

ORBITAL INTEGRALS, ENDOSCOPIC GROUPS AND
L-INDISTINGUISHABILITY FOR REAL GROUPS

D. Shelstad*

1. INTRODUCTION

Our purpose is to discuss the results of [S1], [S2], [S3] and [S4], or rather to provide some background for the discussion of them in [L5].

We begin with the characterization of stable orbital integrals. For a group over any field F of characteristic zero there is a partial ordering on the set of stable conjugacy classes of Cartan subgroups. In the case $F = \mathbb{R}$ the adjacent (classes of) Cartan subgroups are very simply described in terms

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of a suitably general notion of Cayley transform (see (2.2.2)). Perhaps the most significant feature of the characterization theorem for stable orbital integrals (Theorem 2.8.1) is that, besides the transformation and smoothness properties ((2.4.3) - (2.4.5), part (i) of (2.6.6)), we use only some information about the behavior of stable orbital integrals near certain (semi-regular) points γ_0 common to two adjacent Cartan subgroups ... roughly speaking, that around γ_0 the problems introduced by the non-compactness of $G(\mathbb{R})$ are independent of the Cartan subgroup (see (2.6.7)). This is also described by "jump" formulas (see Lemma 2.7.1).

In our preparation for Theorem 2.8.1 we discuss both the jump formulas and the germ expansions (around semi-regular elements only) and their equivalence. Although it is not necessary for the theorem, we study the stable integrals for the stable orbit of an element γ_0 as above; these terms require definition and we follow Kottwitz's suggestion ... it is not appropriate to take the sum of the integrals over the orbits in the stable orbit. We see then that the term Λ_1 from (2.6.2) is "intrinsic to G and its inner forms" and that the term Λ_0 arises from the noncompactness of $G(\mathbb{R})$.

After Theorem 2.8.1 we find that we have assembled enough facts to calculate, for all semisimple stable orbits, the stable integrals for certain functions which appear useful for applications of the trace formula ([L6]). The result is

Lemma 2.9.3. We complete Part 2 by recalling properties of $\mathfrak{A}(T)$, $\mathfrak{E}(T)$, $\mathfrak{K}(T)$ and κ -orbital integrals.

In Part 3 we switch to representations and the explicit parameterization of tempered characters. Central to our point of view are the embedding of the L-group of a maximal torus T in the L-group for G and the expression of a tempered L-packet parameter as the "lift" of a parameter for T (see (3.3) - (3.5)). We obtain various realizations of an L-packet in terms of the Knapp-Zuckerman basic characters.

To pair an L-packet with a finite group is not difficult, but to pair it with the group "S" which is naturally associated to the L-packet (and has analogues for all local and global fields) requires a uniform version of the Knapp-Zuckerman decomposition of unitary principal series. For this reason we include our L-group imitation of the Knapp-Zuckerman decomposition; it rests on Langlands' version of the Knapp R-group. We should note that the pairing and the later factoring of parameters in (5.5) require various choices. Our formulations are simply the most convenient ones and may well require modification for global applications; our aim for the present is to show that pairings exist so that identities as in (5.6) hold.

Our main results on the "internal structure" of L-packets are contained in (3.6); the needed Knapp-Zuckerman theory is described in (3.7).

Part 4 concerns endoscopic groups. We review briefly definitions, constructions and embeddings of L-groups. Our main interest is the normalization of κ -orbital integrals so that they match stable orbital integrals on an endoscopic group. Because of its usefulness elsewhere, we look at the local conditions on the normalization factors and indicate how our data from an embedding of the L-group of an endoscopic group in ${}^L G$ play a central role in "globalization". See (4.5).

Part 5 contains the main conclusions of "L-indistinguishability for the tempered spectrum of a real group."

In an appendix we mention two consequences of the theory. The first concerns Fourier inversion of orbital integrals and the second twisted orbital integrals for complex groups and the associated "twisted endoscopic groups" [S8]. We include also an example for $G = \text{SU}(2,1)$ which is elementary but of interest for both automorphic representations of a unitary group in three variables ([L5] and work of Flicker in progress) and a general local theory of nontempered L-packets.

Throughout the notes we give only outlines of proofs in print, with references. For new material details are included.

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2. ORBITAL INTEGRALS

2.1 Notation

We will follow as far as possible the notation of [L5]. Thus G will denote a connected reductive linear algebraic group defined over \mathbb{R} and $G(\mathbb{R})$ the group of \mathbb{R} -rational points on G ; σ_G will be the Galois automorphism of the group $G(\mathbb{C})$ of \mathbb{C} -rational points, so that $G(\mathbb{R}) = \{g \in G(\mathbb{C}) : \sigma_G(g) = g\}$.

Let T be a maximal torus in G defined over \mathbb{R} . Then $T(\mathbb{R})$ is a Cartan subgroup (CSG) of $G(\mathbb{R})$, and every Cartan subgroup of $G(\mathbb{R})$ is of this form. We denote by S_T , or simply S , the maximal \mathbb{R} -split torus in T , and by M_T or M the centralizer of S_T in G ; $M_T(\mathbb{R})$ is the "cuspidal Levi group in $G(\mathbb{R})$ defined by T ." We regard the set $R(G, T)$ of roots of T in G as a subset of $X^*(T)$, the group of rational characters on T . Recall that $\alpha \in R(G, T)$ is imaginary if $\sigma_T \alpha = -\alpha$, i.e. $\alpha \in R(M, T)$. The imaginary Weyl group of T is the subgroup of the Weyl group $\Omega(G, T)$ of T in G generated by the reflections with respect to the imaginary roots, i.e. the Weyl group $\Omega(M, T)$ of T in M . We may identify $\Omega(G, T)$ with $\text{Norm}(T(\mathbb{C}), G(\mathbb{C})) / T(\mathbb{C})$, where $\text{Norm}(A, B)$ indicates the normalizer of A in B . Then $\omega \in \Omega(G, T)$ is realized in the subset S of $G(\mathbb{C})$ if ω is identified with an element of $ST(\mathbb{C}) / T(\mathbb{C})$.

2.2 Adjacency of Cartan subgroups

If T, T' are maximal tori over \mathbb{R} in G then set $T \leq T'$ (and $T(\mathbb{R}) \leq T'(\mathbb{R})$) if and only if there exists $g \in G(\mathbb{C})$ such that $\text{ad } g^{-1}$ maps T to T' , i.e. $T' = g^{-1}Tg$, and the restriction of $\text{ad } g^{-1}$ to S_T is defined over \mathbb{R} , i.e. $\sigma_G(g)g^{-1} \in M_T$. An argument as in the proof of [S1, Theorem 2.1] shows that $T \leq T'$ if and only if S_T is $G(\mathbb{R})$ -conjugate to a subgroup of $S_{T'}$.

If $T \leq T'$ call T adjacent to T' when T is not $G(\mathbb{R})$ -conjugate to T' and $T \leq T'' \leq T'$ implies that T'' is $G(\mathbb{R})$ -conjugate to one of T, T' .

Lemma 2.2.1

Suppose that $T \leq T'$. Then T is adjacent to T' if and only if there exists an imaginary root α of T and $s \in G(\mathbb{C})$ such that $T' = s^{-1}Ts$ and $\sigma_G(s)s^{-1}$ realizes the Weyl reflection with respect to α .

Proof (outline): If $T \leq T'$ and such α, s exist then it follows that $\dim S_{T'} = 1 + \dim S_T$, and adjacency is immediate. On the other hand, suppose that $T \leq T'$ and that T is adjacent to T' . It is sufficient to consider the case that T is compact modulo the center of G . Then the existence of α, s is established by standard methods. See [S1, Section 2] for similar arguments.

Definition 2.2.2

If α, s are as in the lemma then s is a Cayley transform with respect to α .

Recall that $\alpha \in R(G, T)$ determines a three-dimensional simple complex Lie algebra, namely $\mathbb{C}X_\alpha + \mathbb{C}H_\alpha + \mathbb{C}X_{-\alpha}$, where $X_\alpha, X_{-\alpha}$ are root vectors for $\alpha, -\alpha$ respectively, and $H_\alpha = [X_\alpha, X_{-\alpha}]$. If α is imaginary then this algebra is invariant under σ_G and so its σ_G -fixed points form a three dimensional simple real Lie algebra; α is compact if this algebra is of compact type, and noncompact otherwise.

Definition 2.2.3

An imaginary root α of T is totally compact if every root in the orbit of α under the imaginary Weyl group (or under $\Omega(T)$, see (2.4)) is compact.

Theorem 2.2.4

- (i) There exists a Cayley transform with respect to the imaginary root α if and only if α is not totally compact.
- (ii) If G is quasi-split over \mathbb{R} then no T in G has totally compact roots.

Proof: See [S1, Proposition 4.11] and [S2, Lemma 9.2].

2.3 Orbital integrals

Fix a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ and Haar measures dt on $T(\mathbb{R})$ and dg on $G(\mathbb{R})$. Let G_{reg} denote the set of regular elements in G and $T(\mathbb{R})_{\text{reg}} = T(\mathbb{R}) \cap G_{\text{reg}}$. Fix $\gamma \in T(\mathbb{R})_{\text{reg}}$ and $f \in C_c^\infty(G(\mathbb{R}))$. Then by a compactness principle of Harish-Chandra (see [W2, Theorem 8.1.4.1]) the functions $T(\mathbb{R})g \rightarrow f(g^{-1}\gamma'g)$, for γ' in a suitable neighborhood of γ in $T(\mathbb{R})_{\text{reg}}$, are supported on a common compact subset of

$T(\mathbb{R}) \backslash G(\mathbb{R})$. It then follows that

$$\phi_T(\gamma, f) = \phi_T(\gamma, f; dt, dg) = \int_{T(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\gamma g) \frac{dg}{dt}$$

is well-defined and $\gamma \rightarrow \phi_T(\gamma, f)$ is a C^∞ -function on $T(\mathbb{R})_{\text{reg}}$.

The orbit $O(\gamma)$ of $\gamma \in T(\mathbb{R})_{\text{reg}}$ is the conjugacy class of γ in $G(\mathbb{R})$. Since T is of finite index in the centralizer G_γ of γ in G and $O(\gamma)$ is homeomorphic to $G_\gamma(\mathbb{R}) \backslash G(\mathbb{R})$ via the map $g^{-1}\gamma g \rightarrow G_\gamma(\mathbb{R})g$, $\phi_T(\gamma, f)$ is the integral of f over $O(\gamma)$ relative to a certain $G(\mathbb{R})$ -invariant measure. We will use the term orbital integral for $\phi_T(\gamma, f)$. When multiplied by a suitable function of γ , $\phi_T(\gamma, f)$ becomes Harish-Chandra's $'F_f(\gamma)$ [HC].

Let \mathcal{O} be an open subset of $G(\mathbb{R})$. Then a C^∞ -function on \mathcal{O} is a Schwartz function on \mathcal{O} , i.e. belongs to the Schwartz space $\mathcal{C}(\mathcal{O})$, if it and all its left and right derivatives are rapidly decreasing on \mathcal{O} in the sense of Harish-Chandra [HC]. We will need the results encompassed by Harish-Chandra's theorem on the $'F_f$ transform for $\mathcal{C}(G(\mathbb{R}))$ [HC, Theorem 17.1] (see [W2, Section 8.5]). First is the assertion that for $f \in \mathcal{C}(G(\mathbb{R}))$ the integral $\phi_T(\gamma, f)$, defined formally as for functions of compact support, is absolutely convergent and that the function $\gamma \rightarrow (\prod_{\alpha \in R(G, T)} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi_T(\gamma, f)$, where the product is over all $\alpha \in R(G, T)$, belongs to the Schwartz space of the open subset $T(\mathbb{R})_{\text{reg}}$ of $T(\mathbb{R})$.

2.4 Stable orbital integrals

If G is semisimple and simply-connected then the stable orbit of $\gamma \in T(\mathbb{R})_{\text{reg}}$ is the intersection of $G(\mathbb{R})$ with the orbit of γ in $G(\mathbb{C})$. In general, however, the stable orbit is contained in this intersection. For the precise definition recall that $\mathcal{O}(T) = \mathcal{O}_G(T) = \{g \in G(\mathbb{C}) : g^{-1}T(\mathbb{R})g \subset G(\mathbb{R})\}$ and that $\mathcal{D}(T) = \mathcal{D}_G(T) = T(\mathbb{C}) \backslash \mathcal{O}(T) / G(\mathbb{R})$. Then the stable orbit of γ is $\{w^{-1}\gamma w : w \in \mathcal{O}(T)\}$. If γ is strongly regular, i.e. $G_\gamma = T$, then $\mathcal{D}(T)$ parametrizes the orbits in the stable orbit of γ .

Lemma 2.4.1

There is a bijection between $\mathcal{D}_G(T)$ and $\Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$. Here $\Omega(M(\mathbb{R}), T(\mathbb{R}))$ denotes the subgroup of $\Omega(M, T)$ consisting of elements realized in $M(\mathbb{R})$.

Proof: See[S1, Theorem 2.1]. In summary: (i) every element of $\mathcal{D}_G(T)$ has a representative in M , so that $\mathcal{D}_G(T) = \mathcal{D}_M(T) \dots$ an analogue is true for groups over any field of characteristic zero ... and (ii) the representative from (i) can be chosen to normalize T , because a real group, in this case $M(\mathbb{R})$, has (at most) one conjugacy class of CSG's compact modulo the center. Finally, each element of $\Omega(M, T)$ preserves $T(\mathbb{R})$.

Definition 2.4.2

For $f \in \mathcal{C}(G(\mathbb{R}))$, $\gamma \in T(\mathbb{R})_{\text{reg}}$ and Haar measures dt on $T(\mathbb{R})$ and dg on $G(\mathbb{R})$ set

$$\phi_T^{\text{st}}(\gamma, f) = \phi_T^{\text{st}}(\gamma, f; dt, dg) = [\Omega(M(\mathbb{R}), T(\mathbb{R}))]^{-1} \sum \phi_T(\gamma^w, f; dt, dg),$$

where the summation is over w in $\Omega(M, T)$.

This, clearly, is the stable orbital integral of f in the sense

of Langlands [L5]. Note that by γ^ω we mean $w^{-1}\gamma w$, where $w \in G(\mathbb{C})$ realizes ω . In view of Harish-Chandra's result for $\phi_T(\gamma, f)$ we have immediately that $\phi_T^{\text{st}}(\gamma, f)$ is well-defined and that the function $\gamma \rightarrow (\prod_{\alpha} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi_T^{\text{st}}(\gamma, f)$, where the product is over all $\alpha \in R(G, T)$, belongs to the Schwartz space of $T(\mathbb{R})_{\text{reg}}$.

We now write $\phi^T(\gamma, dt, dg)$ in place of $\phi_T^{\text{st}}(\gamma, f; dt, dg)$.

Then:

$$(2.4.3) \quad \phi^T(\gamma, \alpha dt, \beta dg) = \beta/\alpha \phi^T(\gamma, dt, dg) \quad \text{for } \alpha, \beta > 0,$$

$$(2.4.4) \quad \phi^T(\gamma, dt, dg) = \phi^{T^w}(\gamma^w, (dt)^w, dg) \quad \text{for } w \in \mathcal{O}(T),$$

where $T^w = w^{-1}T w$, etc.,

(2.4.5) $\gamma \rightarrow (\prod_{\alpha} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi^T(\gamma, dt, dg)$ extends smoothly to (i.e. extends to a Schwartz function on)

$$T_{\text{reg}}^I(\mathbb{R}) = \{\gamma \in T(\mathbb{R}) : \alpha(\gamma) \neq 1, \alpha \in R(M, T)\} = T(\mathbb{R}) \cap M_{\text{reg}}.$$

For (2.4.3) and (2.4.4) the proof is immediate; (2.4.5) follows from the analogous result for $\phi_T(\gamma, f)$ which is part of [HC, Theorem 17.1] and is proved, for example, by "descent to M " (see [W2, Section 8.5]).

The boundary of $T_{\text{reg}}^I(\mathbb{R})$ is the union of the "imaginary walls" in $T(\mathbb{R})$, i.e. the union of the kernels in $T(\mathbb{R})$ of the imaginary roots. We will need to know the behavior of ϕ^T near points that lie on a single wall.

2.5 Semi-regular elements

Suppose that $\gamma, \gamma' \in G(\mathbb{R})$ are semisimple, and that $\gamma' = g^{-1}\gamma g$ for some $g \in G(\mathbb{C})$. Then $\sigma_G(g)g^{-1}$ lies in $G_{\gamma}(\mathbb{C})$,

the centralizer of γ in $G(\mathbb{C})$. Following Kottwitz's suggestion we call γ and γ' stably conjugate if g can be chosen so that $\sigma_G(g)g^{-1}$ lies in $G_Y^0(\mathbb{C})$, G_Y^0 indicating the connected component of the identity in G_Y . The stable orbit, or stable conjugacy class, of γ is then the set of all elements stably conjugate to γ . This generalizes the notion for regular semisimple elements.

Lemma 2.5.1

Let $\gamma \in G(\mathbb{R})$ be semisimple. Then:

(i) there exists a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ such that $\gamma \in T(\mathbb{R})$ and if $\gamma \in T'(\mathbb{R})$ then $T \leq T'$. Then say that γ occurs fundamentally in $T(\mathbb{R})$.

(ii) If $\gamma \in T'(\mathbb{R})$ then $\{w^{-1}\gamma w : w \in \mathcal{A}(T')\}$ is contained in the stable orbit of γ .

(iii) If γ occurs fundamentally in $T(\mathbb{R})$ then $\{w^{-1}\gamma w : w \in \mathcal{A}(T)\}$ coincides with the stable orbit of γ .

Proof: For (i), suppose that $\gamma \in T'(\mathbb{R})$. Then $T' \subset G_Y^0$ and T satisfies the conditions of (i) if and only if T is a fundamental maximal torus in G_Y^0 . Thus (i) follows; (ii) is immediate from the definitions. For (iii) we have to show that $\{w^{-1}\gamma w : w \in \mathcal{A}(T)\}$ contains the stable orbit of γ . Suppose that $\gamma' = g^{-1}\gamma g$, where $\sigma_G(g)g^{-1} \in G_Y^0$. Then $\text{ad } g^{-1} : G_Y^0 \rightarrow G_Y^0$, is an inner twist. [S1, Lemma 2.8] implies then that there is h in G_Y^0 , such that $\text{ad } h \circ \text{ad } g^{-1}|_T$ is defined over \mathbb{R} . Let $w = gh^{-1}$. Then $w \in \mathcal{A}(T)$ and $\gamma' = w^{-1}\gamma w$. Thus the lemma is proved.

Returning to the notation of (2.4) we assume that γ_0 is an element of $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$. Fix an imaginary root α such that $\alpha(\gamma_0) = 1$. Assume that γ_0 is semi-regular, i.e. that if $\beta(\gamma_0) = 1$, $\beta \in R(G, T)$, then $\beta = \pm \alpha$, i.e. that G_{γ_0} is of type A_1 . Note that γ_0 occurs fundamentally in $T(\mathbb{R})$.

Call γ_0 of totally compact type if α is totally compact; this means that for γ'_0 in the stable orbit of γ_0 the group $G_{\gamma'_0}$ is anisotropic modulo its center.

Lemma 2.5.2

For γ_0 semi-regular in $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$ we have that:

- (i) the stable orbit of γ_0 meets only CSG's conjugate to $T(\mathbb{R})$ if γ_0 is of totally compact type,
- (ii) the stable orbit of γ_0 meets CSG's conjugate to $T(\mathbb{R})$ and exactly one conjugacy class of CSG's adjacent to $T(\mathbb{R})$ if γ_0 is not of totally compact type.

Proof: (i) is clear from the definitions. For (ii) "at most one" rather than "exactly one" is clear since for γ'_0 in the stable orbit of γ_0 , $G_{\gamma'_0}$ contains at most two conjugacy classes of CSG's and the union of the CSG's in the various $G_{\gamma'_0}$ forms two conjugacy classes in $G(\mathbb{R})$. To produce the "exactly one" adjacent conjugacy class, choose a Cayley transform $s: T \rightarrow T^S$ with respect to α (recall (2.2.2)). Then γ_0^S is contained in $T^S(\mathbb{R})$ and is stably conjugate to γ_0 (see [S1, Section 2]). The conjugacy class of $T^S(\mathbb{R})$ is the desired one.

2.6 Stable orbital integrals (continued)

Fix $f \in \mathcal{C}(G(\mathbb{R}))$ and a CSG $T(\mathbb{R})$. For $\gamma \in T$ we write $|D(\gamma)|$ for $\prod |1 - \alpha(\gamma^{-1})|$, the product being taken over $\alpha \in R(G, T)$. Also we write $\psi^T(\gamma) = \psi^T(\gamma, f; dt, dg)$ for the smooth extension of $|D(\gamma)|^{1/2} \phi^T(\gamma) = |D(\gamma)|^{1/2} \phi_T^{\text{st}}(\gamma, f; dt, dg)$ to $T_{\text{reg}}^I(\mathbb{R})$, and $\mathcal{A}_T(T)$ for the set of elements of $\mathcal{A}(T)$ which normalize T .

Lemma 2.6.1

Let γ_0 be a semi-regular element in $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$. Then there is a neighborhood $N(\gamma_0)$ of γ_0 in $T(\mathbb{R})$ invariant under $\mathcal{A}_T(T)$, and on $N(\gamma_0)$ C^∞ -functions $\gamma \rightarrow \Lambda_i^T(\gamma_0, \gamma)$, $i = 1, 2$, (depending on f, dt, dg also and written $\Lambda_i^T(\gamma_0, \gamma, f; dt, dg)$ when the occasion demands) such that:

$$(2.6.2) \quad \psi^T(\gamma) = \Lambda_1^T(\gamma_0, \gamma) |D(\gamma)|^{1/2} + \Lambda_0^T(\gamma_0, \gamma)$$

for all $\gamma \in N(\gamma_0) \cap T_{\text{reg}}^I(\mathbb{R})$.

Proof: See (2.7).

Let $\alpha(\gamma_0) = 1$; recall that the root α is unique up to sign. The Weyl reflection ω_α is realized in $\mathcal{A}_T(T)$. Thus if X is an invariant differential operator on T odd with respect to ω_α we have, with the obvious abuse of notation,

$$(2.6.3) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0) = 0.$$

Assume that X is fixed by ω_α . Set $\gamma_\nu = \gamma_0 \exp \nu \alpha^\vee$. Then for ν in some deleted neighborhood of zero $\gamma_\nu \in T_{\text{reg}}^I(\mathbb{R})$. Applying X to (2.6.2) we obtain easily that:

$$(2.6.4) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0) = \lim_{\nu \rightarrow 0} (X\psi^T)(\gamma_\nu).$$

On the other hand, $(X\Lambda_0^T)(\gamma_0, \gamma_0)$ is computed explicitly by, for example, reduction to SL_2 (see references next page); the answer involves integrals along non-semisimple orbits. None of this is needed here. Nor do we need further information on $(X\Lambda_1^T)(\gamma_0, \gamma_0)$, although we will stop to mention $\Lambda_1^T(\gamma_0, \gamma_0)$ as an illustration of Kottwitz's suggestion about stable orbital integrals for non-regular semisimple orbits [Ko] and as pre-
amble to (2.9). Note that (2.6.2) gives:

$$(2.6.5) \quad \Lambda_1^T(\gamma_0, \gamma_0) = 1/2 \lim_{v \rightarrow 0} (H_\alpha(|1 - \alpha^{-1}| \phi^T))(\gamma_v),$$

where H_α denotes the coroot of α regarded as element of the Lie algebra of T .

There are two properties of $X\Lambda_0^T$ which will be needed:

Lemma 2.6.6

Assume that X is fixed by ω_α .

(i) If γ_0 is of totally compact type then $(X\Lambda_0^T)(\gamma_0, \gamma_0) = 0$.

(ii) If γ_0 is not of totally compact type then $(X\Lambda_0^T)(\gamma_0, \gamma_0)$

is independent of T in the following sense. Suppose that

$\gamma_0 \in T'$. This implies that T' is either $G(\mathbb{R})$ -conjugate to

or adjacent to T . Then $T' = s^{-1}Ts$, with s either \mathbb{R} -

rational or a Cayley transform with respect to the root α

which annihilates γ_0 and:

$$(2.6.7) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0, f; dt, dg) = (X^s \Lambda_0^{Ts})(\gamma^s, \gamma_0^s, f; (dt)^s, dg),$$

where in the case that s is a Cayley transform the right side

is to be interpreted as $(X^s \Psi^{Ts})(\gamma_0^s)$.

Proof: See (2.7).

Note that $\gamma_0^s \in (T^s)_{\text{reg}}^I(\mathbb{R})$, that $(dt)^s$ is defined as in [S1]

and that the right side of (2.6.7) is independent of the choice of s (see [S1, Proposition 2.7]).

To compute $\Lambda_1^T(\gamma_0, \gamma_0)$ we first assume that $f \in C_c^\infty(G(\mathbb{R}))$. Suppose that γ_0 is of totally compact type. Recall that this means that G_δ is anisotropic modulo center, for all δ in the stable orbit of γ_0 . Part (i) of the lemma above says simply that ϕ^T itself extends smoothly to γ_0 , so that $\Lambda_1^T(\gamma_0, \gamma_0)$ is nothing but $\lim_{\gamma \rightarrow \gamma_0} \phi^T(\gamma)$; i.e. the behavior is as if $G(\mathbb{R})$ were compact. Reduction to $SL_2/SU(2)$ (... in this case $SU(2)$) via Harish-Chandra's compactness principle (see [W2, Section 8.5] or [S1, page 17], for example) then shows that $\Lambda_1^T(\gamma_0, \gamma_0)$ is

the sum over (representatives δ for the elements of) $\mathcal{Q}(T)$ of

$$\text{vol}(T(\mathbb{R}) \backslash G_\delta^0(\mathbb{R})) \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where dh_δ is a Haar measure on $G_\delta^0(\mathbb{R})$ to be used also in the calculation of the volume (so that the choice is of no consequence).

If $\delta = w\gamma_0$, with w in $\mathcal{Q}(T)$ then $\text{ad}w: G_\delta^0 \rightarrow G_{\gamma_0}^0$ is an inner twist (recall the proof of Lemma 2.5.1) and, in fact, an \mathbb{R} -isomorphism since both groups are anisotropic modulo center.

Fix a Haar measure dh on $G_{\gamma_0}^0(\mathbb{R})$ and take dh_δ to be the twist of dh by w . This allows us to rewrite the sum as

$$\text{vol}(T(\mathbb{R}) \backslash G_{\gamma_0}^0(\mathbb{R})) \sum_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where the summation remains over $\mathcal{Q}(T)$.

This, however, is misleading for a general definition of "stable orbital integral of f relative to γ_0 ." Indeed,

suppose that γ_0 is not of totally compact type. Then by reduction to $SL_2/SU(2)$ we obtain that the right side of (2.6.5) equals

$$\text{vol}(\bar{T}(\mathbb{R}) \backslash \bar{G}_{\gamma_0}^0(\mathbb{R})) \sum (-1)^{q(G_\delta^0)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where the summation is now over $\Omega(G_{\gamma_0}^0, T) \backslash \Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$ [$\delta = \gamma_0^w$; $w \in \mathcal{O}(T)$ realizes an element ω of $\Omega(M, T)$ and then ω represents an element of the double quotient]; $\bar{\quad}$ indicates passage by an inner twist to the form anisotropic modulo center [L3, Chapter 6]. The volume is to be calculated using the twists of dh, dt to \bar{G}, \bar{T} respectively; dh_δ is as before. As in [Ko], $q(G_\delta^0)$ is one-half the dimension of the symmetric space attached to G_δ^0 , i.e. we take $+1$ for $(-1)^q$ if G_δ^0 is anisotropic modulo center, but -1 otherwise.

For a Haar measure dh on $G_{\gamma_0}^0(\mathbb{R})$ we set

$$O^{\text{st}}(\gamma_0, f) = O^{\text{st}}(\gamma_0, f; dh, dg) = \sum (-1)^{q(G_\delta^0)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

the summation being over $\Omega(G_{\gamma_0}^0, T) \backslash \Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$, as above. The conventions for measures remain the same also. Then $O^{\text{st}}(\gamma_0, f)$ is well-defined for all $f \in \mathcal{C}(G(\mathbb{R}))$ (see [W2, Section 9.3.1]) and $f \mapsto O^{\text{st}}(\gamma_0, f)$ is a stable tempered distribution in the sense of [S1, Section 6] (see (2.9)).

Lemma 2.6.8

$\Lambda_1^T(\gamma_0, \gamma_0, f; dt, dg) = \text{vol}(\bar{T}(\mathbb{R}) \backslash \bar{G}_{\gamma_0}^0(\mathbb{R})) O^{\text{st}}(\gamma_0, f; dh, dg)$
for all $f \in \mathcal{C}(G(\mathbb{R}))$.

Note that when γ_0 is of totally compact type the double coset space in the summation coincides with $\mathfrak{A}(T)$ itself. Thus our remarks above prove the lemma when $f \in C_c^\infty(G(\mathbb{R}))$. A simple continuity argument extends the formula to the Schwartz space.

2.7 Jump formulas

The results of Lemmas 2.6.1 and 2.6.6 can be expressed in terms of "jump" formulas analogous to those for $'F_f$ given by Harish-Chandra. It is these formulas rather than the expansions (2.6.2) which guide us to the construction in (4.5) of the "transfer factors" for the main Matching Theorem.

First we will change ψ^T to a function $\bar{\psi}^T$ by replacing $|D(\gamma)|^{1/2}$ with the normalizing factor for $'F_f$. This factor will depend on our choice of a positive system for the imaginary roots of T ; care in that choice will greatly simplify the results.

For any choice of positive imaginary roots the function $\gamma \rightarrow \prod_{\alpha \in I^+} (1 - \alpha(\gamma^{-1})) \prod_{\alpha \in I^-} |1 - \alpha(\gamma^{-1})|^{1/2} \phi^T(\gamma)$, where the first product is over all positive imaginary roots and the second over all non-imaginary roots, is defined on $T(\mathbb{R})_{\text{reg}}$ and equals ψ^T up to a factor which is bounded and has a C^∞ -extension to $T_{\text{reg}}^I(\mathbb{R})$. Thus it extends to a Schwartz function on $T_{\text{reg}}^I(\mathbb{R})$; this function will be denoted $\bar{\psi}^T$, mention of the choice of positive imaginary roots being omitted in notation. In (2.7.2) any choice will do. For (2.7.3) we need some preparation.

Let α be an imaginary root of T . Then a positive system for the imaginary roots is adapted to α if all imaginary roots β for which $\langle \alpha, \beta^\vee \rangle > 0$ are positive. If I^+ is such a system and s is a Cayley transform with respect to α then there is a unique positive system for the imaginary

roots of T^S transported into I^+ by ads ; denote this system by I_S^+ . In (2.7.3) we use a system I^+ adapted to either one of the two roots annihilating γ_0 and the related system I_S^+ for T^S . Also, denote by $X \rightarrow X'$ the automorphism of the algebra of invariant differential operators on T induced by the map $H \rightarrow H + \nu(H)I$, for H in the Lie algebra of T , where ν is one-half the sum of the roots in I^+ ; let $X \rightarrow X''$ be the automorphism of the operators on T^S induced by the map $H \rightarrow H + \nu_S(H)I$, for H in the Lie algebra of T^S , with ν_S one-half the sum of the roots in I_S^+ .

Lemma 2.7.1

Suppose that $\{\gamma \rightarrow \phi^T(\gamma) = \phi^T(\gamma, dt, dg)\}$ is a family of functions satisfying (2.4.3), (2.4.4) and (2.4.5). Form $\psi^T, \bar{\psi}^T$ as above.

Then $\{\psi^T\}$ satisfies the statements of Lemmas 2.6.1 and 2.6.6 for all semi-regular elements γ_0 in $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$ if and only if for all such γ_0 we have:

(2.7.2) $\lim_{\nu \rightarrow 0} (X\bar{\psi}^T)(\gamma_\nu) = 0$ if γ_0 is of totally compact type
and

(2.7.3) $\lim_{\nu \rightarrow 0} (X'\bar{\psi}^T)(\gamma_\nu) = i ((X^S)''\bar{\psi}^{T^S})(\gamma_0^S)$ if γ_0 is not of
totally compact type,

for all invariant differential operators X on T .

Proof: Note that for X odd with respect to the Weyl reflection ω_α , α a root annihilating γ_0 , (2.7.2) and (2.7.3) are automatically satisfied. Thus in the statement of the lemma we may as well take X fixed by ω_α . Then (2.7.2) follows from (2.6.4) and (i) in Lemma 2.6.6; (2.7.3) follows from (2.6.4) and (ii)

in Lemma 2.6.6, although now some (elementary) calculations are needed. We omit the details (see [S1, page 27] regarding a crucial property of I^+).

For the converse, assume (2.7.2) and (2.7.3). We have just to verify the existence of expansions (2.6.2) for then (2.6.4) is true and reversing the arguments of the first part of the proof yields Lemma 2.6.6. Fix T and γ_0 . If γ_0 is of totally compact type then (2.7.2) implies that $\bar{\psi}^T$ is C^∞ around γ_0 . It then follows easily that ϕ^T itself is C^∞ around γ_0 also, and (2.6.2) is true.

Now assume that γ_0 is not of totally compact type. We will give just an outline of the steps involved in verifying an expansion (2.6.2). We may assume that G is simply-connected and semisimple, first by a routine reduction to $M = M_T$ and secondly by reduction to the simply-connected covering of the derived group of M . This second reduction rests on the definition of stable conjugacy. Next we define Λ_0^T . Fix a Cayley transform s with respect to a root α annihilating γ_0 . Then the restriction of $\bar{\psi}^{T^S}$ to a suitable neighborhood of γ_0^S in $T^S(\mathbb{R})$ can be extended to an $\Omega(G, T)$ -invariant C^∞ -function on a neighborhood of γ_0^S in $T^S(\mathbb{C})$. Transport this function to $T(\mathbb{C})$ by s and restrict to (a neighborhood of γ_0 in) $T(\mathbb{R})$. This defines our Λ_0^T . Note that (2.6.7) is then automatic. Finally we use (2.7.3) to show that ψ^T is C^∞ -divisible by $|D(\gamma)|^{1/2}$ around γ_0 . See [L3, Chapter 6] for a similar style of argument.

It remains then to recall the proofs of Lemmas 2.6.1 and 2.6.6 and the formulas (2.7.2) and (2.7.3) for $\phi^T(\gamma) = \phi_T^{\text{st}}(\gamma, f)$. The formulas are proved in Section 4 of [S1]. The arguments combine those of Harish-Chandra for $'F_f$ (involving reduction to $SL_2/SU(2)$) with facts about the orbits in a stable orbit. For example, the left side of (2.7.3) involves $[\mathfrak{L}(T)]$ orbits and the right side $[\mathfrak{L}(T^S)]$ orbits, but only either $[\mathfrak{L}(T^S)]$ or $2[\mathfrak{L}(T^S)]$ orbits on the left contribute to the limit. For the proofs of Lemmas 2.6.1 and 2.6.6 we can invoke the last lemma, or argue directly by reduction to $SL_2/SU(2)$, recalling again [L3, Chapter 6] and [S1, Section 4].

2.8 Characterization of stable orbital integrals

Theorem 2.8.1

Suppose that $\{\gamma \rightarrow \phi^T(\gamma, dt, dg)\}$ is a family satisfying (2.4.3), (2.4.4), (2.4.5), (2.7.2) and (2.7.3). Then there exists $f \in \mathcal{C}(G(\mathbb{R}))$ such that:

$$\phi^T(\gamma, dt, dg) = \phi_T^{\text{st}}(\gamma, f; dt, dg), \quad \gamma \in T(\mathbb{R})_{\text{reg}},$$

for all T , dt and dg .

Proof: See [S1, Theorem 4.7]. Suppose that $\phi^{T'} = 0$ whenever $T' \not\geq T$ and T' is not $G(\mathbb{R})$ -conjugate to T . Then $\bar{\psi}^T$ extends smoothly to the union of the regular and the semi-regular elements in $T(\mathbb{R})$ and thence, by a well-known principle of Harish-Chandra, smoothly to all of $T(\mathbb{R})$, i.e. $\bar{\psi}^T$ extends to a Schwartz function on $T(\mathbb{R})$. An inductive argument then

shows that to prove the theorem we need only verify the following for each Cartan subgroup $T(\mathbb{R})$:

(2.8.2) if $\bar{\psi}^T$ extends to a Schwartz function on $T(\mathbb{R})$ then there exists $f \in \mathcal{C}(G(\mathbb{R}))$ such that $\phi_T^{\text{st}}(\cdot, f) \equiv \phi^T$ and $\phi_{T'}^{\text{st}}(\cdot, f) \equiv 0$ unless $T' \leq T$.

In [S1] we construct f as a sum of (scalar projections of) wave-packets. In the case that $T(\mathbb{R})$ is compact the constructed f is a sum of K -finite matrix coefficients of discrete series representations, with some constraints on the K -types involved.

2.9 A limit formula

The following is a short exercise based on our discussion so far (and some fundamental results of Harish-Chandra for which we refer to [W2]), and concerns one of the simplest problems associated with applications of the trace formula ([L4], [L6]).

Suppose that $G(\mathbb{R})$ has Cartan subgroups compact modulo the center of $G(\mathbb{R})$ (CCSG's). We consider a function f with the following property:

$$(2.9.1) \quad \phi_{T'}^{\text{st}}(\cdot, f) \equiv 0 \text{ unless } T'(\mathbb{R}) \text{ is a CCSG.}$$

In [L4] (see also [L6]) the constraints on f are greater; moreover, $\phi_T^{\text{st}}(\cdot, f)$ is specified for $T(\mathbb{R})$ a CCSG, and f is to be C_c^∞ . This will not concern us here.

Recall that (2.9.1) implies that $\bar{\psi}^T$ extends to a Schwartz function on $T(\mathbb{R})$, for $T(\mathbb{R})$ a CCSG.

Lemma 2.9.2

Assume (2.9.1). Then $\gamma \mapsto \Phi_T^{\text{St}}(\gamma, f)$ extends smoothly to $T(\mathbb{R})$; for each CCSG $T(\mathbb{R})$.

Proof: After a straightforward reduction to the simply-connected covering of the derived group of G , we find that it is sufficient to show the following. Suppose that G is simply connected and semisimple and that $T(\mathbb{R})$ is a CCSG. Then a C^∞ -function F on $T(\mathbb{R})$ satisfying $F(\gamma^\omega) = \det \omega F(\gamma)$, $\omega \in \Omega(G, T)$, is C^∞ -divisible by Δ , where, as usual,

$$\Delta(\gamma) = \prod_{\alpha \in \Sigma^+} (1 - \alpha(\gamma^{-1})),$$

α denoting one-half the sum of the roots in some positive system for $R(G, T)$ and the product being taken over these roots. But this follows easily from rearrangement of the Fourier series for F .

For any $\gamma_0 \in T(\mathbb{R})$ we define $O^{\text{St}}(\gamma_0, f; dh, dg)$ by the formula of (2.6). Thus:

$$O^{\text{St}}(\gamma_0, f; dh, dg) = \sum (-1)^{q(\delta)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where $q(\delta) = q(G_\delta^0)$ is well-defined since $G(\mathbb{R})$ has CCSG's, δ is of the form γ_0^w , with w in $\mathcal{U}(T)$, dh_δ is the twist of dh (a Haar measure on $G_{\gamma_0}^0$) by w , and the summation is over w in a set of representatives for

$$\Omega(G_{\gamma_0}^0, T) \backslash \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R}));$$

more precisely, we choose representatives \underline{w} in $\Omega(G, T)$ for these double cosets and then for each \underline{w} an element w

in G (i.e. in $\tilde{\mathcal{A}}_T(T)$) realizing \underline{w} . Note that relative to the natural projection of $\mathcal{D}(T) = \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R}))$ onto $\Omega(G_{\gamma_0}^0, T) \setminus \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R}))$ the preimage of the double coset of $\underline{w} \in \Omega(G, T)$ contains $[\Omega(G_\delta^0, T) / \Omega(G_\delta^0(\mathbb{R}), T(\mathbb{R}))]$ = $[\mathcal{D}_{G_\delta}^0(T)]$ elements, where $\delta = (\gamma_0)^{\underline{w}}$.

Lemma 2.9.3

Assume (2.9.1). Then:

$$\lim_{\gamma \rightarrow \gamma_0} \phi_T^{\text{st}}(\gamma, f; dt, dg) = \text{vol}(\bar{T}(\mathbb{R}) \setminus \bar{G}_{\gamma_0}^0(\mathbb{R})) O^{\text{st}}(\gamma_0, f; dh, dg)$$

for each CCSG $T(\mathbb{R})$.

Here, as in (2.6), $\bar{\quad}$ indicates passage to the inner form anisotropic modulo center, and the volume is to be calculated relative to the twists of dh, dt (so that the right side of our formula, like the left side, is independent of the choice for dh).

We will include a proof of the lemma just for the case G simply-connected and semisimple, where a CCSG $T(\mathbb{R})$ is compact and we have at our disposal the normalizing factor Δ from the proof of Lemma 2.9.2. The general case involves only minor modification of the arguments, but considerably more notation.

For the rest of this section, then, G will be simply connected and semisimple and $T(\mathbb{R})$ compact. If $\delta = \gamma_0^{\underline{w}}$, with $\underline{w} \in \tilde{\mathcal{A}}_T(T)$, then we write $H_{\underline{w}}$ for G_δ^0 ; in particular, $H_1 = G_{\gamma_0}^0$. The lemma has only to be proved for one choice of dg, dh and dt . We fix maximal compact subgroups

K, K_w for $G(\mathbb{R}), H_w(\mathbb{R})$ containing $T(\mathbb{R})$ and denote by dg, dh_w, dt the standard Haar measures on $G(\mathbb{R}), H_w(\mathbb{R}), T(\mathbb{R})$, all as in [HC, Sections 3,7]. Note that $\text{vol}(\overline{T}(\mathbb{R}))$ is now 1. Fix a positive system for $R(G,T)$ and define the factor Δ relative to this system. Also set $\Delta^*(\gamma_0) = \prod_{\alpha \in R(H_1, T)} (1 - \alpha(\gamma_0^{-1}))$, where the product is over positive roots not in $R(H_1, T)$. The positive system for $R(H_1, T)$ will consist of the roots positive for $R(G, T)$, and the system for $R(H_w, T)$ will be the twist of this one by w . We define ν_1, ν_w and the operators ϖ_1, ϖ_w (see [HC, Section 17]) accordingly.

Fix $w \in \mathcal{O}_T(T)$ and let $\delta = \gamma_0^w$. From [HC, Section 37] we have that dh , the twist of dh by w , is $v(K_w)/v(K_1) dh_w$, where v is as defined in the reference. Thus, for any Schwartz function f we have:

$$\begin{aligned}
 O(\delta, f; dh_\delta, dg) &= C_w \lim_{\gamma \rightarrow \gamma_0} (\varpi_w(\Delta^w \phi_T(\cdot, f; dt, dg)))(\gamma^w), \text{ where } C_w \\
 &= (\Delta^*(\gamma_0))^{-1} (-1)^{q(w)} v(K_1) (v(T(\mathbb{R})))^{-1} [\Omega(H_w(\mathbb{R}), T(\mathbb{R}))]^{-1} (2\pi)^{-r(w)}, \\
 & q(w) \text{ denoting } q(H_w) \text{ and } r(w) = r(1) \text{ one-half the dimension} \\
 & \text{of } H_w/T \text{ (see [W2, Section 9.3.1, Theorem 8.5.1.6, Section 8.1.3]} \\
 & \text{and [HC, Lemma 37.4])}. \text{ To form } O^{\text{st}}(\gamma_0, f) \text{ we have to sum} \\
 & \text{over } w \text{ for which } \underline{w} = wT(\mathbb{C}) \text{ form a set of representatives} \\
 & \text{for } \Omega(H_1, T) \setminus \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R})). \text{ If we replace } w \text{ by } w' \\
 & \text{such that } \underline{w}' \text{ lies in the preimage in } \mathcal{O}(T) \text{ of the double coset} \\
 & \text{of } \underline{w} \text{ then the expression above does not change. Thus we have} \\
 & \text{that } O^{\text{st}}(\gamma_0, f; dh, dg) =
 \end{aligned}$$

$$C \lim_{\gamma \rightarrow \gamma_0} (\alpha_1(\Delta \phi_T^{st}(\cdot, f; dt, dg)))(\gamma),$$

where $C = (\Delta^*(\gamma_0))^{-1} v(K_1)(v(T(\mathbb{R}))^{-1} [\Omega(H_1, T)]^{-1} (2\pi)^{-r(1)})$.

Note that this implies that the tempered distribution $f + O^{st}(\gamma_0, f)$ is stable in the sense of [S1, Section 6]. We may therefore argue that to prove the lemma for a given f satisfying (2.9.1) we may replace f by any Schwartz function f' with same stable orbital integrals, i.e. $\phi_{T'}^{st}(\cdot, f) \equiv \phi_{T'}^{st}(\cdot, f')$ for all CSG's $T'(\mathbb{R})$. We choose f' to be the function constructed in our proof of Theorem 2.8.1 (see [S1, Section 4]). It then follows that we have only to verify the formula in the statement of the lemma for the case that f is a K -finite matrix coefficient of a discrete series representation of $G(\mathbb{R})$.

Let Λ be a regular element in the positive chamber for $X^*(T)$ and $\pi(\Lambda)$ be the discrete series (class of) representation(s) attached to Λ . Let $\theta_\Lambda^* = \sum \det w \Lambda / \sum \det w w_1$, so that the stabilized character of $\pi(\Lambda)$ is $(-1)^{q(G)} \theta_\Lambda^*$. If f is a K -finite matrix coefficient for $\pi(\Lambda)$ then $\phi_T^{st}(\gamma, f; dt, dg) = (-1)^{q(G)} f(1) d_\Lambda^{-1} \theta_\Lambda^*(\gamma)$, where d_Λ is the formal degree of $\pi(\Lambda)$ relative to dg (see [W2]).

Thus $\lim_{\gamma \rightarrow \gamma_0} \phi_T^{st}(\gamma, f; dt, dg) =$

$$(-1)^{q(G)} f(1) d_\Lambda^{-1} \alpha_1(\sum \det w \Lambda (\Delta^*(\gamma_0) \alpha_1(1_1) [\Omega(H_1, T)]))^{-1}$$

(with the usual abuse of notation). This equals $O^{st}(\gamma_0, f; dh, dg)$

up to a constant which Lemma 37.4 of [HC] shows to be

$\text{vol}(\bar{T}(\mathbb{R}) \setminus \bar{H}_1(\mathbb{R}))$. Hence Lemma 2.9.3 is proved.

Similar arguments show that $d_\Lambda = \theta_\Lambda^*(1) / \text{vol}(\bar{G}(\mathbb{R}))$, where the volume is to be calculated relative to the twist of the

measure used to define d_Λ , i.e. formal degree is "invariant under inner twisting." (See [W2, Theorem 10.2.4.1] for Harish-Chandra's formula for d_Λ).

2.10 $\mathfrak{D}(T)$, $\mathfrak{E}(T)$ and κ .

This section is preparation for the definition of "unstable", i.e. "k-", orbital integrals. Recall $\mathfrak{D}(T)$ as $T(\mathbb{C}) \backslash \mathcal{O}(T) / G(\mathbb{R})$. If $w \in \mathcal{O}(T)$ then $\sigma_G(w)w^{-1} \in T(\mathbb{C})$ and the map $w \rightarrow \{1 \rightarrow 1, \sigma \rightarrow \sigma_G(w)w^{-1}\}$ induces an embedding of $\mathfrak{D}(T)$ in $H^1(T) = H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), T(\mathbb{C}))$. The image of $\mathfrak{D}(T)$ is contained in the subgroup $\mathfrak{E}(T)$ of [L2]. In fact, $\mathfrak{E}(T)$ is the smallest subgroup of $H^1(T)$ containing $\mathfrak{D}(T)$ and "invariant under inner twisting" in the obvious sense (we leave a proof of this to the reader). Following [L2], we regard $\mathfrak{E}(T)$ as a quotient of the span of the imaginary coroots; more precisely, as $\mathbb{Z}[R^\vee(M, T)] / \mathbb{Z}[R^\vee(M, T)] \cap \{\lambda^\vee - \sigma_T \lambda^\vee : \lambda^\vee \in X_*(T)\}$. To see how $\mathfrak{D}(T)$ appears in this realization we return to $\mathfrak{D}(T)$ as $\Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$. Consider the coset of ω_α , for α an imaginary root. If α is compact then ω_α is realized in $G(\mathbb{R})$ so that the coset of ω_α is trivial and the corresponding element of $\mathfrak{E}(T)$ is trivial also. If α is noncompact then the coset of ω_α corresponds to the coset of α^\vee . More generally, there is the following (unpublished) result of Langlands. First, an inductive argument shows that there exists $\Lambda^\vee \in \mathbb{Z}[R^\vee(M, T)]$ such that $\langle \Lambda^\vee, \alpha \rangle$ is even if α is compact and $\langle \Lambda^\vee, \alpha \rangle$ is odd if α is noncompact.

Then $\omega \in \Omega(M(\mathbb{R}), T(\mathbb{R})) \in \mathfrak{D}(T)$ is identified with the coset of $\lambda^\vee - \omega \lambda^\vee$.

A character κ on $\mathfrak{E}(T)$ has order two and so defines by restriction a function on $\mathfrak{D}(T)$ assuming the values ± 1 . Recall that the source of κ -orbital integrals is the trace formula [L5, Chapter 8]. There κ appears as the restriction to $\mathfrak{D}(T)$ of a character on all coroots of (G, T) . Thus, following Langlands, we will assume that κ is a quasicharacter on $\mathbb{Z}[R^\vee(G, T)] / \mathbb{Z}[R^\vee(G, T)] \cap \{\lambda^\vee - \sigma_T \lambda^\vee : \lambda^\vee \in X_*(T)\}$, i.e. that $\kappa \in \bar{K}(T/\mathbb{R})$ as in [L5]. If T is anisotropic modulo center then this quotient is $\mathfrak{E}(T)$, so there is no change, but if T is split over \mathbb{R} then we will be allowing any quasicharacter on $\mathbb{Z}[R^\vee(G, T)]$.

An element w of $\mathfrak{A}(T)$ defines a map $\kappa \rightarrow \kappa^w$ from $\bar{K}(T) = \bar{K}(T/\mathbb{R})$ to $\bar{K}(T^w)$ in the obvious way. Let $\Omega_0(G, T)$ be the subgroup of $\Omega(G, T)$ consisting of the elements realized in $\mathfrak{A}(T)$, i.e. $\Omega_0(G, T) = \{\omega \in \Omega(G, T) : \omega \sigma_T = \sigma_T \omega\}$. Then $\Omega_0(G, T)$ acts on $\bar{K}(T)$; if w realizes ω then $\kappa^\omega = \kappa^w$. Also we write $\kappa(\omega) = \kappa(w)$ for the value of κ on the element of $\mathfrak{D}(T)$ determined by ω and w . Then, regarding the (Weyl group-) product in $\Omega_0(G, T)$, we have

$$(2.10.1) \quad \kappa(\omega \omega') = \kappa(\omega) \kappa^\omega(\omega')$$

for all $\omega, \omega' \in \Omega_0(G, T)$.

Suppose that s is a Cayley transform with respect to the imaginary root α of T . Suppose also that $\kappa(\alpha^\vee) = 1$. Then a calculation shows that the map $\mathbb{Z}[R^\vee(G, T)] \rightarrow \mathbb{Z}[R^\vee(G, T^S)]$

determined by $\text{ad } s^{-1}$ induces a map $\kappa \rightarrow \kappa^S$ of $\bar{K}(T)$ to $\bar{K}(T^S)$, i.e. we may propagate a given κ to certain adjacent CSG's. (See [S2, Section 3] for this and some related observations. Note that there is an error in the statement of part (iii) of Proposition 3.3: " $\omega \in \mathcal{A}(T_S)$ which normalizes T_S " should read " ω in the imaginary Weyl group of T_S ." The error is of no consequence for the rest of the paper.)

Fix $\kappa \in \bar{K}(T)$. Suppose that H is a group (connected, reductive, defined over \mathbb{R}) such that:

- (i) T is contained in H ,
- (ii) $R^\vee(H, T) = \{\alpha^\vee \in R^\vee(G, T) : \kappa(\alpha^\vee) = 1\}$.

Then a review of the axioms for root systems shows:

(2.10.2) $\Omega(H, T)$ is naturally embedded in $\Omega(G, T)$ as the subgroup generated by the reflections ω_α , where $\alpha^\vee \in R^\vee(H, T)$, and

(2.10.3) on fixing a nondegenerate symmetric bilinear form on $X_*(T)$ invariant under $\Omega(G, T)$, we may identify $R(H, T) = (R^\vee(H, T))^\vee$ with a subset (but, in general, not a subsystem) of $R(G, T) = (R^\vee(G, T))^\vee$.

Note that in both (2.10.2) and (2.10.3) the Galois action is respected.

If $\omega \in \Omega(G, T)$ lies in the embedded $\Omega(H, T)$ then we say that ω is "from H ." Similarly, a root of $\Omega(G, T)$ is "from H " if it is in the embedded $R(H, T)$ (some suitable bilinear form will be assumed fixed). These notions are compatible

in the sense that a reflection from H is with respect to a root from H ; also, imaginary roots come only from imaginary roots of H and elements of $\Omega_0(G, T)$ only from $\Omega_0(H, T)$.

Suppose that $\omega \in \Omega_0(G, T)$ is from H . Then for λ^V in $\mathbb{Z}[R^V(G, T)]$ we have that $\lambda^V - \omega^{-1}\lambda^V$ lies in $\mathbb{Z}[R^V(H, T)]$ and so is annihilated by κ . We conclude then that:

$$(2.10.4) \quad \kappa^\omega = \kappa.$$

Note that (2.10.4) does not characterize the elements from $\Omega_0(H, T)$.

Lemma 2.10.5

$\kappa(\omega) = 1$ provided that ω is from the imaginary Weyl group of T in H .

Proof: See [S2, Proposition 7.4]. This is a simple exercise: if α is a compact root then $\kappa(\omega_\alpha) = 1$, and if α is noncompact then $\kappa(\omega_\alpha) = \kappa(\alpha^V)$ so that $\kappa(\omega_\alpha) = 1$ if ω is from H . Thus the lemma is true for reflections. Next, (2.10.4) implies that κ is multiplicative on the elements from $\Omega_0(H, T)$ and the proof is completed by an inductive argument.

For a general element of $\Omega_0(G, T)$ from H Lemma 2.10.5 is false. Such an element ω has two signatures, $\text{sgn}_H(\omega)$ defined with respect to the imaginary roots from H and $\text{sgn}_G(\omega)$ defined with respect to all imaginary roots of T in G .

Lemma 2.10.6

$$\kappa(\omega) = \text{sgn}_G(\omega) / \text{sgn}_H(\omega).$$

Proof: See [S2, Section 7 and Lemma 8.2]. The proof is quite long as we need to consider generators for $\Omega_0(G, T)$. In view of the proof of Lemma 2.4.1 there is an exact sequence:

$$1 \rightarrow \Omega(M, T) \rightarrow \Omega_0(G, T) \rightarrow W_T \rightarrow 1,$$

where W_T is the relative Weyl group of T , i.e. $W_T = \text{Norm}(M(\mathbb{R}), G(\mathbb{R})) / M(\mathbb{R})$. The results of [Kn] give generators for W_T . We then form generators for $\Omega_0(G, T)$ in the obvious way. The proof of the lemma consists of reduction to and examination of certain "types" of generators. Details are given in the reference.

2.11 κ -orbital integrals

Let f be a Schwartz function on $G(\mathbb{R})$, $T(\mathbb{R})$ be a CSG and κ an element of $\mathcal{K}(T)$. Then, as in [L5], we set:

$$\phi_T^K(\gamma, f) = \phi_T^K(\gamma, f; dt, dg) = \sum \kappa(w) \phi_{T^w}^K(\gamma^w, f; (dt)^w, dg),$$

$\gamma \in T(\mathbb{R})_{\text{reg}}$, where the summation is over representatives w for the elements of $\mathcal{Q}(T)$. Then:

$$(2.11.1) \quad \phi_T^K(\gamma, f; \alpha dt, \beta dg) = \beta/\alpha \phi_T^K(\gamma, f; dt, dg),$$

$$(2.11.2) \quad \kappa(w) \phi_{T^w}^K(\gamma^w, f; (dt)^w, dg) = \phi_T^K(\gamma, f; dt, dg)$$

for $w \in \mathcal{A}(T)$.

We normalize ϕ_T^K by the factor from (2.7), following the same conventions for positive imaginary roots in (2.11.5) below. Let $\bar{\Psi}^K$ be the function so obtained (so that $\bar{\Psi}^1 = \bar{\Psi}^T$). Then:

$$(2.11.3) \quad \bar{\Psi}^K \text{ extends to a Schwartz function on}$$

$$T_{\text{reg}}^{I, K}(\mathbb{R}) = \{\gamma \in T(\mathbb{R}) : \alpha(\gamma) \neq 1 \text{ for imaginary roots } \alpha \text{ such}$$

that $\kappa(\alpha^V) = 1$).

This follows from the results for ordinary orbital integrals and the observation that (anti-)symmetry forces $\bar{\psi}^K$ to be smooth across the semi-regular elements on walls $\alpha = 1$ where α is not totally compact and $\kappa(\alpha^V) = -1$. (See [S2, Lemma 4.3]).

We now follow the notation of Lemma 2.7.1. Suppose that α is an imaginary root for which $\kappa(\alpha^V) = 1$. Then:

$$(2.11.4) \quad \lim_{v \rightarrow 0} (X\bar{\psi}^K)(\gamma_v) = 0 \quad \text{for all semi-regular}$$

elements γ_0 annihilated by α , if α is totally compact, and

$$(2.11.5) \quad \lim_{v \rightarrow 0} (X'\bar{\psi}^K)(\gamma_v) - \lim_{v \rightarrow 0} (X'\bar{\psi}^K)(\gamma_v) =$$

$$2i \varepsilon_\kappa(s) ((X^S)\bar{\psi}^{K^S})(\gamma_0^S) \quad \text{for all semi-regular}$$

elements γ_0 annihilated by α , if α is not totally compact.

The new term $\varepsilon_\kappa(s)$ is the κ -signature of s defined in Section 4 of [S2]; it assumes the values ± 1 ... this is all that concerns us for the present. For a proof of (2.11.4) and (2.11.5) see [S2, Section 4].

In Part 4 we will take up one of the themes of endoscopic groups, that κ -orbital integrals are "stable orbital integrals on lower-dimensional groups." Note that if H is as in (2.10) then:

$$(i) \quad T_{\text{reg}}^{I, \kappa}(\mathbb{R}) \quad \text{is} \quad T_{\text{reg}}^I(\mathbb{R}) \quad \text{relative to } H, \quad \text{and}$$

$$(ii) \quad \phi_T^K(\gamma^\omega, f) = \text{sgn}_H(\omega) / \text{sgn}_G(\omega) \phi_T^K(\gamma, f), \quad \gamma \in T(\mathbb{R})_{\text{reg}},$$

for all $\omega \in \Omega_0(G, T)$ which are "from H ."

3. TEMPERED SPECTRUM AND L-GROUPS

3.1 Notation

For L-group data we will follow the notation of [S4] which is essentially that of [L5]. Thus G remains a connected, reductive group over \mathbb{R} ; $\psi: G \rightarrow G^*$ is an inner twist from G to a quasi-split form G^* , and $L_G = L_G^0 \rtimes W$ is the L-group of G^* and of G . We realize the Weil group W as $\{z \times \tau: z \in \mathbb{C}^\times, \tau \in \text{Gal}(\mathbb{C}/\mathbb{R})\}$, with $(z \times \tau)(z' \times \tau') = z\tau(z')a_{\tau, \tau'} \times \tau\tau'$, where $a_{\tau, \tau'} = 1$ unless $\tau = \tau' = \sigma$, and $a_{\sigma, \sigma} = -1$. Also included in the "L-group data" is a Borel subgroup B^* over \mathbb{R} in G^* ; B^* contains the maximal torus T^* over \mathbb{R} ; L_G^0 , L_B^0 and L_T^0 are the "dual" complex groups. The group W acts on L_G^0 via a homomorphism $\rho_G: W \xrightarrow{\text{proj}} \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow \text{Aut}(L_G^0, L_B^0, L_T^0, \{X_{\alpha^v}\})$. The action is completely specified by the automorphism $\sigma_G = \rho_G(1 \times \sigma)$ which is the "algebraic dual" of the Galois automorphism of G^* . Recall that X_{α^v} is a root vector for the simple root α^v of L_T^0 in L_B^0 .

3.2 L-groups for maximal tori and Levi groups

Let T be a maximal torus over \mathbb{R} in G^* . We will call T standard if S_T is contained in T^* (i.e. in S_{T^*}). Every maximal torus over \mathbb{R} in G^* is $G^*(\mathbb{R})$ -conjugate to a standard

one. Suppose then that T is standard. We may choose $m \in M_T$ so that $m^{-1}Tm = T^*$. The automorphism $\text{adm}^{-1} \circ \sigma_T \circ \text{adm}$ of T^* (and hence of L_{T^0}) is independent of the choice for m ; we denote the automorphism by σ_T again. For L_T we take $L_{T^0} \rtimes W$, with W -action given by $\rho_T(1 \times \sigma) = \sigma_T$. Moreover for $L_M = L_{(M_T)}$ we may take $L_{M^0} \rtimes W$, where L_{M^0} contains L_{T^0} , $R(L_{M^0}, L_{T^0}) = \{\alpha^\vee \in R(L_G^0, L_{T^0}) : \sigma_T \alpha^\vee = -\alpha^\vee\}$, and W acts by restriction of ρ_G to L_{M^0} , i.e. σ_M is the restriction of σ_G to L_{M^0} .

Suppose now that T is a maximal torus over \mathbb{R} in G . Fix a standard torus T^{**} in G^* and $x \in G^*$ such that $\psi_x = \text{ad } x^{-1} \circ \psi$ maps T to T^{**} over \mathbb{R} (see [L1]). Fix $m \in M_{T^{**}}$ such that $m^{-1}(T^{**})m = T^*$. We write η for the pair (ψ_x, m) and call it a pseudodiagonalization (p-d.) of T . Via $\text{adm}^{-1} \circ \psi_x$, or more informally "via η ", we transport σ_T to an automorphism $\sigma_{(T, \eta)}$ of T^* and hence of L_{T^0} . Clearly $\sigma_{(T, \eta)} = \sigma_{T^{**}}$. We have then a realization of the L -group for T , namely $L_T = L_{(T^{**})}$. Since ψ_x is an inner twist from M_T to $M_{T^{**}}$, we have also the realization of $L_M = L_{(M_T)}$ as $L_{(M_{T^{**}})}$. Whenever η has been given we will assume L_T and L_M chosen in this way. Note that if T is anisotropic modulo the center of G , so that $L_M = L_G$, then L_T is independent of the choice for η .

Remark: We make this definition of p-d. for use in parts 3 and 4 only, and only for the purpose of fixing L -groups. It will simplify our discussion of some constructions. Later (see (4.4)) we will allow any map from T to T^* of the form $\text{ad } y^{-1} \circ \psi$, with y in G^* , as is appropriate for Langlands' notion of "diagram" [L5].

3.3 Embedding L_T in L_G

We recall some material from [S3, Section 1.3]. Fix a maximal torus T over \mathbb{R} in G and p -d. η of T . We have L_M naturally embedded in L_G . To embed L_T in L_G it is then sufficient to embed L_T in L_M . We therefore assume that T is anisotropic modulo the center of G . An admissible embedding $\xi: L_T \rightarrow L_G$ will be a homomorphism $\xi(t \times w) = t\xi(w)$, $t \in L_T^0$, $w \in W$, where $\xi(w)$ is of the form $\xi_0(w) \times w \in L_G^0 \times w$, $w \in W$. Since $\xi(\mathbb{C}^\times \times 1)$ must act (by conjugation) trivially on L_T^0 we have the restriction of ξ_0 to $\mathbb{C}^\times \times 1$ is a homomorphism into L_T^0 . On the other hand, $\xi(1 \times \sigma)$ must act on L_T^0 as $\sigma_{(T, \eta)}$; $1 \times (1 \times \sigma)$ preserves the positive roots of L_T^0 (i.e. the roots of L_T^0 in L_B^0) and $\sigma_{(T, \eta)}$ acts as -1 on all roots. Thus we conclude that $\xi_0(1 \times \sigma)$ must be an element of $\text{Norm}(L_T^0, L_G^0)$ which takes the positive roots to negative roots. Finally, $(1 \times \sigma)^2 = -1 \times 1$ requires that $\xi_0(1 \times \sigma) \sigma_G(\xi_0(1 \times \sigma)) = \xi_0(-1 \times 1)$.

Lemma 3.2 of [L1] shows how to construct ξ . (The proof of this lemma is reproduced in an appendix to [S3].) In [L1] the lemma is the crucial step in constructing parameters for discrete series representations. Here we construct embeddings of L_T and the parameters will appear simply as "lifts" from T .

Let λ be one-half the sum of the coroots for the roots of L_T^0 in L_B^0 . Define ξ_0 on $\mathbb{C}^\times \times 1$ by $\lambda^v(\xi_0(z \times 1)) = (z/\bar{z})^{\langle 1, \lambda^v \rangle}$, $\lambda^v \in X^*(L_T^0)$. For $\xi_0(1 \times \sigma)$ take any element in the normalizer of L_T^0 in the derived group of L_G^0 which maps the positive roots to negative roots (such elements do

exist [L1], see [S3, Appendix]). Then the "Lemma 3.2" implies that we may reverse the steps above and extend ξ_0 to an admissible embedding ξ of L_T in L_G . If ξ' is another admissible embedding of L_T in L_G then $\xi'(t \times w) = a(w)\xi(t \times w)$, $t \in L_T^0$, $w \in W$, where $a(w)$ is a 1-cocycle of W in L_T^0 with the action of W determined by $\sigma_{(T,\eta)}$. In particular, the image of L_T under ξ' is the same as that under ξ . Also it follows that in replacing ξ by ξ' we replace τ by, roughly speaking, τ plus the data for a quasi-character on $T(\mathbb{R})$, i.e. there is an essential "twist" by τ when we embed L_T in L_G ... we could state this more precisely following [S3, Section 1.3] and the discussion of the next section.

The existence of an admissible embedding of L_T in L_G , for arbitrary T, η can also be verified directly from [L2, Lemma 4].

3.4 Langlands parameters

Recall that an admissible homomorphism $\phi: W \rightarrow L_G$ is of the form $\phi(w) = \phi_0(w) \times w$, $w \in W$, where $\phi_0(w)$ is a semi-simple element of L_G^0 ; ϕ is equivalent to ϕ' if $\phi' = \text{ad}_g \circ \phi$, for some g in L_G^0 . When it is necessary to distinguish in notation between a homomorphism and its equivalence class we will use $\{\phi\}$ for the class of ϕ . The set $\phi(G^*)$ is the collection of all equivalence classes of admissible $\phi: W \rightarrow L_G$, and $\phi(G)$ consists of those classes "relevant to G ." See [B].

Let $\phi:W \rightarrow L_G$ be admissible. If $\phi_0(\mathbb{C}^\times \times 1)$ is contained in L_T^0 and $\phi(1 \times \sigma)$ normalizes L_T^0 and acts on L_T^0 as $c_{T^{**}}$, for some standard torus T^{**} in G^* , then $\phi(W)$ is contained in the image of $L_{T^{**}}$ under the admissible embeddings of $L_{T^{**}}$ in $L_{M_{T^{**}}} \subseteq L_G$. Fix such an embedding ξ . Then clearly $\phi = \xi \circ \phi'$, for some admissible homomorphism $\phi':W \rightarrow L_{T^{**}}$, i.e. $\{\phi\}$ is contained in the image of the map from $\phi(T^{**})$ to $\phi(G^*)$ induced by ξ . Since the image of $\phi(T^{**})$ is independent of the choice of ξ we will say simply that $\{\phi\}$ "factors through $\phi(T^{**})$." Suppose now that T is a maximal torus over \mathbb{R} in G . Fix a p-d. $\eta = (\psi_x, m)$ of T . Let $T^{**} = \psi_x(T)$. Then $\phi(T) = \phi(T^{**})$ and from [L1] (see [B] also) we may check that $\text{Image}(\phi(T)) = \text{Image}(\phi(T^{**}))$ is independent of the choice for η and:

$$(3.4.1) \quad \phi(G) = \bigcup_T \text{Image}(\phi(T)) \cap \phi(G).$$

Note that in our terms "relevant to G ", i.e. " $\in \phi(G)$ ", means simply $\{\phi\}$ factors through only those $\phi(T^{**})$ for which T^{**} "originates in G " [S1, Section 2], i.e. T^{**} is of the form $\psi_x(T)$.

We will consider only bounded parameters, i.e. classes with representatives ϕ such that $\phi_0(W)$ is bounded. Let $\phi_0(G^*)$ be the set of all bounded parameters and $\phi_0(G) = \phi_0(G^*) \cap \phi(G)$. As long as we allow only embeddings ξ of L_T for which $\xi_0(W)$ is bounded we have:

$$(3.4.2) \quad \phi_0(G) = \bigcup_T \text{Image}(\phi_0(T)) \cap \phi(G).$$

The discrete parameters in $\phi_0(G)$ are those which factor through only the $\phi_0(T)$ for T anisotropic modulo the center

of G . There are other parameters in the image of $\phi_0(T)$, for such T . They are not necessarily relevant to G ; we call them "limits of discrete parameters." A more concrete description is given as follows.

Fix a torus T anisotropic modulo the center of G , and a p -d. η of T . We embed L_T in L_G via the homomorphism ξ constructed in (3.3). A class factoring through $\phi_0(T)$ has a representative ϕ such that $\phi_0(\mathbb{C}^\times \times 1)$ is contained in L_T^0 and $\phi(1 \times \sigma) = n \times (1 \times \sigma)$, where $n \in L_G^0$ normalizes L_T^0 and maps the positive roots to negative roots. We write $\phi = \phi(\mu, \lambda)$ if $\lambda^V(\xi_0(z \times 1)) = z^{\langle \mu, \lambda^V \rangle} \bar{z}^{\langle \bar{\sigma}\mu, \lambda^V \rangle}$, $\lambda^V \in X^*(L_T^0)$, where $\bar{\sigma}$ abbreviates $\sigma_{(T, \eta)}$, and $\lambda^V(n) = e^{2\pi i \langle \lambda, \lambda^V \rangle}$, $\lambda^V \in X^*(L_G^0)$. Both μ and λ are elements of $X_*(L_T^0) \otimes \mathbb{C}$ which we can identify with $X^*(T) \otimes \mathbb{C}$ via η . While μ is uniquely determined by ϕ , there is ambiguity in λ (which we will ignore); see [S4, Section 4.3] for details. Note that $\phi(\mu, \lambda)$ is equivalent to $\phi(\mu', \lambda')$ if and only if μ' is in the orbit of μ under $\Omega(L_G^0, L_T^0)$ and λ equals λ' up to the allowed ambiguity [S4, Lemma 4.3.1]. The class of $\phi(\mu, \lambda)$ forms a discrete parameter if and only if $\langle \mu, \alpha^V \rangle \neq 0$ for all roots α^V of L_T^0 ([L1], see also [B]). In any case we can choose for a class $\{\phi\}$ a representative $\phi(\mu, \lambda)$ such that $\langle \mu, \alpha^V \rangle \geq 0$ for all roots α^V of L_T^0 in L_B^0 , i.e. with μ dominant. Then μ is uniquely determined by $\{\phi\}$.

Clearly we can write any $\phi(\mu, \lambda)$ as $\xi \circ \phi'$, where ϕ' has parameters $(\mu - \nu, \lambda)$ relative to T . By the Langlands correspondence for tori we have then that $(\mu - \nu, \lambda)$ is datum for a character on $T(\mathbb{R})$ (see [S3, Section 4.1] for a review of the correspondence); so also is $(\omega\mu - \omega\nu, \lambda)$, $\omega \in \Omega(G, T)$.

3.5 L-packets

We recall now the Langlands correspondence for tempered representations of G , i.e. the assignment to an element of $\Phi_0(G)$ of an L-packet of (equivalence classes of) tempered representations of $G(\mathbb{R})$.

Suppose that $\{\phi\}$ is discrete. Choose a maximal torus T compact modulo center and a p-d. η of T . Let $\phi = \phi(\mu, \lambda)$, with μ dominant, represent $\{\phi\}$. Transport μ, λ to T via η . Then for each $\omega \in \Omega(G, T)$ the pair $(\omega\mu, \lambda)$ is datum for a unique (equivalence class of) discrete series representation(s); note that $\omega\mu$ is strictly dominant with respect to the ordering obtained from (L_B^0, L_T^0) by transport by η and then ω . We denote this representation, and its character, by $\theta_\omega(\mu, \lambda)$. The L-packet attached to $\{\phi\}$ is $\{\theta_\omega(\mu, \lambda) : \omega \in \Omega(G, T)\}$. It is independent of the choice for T and η .

In general, we can find a pair (T, η) so that $\{\phi\}$ has a representative ϕ where $\phi(W)$ is contained in L_M and the class of ϕ in $\Phi(M)$ is discrete. The L-packet attached to $\{\phi\}$ is the set of constituents of the unitary principal