalized pseudo-coefficients.

There is a preferred choice of parameter φ_1 for those π_1 which are related to π, *i.e.* for which $\Delta(\pi_1, \pi) \neq 0$. Namely, we have introduced the notion of φ_1 being *well-positioned* for (the parameter φ attached to) the packet Π of π (see [S5]). First to transport data from groups to L-groups and vice-versa, we align, by choice of $\mathfrak{e}$ within its isomorphism class, Γ-splittings for the duals of G and H_1 as in Section 7 of [S2]; here $\Gamma = Gal(\mathbb{C}/\mathbb{R})$. Then we may define a representative for a regular elliptic parameter that is canonical for the relevant Γ-splitting (up to a conjugation which does not affect attached data). The parameter φ_1 is well-positioned for φ if it has as canonical representative $\phi_1 = \xi_1 \circ \phi$, where ϕ is canonical for φ. Here if $\mathfrak{e} = (H, \mathcal{H}, \mathfrak{s})$ denotes our set of endoscopic data then it has been arranged that the image of ϕ lies in $\mathcal{H}$, and ξ_1, from our chosen z-pair, is an L-homomorphism of $\mathcal{H}$ into ${}^L H_1$. It is convenient for calculations to fix also an $\mathbb{R}$-splitting of the quasi-split form G^*.

Lemma 3.4. *Suppose that φ is regular elliptic. Then there exists a unique parameter φ_1 well-positioned for φ. If φ_1, φ'_1 are well-positioned for the packets of discrete series representations π, π' respectively, then the relative transfer factor*

$$\Delta(\pi_1, \pi; \pi'_1, \pi')$$

reduces to the term

$$\Delta_{III}(\pi_1, \pi; \pi'_1, \pi'),$$

provided that we use the same data (a-, χ- and toral) in the construction of the terms $\Delta_I, \Delta_{II}, \Delta_{III}$ for both (π_1, π) and (π'_1, π').

Proof. For existence and uniqueness, see the comment and reference after Theorem 3.6 below. For the rest, we need only observe from the constructions in [S2] that, for the pairs (π_1, π) and (π'_1, π') as given, the relative terms Δ_I, Δ_{II} are trivial when the same *a*-, χ- and toral data are used. □

Suppose now that π, π' belong to the same packet Π. Assume only that $\Delta(\pi_1, \pi')$ is nonzero (so that φ_1 need not be well-positioned). We may compute the sign $\Delta_{III}(\pi_1, \pi; \pi'_1, \pi')$ internally in Π in the following sense. There are unique toral data for which $\pi = \pi(1)$ in the sense of Section 7 of [S2]. Continuing with the same reference, we write π' as $\pi(\omega)$, for some Weyl element ω. Then we define $inv(\pi, \pi')$ to be the cohomology class $inv(\pi(1), \pi(\omega))$ defined there. It belongs to $H^1(\Gamma, T_{sc})$, where T_{sc} is a maximal torus anisotropic over $\mathbb{R}$ in the simply-connected covering of the derived group of G, and its image in $H^1(\Gamma, T)$ is independent of the choice of ω. Finally, the prescribed toral data are used to obtain $\mathfrak{s}_\pi$ from the endoscopic datum $\mathfrak{s}$. Using the Tate-Nakayama pairing we obtain the following.

Lemma 3.5.

$$\Delta_{III}(\pi_1, \pi; \pi'_1, \pi') = \langle inv(\pi, \pi'), \mathfrak{s}_\pi \rangle^{-1}$$

Proof. Notice that the right side is simply a sign and that the formula is similar to (1) in Theorem 5.1.D of [KS] for conjugacy classes in a stable conjugacy class. We apply the argument for that case instead to the spectrally constructed objects, and notice that the discussion in Section 5.1 of [KS] simplifies since there is no twisting. See Section 9 of [S2]. □

What we have done so far applies to all compatible normalizations of transfer factors, with G arbitrary. Consider now the case that the group G is quasi-split over $\mathbb{R}$. Choose a $G(\mathbb{R})$-conjugacy class wh of Whittaker data for G. As we have already recalled, the attached normalizations Δ_{wh} are compatible.

Theorem 3.6. *([S3]) Suppose that φ_1 is the unique parameter well-positioned for Π and that π_{wh} is the unique member of Π that is generic for wh. Then*

$$\Delta_{wh}(\pi_1, \pi_{wh}) = 1.$$

This is Theorem 11.1 of [S3]. The parameter φ^{s} for π_1 that is constructed there (see Section 7 of [S3]) is clearly the well-positioned one (we may assume, without harm, that the endoscopic datum $\mathfrak{s}$ centralizes the image of a canonical representative for φ).

Lemma 3.7. *Continue the setting of the last lemma and, in particular, assume that φ_1 is well-positioned for Π. Consider the transfer c_1 of a pseudo-coefficient c for a given $\pi \in \Pi$. Then*

$$\textit{St-Trace } \pi_1(c_1) = \langle inv(\pi_{wh}, \pi), \mathfrak{s}_{\pi_{wh}} \rangle.$$

Proof. Since

$$\Delta_{wh}(\pi_1, \pi_{wh}) = \Delta(\pi_1, \pi_{wh}; \pi_1, \pi)\ \Delta_{wh}(\pi_1, \pi),$$

this follows from Lemmas 3.4 and 3.5, along with Theorem 3.6. □

The right side in Lemma 3.7 is calculated easily by first using some remarks of Langlands (partly described in [S6]) that provide a quick passage from the definition of $inv(\pi_{wh}, \pi)$ by Weyl group elements to a cocharacter. Then the pairing is given by evaluation on the transport $\mathfrak{s}_{\pi_{wh}}$ of the endoscopic datum $\mathfrak{s}$.

4. Other cases

The Whittaker normalizations, and the results above, extend to K-groups. In particular, we may extend Lemmas 3.4, 3.5 and 3.7 to certain inner forms (G, ψ) of a cuspidal G^* quasi-split over $\mathbb{R}$, those for which the (inner class of the) twist ψ is specified by the choice u of an element of the set $H^1(\Gamma, G^*_{sc})$. Here we take π in the packet Π^* for $G^*(\mathbb{R})$ attached to φ and π' in the packet Π for $G(\mathbb{R})$ also attached to φ; in our application π is chosen to be π_{wh}, and π' is then written π.

For Lemma 3.4, the term $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$, and then also $\Delta(\pi_1, \pi; \pi_1', \pi')$ itself, is defined in [S3] for any K-group; the definition parallels that of Kottwitz described in [A] for the geometric transfer factors. Since we limit our attention to a K-group of quasi-split type, this relative term is naturally a quotient. In Lemmas 3.5, 3.7, the pairing again yields a sign. That sign may then be written a product of two signs, one for position in Π (with basepoint fixed by means of π_{wh} and ψ) and one attached to u (see [S3]).

For the case that φ_1 is not well-positioned for Π, there is an (easy) explicit description of all parameters φ_1 related to Π which is convenient for handling the additional sign in $\Delta(\pi_1, \pi)$ produced by Δ_{II}; that sign is described in Section 9 of [S2].

For general (G, ψ), we may either work with a local hypothesis or return to the definition of the spectral transfer factors in terms of the relative factor $\Delta = \Delta_I \Delta_{II} \Delta_{III}$, noting that each relative term is a sign [S2] and that we may ignore Δ_I if we use a fixed related pair (π_1, π), where π is in the discrete series, to fix compatible normalizations and then follow the conventions of Lemma 3.4.

5. Alternative definition of spectral factors

We may of course use Lemma 3.1 for an alternative and simpler a priori definition, also available in other settings, for compatible spectral factors $\Delta(\pi_1, \pi)$, where π belongs to the discrete series. We proceed as follows.

Suppose that the strongly regular geometric transfer identity, *i.e.* the first half of Theorem 2.1, has been proved for the transfer factors Δ_{geom} from [LS]. Given a discrete series representation π of $G(\mathbb{R})$, choose a normalized (pseudo-) coefficient c as in Section 2, and define a measure c_1 by geometric transfer of c. If now π_1 is a tempered irreducible representation of $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1 then, as noted in Section 2, $St\text{-}Trace\ \pi_1(c_1)$ depends only on the pair (π_1, π) and the choice of normalization for Δ_{geom}. Define

$$\Delta_{new}(\pi_1, \pi) = St\text{-}Trace\ \pi_1(c_1).$$

To show that $\Delta_{new}(\pi_1, \pi)$ may replace $\Delta_{spec}(\pi_1, \pi)$ in the dual spectral transfer, *i.e.* to prove the second half of Theorem 2.1 when π is in the discrete series, we may invoke such transfer for the factors $\Delta_{spec}(\pi_1, \pi)$. It is then immediate from the transfer statement that

$$\Delta_{new}(\pi_1, \pi) = \Delta_{spec}(\pi_1, \pi)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ with $Z_1(\mathbb{R})$ acting by λ_1. At the same time we have available the various properties of $\Delta_{spec}(\pi_1, \pi)$, especially the adjoint relations that provide structure on the extended discrete series packets (see [S3]) as well as the simple formulas of the present note. To complete the definition of $\Delta_{new}(\pi_1, \pi)$ or $\Delta_{spec}(\pi_1, \pi)$ for all tempered π, we check first that both endoscopic transfer and tempered Langlands parameters behave well for (non-degenerate) coherent continuation to the wall and for parabolic induction. Then we make the definitions as we must for functoriality; see [S2].

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