

ORBITAL INTEGRALS, ENDOSCOPIC GROUPS AND  
L-INDISTINGUISHABILITY FOR REAL GROUPS

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1. INTRODUCTION

Our purpose is to discuss the results of [S1], [S2], [S3] and [S4], or rather to provide some background for the discussion of them in [L5].

We begin with the characterization of stable orbital integrals. For a group over any field  $F$  of characteristic zero there is a partial ordering on the set of stable conjugacy classes of Cartan subgroups. In the case  $F = \mathbb{R}$  the adjacent (classes of) Cartan subgroups are very simply described in terms

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of a suitably general notion of Cayley transform (see (2.2.2)). Perhaps the most significant feature of the characterization theorem for stable orbital integrals (Theorem 2.8.1) is that, besides the transformation and smoothness properties ((2.4.3) - (2.4.5), part (i) of (2.6.6)), we use only some information about the behavior of stable orbital integrals near certain (semi-regular) points  $\gamma_0$  common to two adjacent Cartan subgroups ... roughly speaking, that around  $\gamma_0$  the problems introduced by the non-compactness of  $G(\mathbb{R})$  are independent of the Cartan subgroup (see (2.6.7)). This is also described by "jump" formulas (see Lemma 2.7.1).

In our preparation for Theorem 2.8.1 we discuss both the jump formulas and the germ expansions (around semi-regular elements only) and their equivalence. Although it is not necessary for the theorem, we study the stable integrals for the stable orbit of an element  $\gamma_0$  as above; these terms require definition and we follow Kottwitz's suggestion ... it is not appropriate to take the sum of the integrals over the orbits in the stable orbit. We see then that the term  $\Lambda_1$  from (2.6.2) is "intrinsic to  $G$  and its inner forms" and that the term  $\Lambda_0$  arises from the noncompactness of  $G(\mathbb{R})$ .

After Theorem 2.8.1 we find that we have assembled enough facts to calculate, for all semisimple stable orbits, the stable integrals for certain functions which appear useful for applications of the trace formula ([L6]). The result is

Lemma 2.9.3. We complete Part 2 by recalling properties of  $\mathfrak{A}(T)$ ,  $\mathfrak{E}(T)$ ,  $\mathfrak{K}(T)$  and  $\kappa$ -orbital integrals.

In Part 3 we switch to representations and the explicit parameterization of tempered characters. Central to our point of view are the embedding of the L-group of a maximal torus  $T$  in the L-group for  $G$  and the expression of a tempered L-packet parameter as the "lift" of a parameter for  $T$  (see (3.3) - (3.5)). We obtain various realizations of an L-packet in terms of the Knapp-Zuckerman basic characters.

To pair an L-packet with a finite group is not difficult, but to pair it with the group "S" which is naturally associated to the L-packet (and has analogues for all local and global fields) requires a uniform version of the Knapp-Zuckerman decomposition of unitary principal series. For this reason we include our L-group imitation of the Knapp-Zuckerman decomposition; it rests on Langlands' version of the Knapp R-group. We should note that the pairing and the later factoring of parameters in (5.5) require various choices. Our formulations are simply the most convenient ones and may well require modification for global applications; our aim for the present is to show that pairings exist so that identities as in (5.6) hold.

Our main results on the "internal structure" of L-packets are contained in (3.6); the needed Knapp-Zuckerman theory is described in (3.7).

Part 4 concerns endoscopic groups. We review briefly definitions, constructions and embeddings of L-groups. Our main interest is the normalization of  $\kappa$ -orbital integrals so that they match stable orbital integrals on an endoscopic group. Because of its usefulness elsewhere, we look at the local conditions on the normalization factors and indicate how our data from an embedding of the L-group of an endoscopic group in  ${}^L G$  play a central role in "globalization". See (4.5).

Part 5 contains the main conclusions of "L-indistinguishability for the tempered spectrum of a real group."

In an appendix we mention two consequences of the theory. The first concerns Fourier inversion of orbital integrals and the second twisted orbital integrals for complex groups and the associated "twisted endoscopic groups" [S8]. We include also an example for  $G = \text{SU}(2,1)$  which is elementary but of interest for both automorphic representations of a unitary group in three variables ([L5] and work of Flicker in progress) and a general local theory of nontempered L-packets.

Throughout the notes we give only outlines of proofs in print, with references. For new material details are included.

## 2. ORBITAL INTEGRALS

### 2.1 Notation

We will follow as far as possible the notation of [L5]. Thus  $G$  will denote a connected reductive linear algebraic group defined over  $\mathbb{R}$  and  $G(\mathbb{R})$  the group of  $\mathbb{R}$ -rational points on  $G$ ;  $\sigma_G$  will be the Galois automorphism of the group  $G(\mathbb{C})$  of  $\mathbb{C}$ -rational points, so that  $G(\mathbb{R}) = \{g \in G(\mathbb{C}) : \sigma_G(g) = g\}$ .

Let  $T$  be a maximal torus in  $G$  defined over  $\mathbb{R}$ . Then  $T(\mathbb{R})$  is a Cartan subgroup (CSG) of  $G(\mathbb{R})$ , and every Cartan subgroup of  $G(\mathbb{R})$  is of this form. We denote by  $S_T$ , or simply  $S$ , the maximal  $\mathbb{R}$ -split torus in  $T$ , and by  $M_T$  or  $M$  the centralizer of  $S_T$  in  $G$ ;  $M_T(\mathbb{R})$  is the "cuspidal Levi group in  $G(\mathbb{R})$  defined by  $T$ ." We regard the set  $R(G, T)$  of roots of  $T$  in  $G$  as a subset of  $X^*(T)$ , the group of rational characters on  $T$ . Recall that  $\alpha \in R(G, T)$  is imaginary if  $\sigma_T \alpha = -\alpha$ , i.e.  $\alpha \in R(M, T)$ . The imaginary Weyl group of  $T$  is the subgroup of the Weyl group  $\Omega(G, T)$  of  $T$  in  $G$  generated by the reflections with respect to the imaginary roots, i.e. the Weyl group  $\Omega(M, T)$  of  $T$  in  $M$ . We may identify  $\Omega(G, T)$  with  $\text{Norm}(T(\mathbb{C}), G(\mathbb{C}))/T(\mathbb{C})$ , where  $\text{Norm}(A, B)$  indicates the normalizer of  $A$  in  $B$ . Then  $\omega \in \Omega(G, T)$  is realized in the subset  $S$  of  $G(\mathbb{C})$  if  $\omega$  is identified with an element of  $ST(\mathbb{C})/T(\mathbb{C})$ .

## 2.2 Adjacency of Cartan subgroups

If  $T, T'$  are maximal tori over  $\mathbb{R}$  in  $G$  then set  $T \leq T'$  (and  $T(\mathbb{R}) \leq T'(\mathbb{R})$ ) if and only if there exists  $g \in G(\mathbb{C})$  such that  $\text{ad } g^{-1}$  maps  $T$  to  $T'$ , i.e.  $T' = g^{-1}Tg$ , and the restriction of  $\text{ad } g^{-1}$  to  $S_T$  is defined over  $\mathbb{R}$ , i.e.  $\sigma_G(g)g^{-1} \in M_T$ . An argument as in the proof of [S1, Theorem 2.1] shows that  $T \leq T'$  if and only if  $S_T$  is  $G(\mathbb{R})$ -conjugate to a subgroup of  $S_{T'}$ .

If  $T \leq T'$  call  $T$  adjacent to  $T'$  when  $T$  is not  $G(\mathbb{R})$ -conjugate to  $T'$  and  $T \leq T'' \leq T'$  implies that  $T''$  is  $G(\mathbb{R})$ -conjugate to one of  $T, T'$ .

### Lemma 2.2.1

Suppose that  $T \leq T'$ . Then  $T$  is adjacent to  $T'$  if and only if there exists an imaginary root  $\alpha$  of  $T$  and  $s \in G(\mathbb{C})$  such that  $T' = s^{-1}Ts$  and  $\sigma_G(s)s^{-1}$  realizes the Weyl reflection with respect to  $\alpha$ .

Proof (outline): If  $T \leq T'$  and such  $\alpha, s$  exist then it follows that  $\dim S_{T'} = 1 + \dim S_T$ , and adjacency is immediate. On the other hand, suppose that  $T \leq T'$  and that  $T$  is adjacent to  $T'$ . It is sufficient to consider the case that  $T$  is compact modulo the center of  $G$ . Then the existence of  $\alpha, s$  is established by standard methods. See [S1, Section 2] for similar arguments.

### Definition 2.2.2

If  $\alpha, s$  are as in the lemma then  $s$  is a Cayley transform with respect to  $\alpha$ .

Recall that  $\alpha \in R(G, T)$  determines a three-dimensional simple complex Lie algebra, namely  $\mathbb{C}X_\alpha + \mathbb{C}H_\alpha + \mathbb{C}X_{-\alpha}$ , where  $X_\alpha, X_{-\alpha}$  are root vectors for  $\alpha, -\alpha$  respectively, and  $H_\alpha = [X_\alpha, X_{-\alpha}]$ . If  $\alpha$  is imaginary then this algebra is invariant under  $\sigma_G$  and so its  $\sigma_G$ -fixed points form a three dimensional simple real Lie algebra;  $\alpha$  is compact if this algebra is of compact type, and noncompact otherwise.

### Definition 2.2.3

An imaginary root  $\alpha$  of  $T$  is totally compact if every root in the orbit of  $\alpha$  under the imaginary Weyl group (or under  $\Omega(T)$ , see (2.4)) is compact.

### Theorem 2.2.4

- (i) There exists a Cayley transform with respect to the imaginary root  $\alpha$  if and only if  $\alpha$  is not totally compact.
- (ii) If  $G$  is quasi-split over  $\mathbb{R}$  then no  $T$  in  $G$  has totally compact roots.

Proof: See [S1, Proposition 4.11] and [S2, Lemma 9.2].

## 2.3 Orbital integrals

Fix a Cartan subgroup  $T(\mathbb{R})$  of  $G(\mathbb{R})$  and Haar measures  $dt$  on  $T(\mathbb{R})$  and  $dg$  on  $G(\mathbb{R})$ . Let  $G_{\text{reg}}$  denote the set of regular elements in  $G$  and  $T(\mathbb{R})_{\text{reg}} = T(\mathbb{R}) \cap G_{\text{reg}}$ . Fix  $\gamma \in T(\mathbb{R})_{\text{reg}}$  and  $f \in C_c^\infty(G(\mathbb{R}))$ . Then by a compactness principle of Harish-Chandra (see [W2, Theorem 8.1.4.1]) the functions  $T(\mathbb{R})g \rightarrow f(g^{-1}\gamma'g)$ , for  $\gamma'$  in a suitable neighborhood of  $\gamma$  in  $T(\mathbb{R})_{\text{reg}}$ , are supported on a common compact subset of

$T(\mathbb{R}) \backslash G(\mathbb{R})$ . It then follows that

$$\phi_T(\gamma, f) = \phi_T(\gamma, f; dt, dg) = \int_{T(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\gamma g) \frac{dg}{dt}$$

is well-defined and  $\gamma \rightarrow \phi_T(\gamma, f)$  is a  $C^\infty$ -function on  $T(\mathbb{R})_{\text{reg}}$ .

The orbit  $O(\gamma)$  of  $\gamma \in T(\mathbb{R})_{\text{reg}}$  is the conjugacy class of  $\gamma$  in  $G(\mathbb{R})$ . Since  $T$  is of finite index in the centralizer  $G_\gamma$  of  $\gamma$  in  $G$  and  $O(\gamma)$  is homeomorphic to  $G_\gamma(\mathbb{R}) \backslash G(\mathbb{R})$  via the map  $g^{-1}\gamma g \rightarrow G_\gamma(\mathbb{R})g$ ,  $\phi_T(\gamma, f)$  is the integral of  $f$  over  $O(\gamma)$  relative to a certain  $G(\mathbb{R})$ -invariant measure. We will use the term orbital integral for  $\phi_T(\gamma, f)$ . When multiplied by a suitable function of  $\gamma$ ,  $\phi_T(\gamma, f)$  becomes Harish-Chandra's  $'F_f(\gamma)$  [HC].

Let  $\mathcal{O}$  be an open subset of  $G(\mathbb{R})$ . Then a  $C^\infty$ -function on  $\mathcal{O}$  is a Schwartz function on  $\mathcal{O}$ , i.e. belongs to the Schwartz space  $\mathcal{C}(\mathcal{O})$ , if it and all its left and right derivatives are rapidly decreasing on  $\mathcal{O}$  in the sense of Harish-Chandra [HC]. We will need the results encompassed by Harish-Chandra's theorem on the  $'F_f$  transform for  $\mathcal{C}(G(\mathbb{R}))$  [HC, Theorem 17.1] (see [W2, Section 8.5]). First is the assertion that for  $f \in \mathcal{C}(G(\mathbb{R}))$  the integral  $\phi_T(\gamma, f)$ , defined formally as for functions of compact support, is absolutely convergent and that the function  $\gamma \rightarrow (\prod_{\alpha \in R(G, T)} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi_T(\gamma, f)$ , where the product is over all  $\alpha \in R(G, T)$ , belongs to the Schwartz space of the open subset  $T(\mathbb{R})_{\text{reg}}$  of  $T(\mathbb{R})$ .

## 2.4 Stable orbital integrals

If  $G$  is semisimple and simply-connected then the stable orbit of  $\gamma \in T(\mathbb{R})_{\text{reg}}$  is the intersection of  $G(\mathbb{R})$  with the orbit of  $\gamma$  in  $G(\mathbb{C})$ . In general, however, the stable orbit is contained in this intersection. For the precise definition recall that  $\mathcal{O}(T) = \mathcal{O}_G(T) = \{g \in G(\mathbb{C}) : g^{-1}T(\mathbb{R})g \subset G(\mathbb{R})\}$  and that  $\mathcal{D}(T) = \mathcal{D}_G(T) = T(\mathbb{C}) \backslash \mathcal{O}(T) / G(\mathbb{R})$ . Then the stable orbit of  $\gamma$  is  $\{w^{-1}\gamma w : w \in \mathcal{O}(T)\}$ . If  $\gamma$  is strongly regular, i.e.  $G_\gamma = T$ , then  $\mathcal{D}(T)$  parametrizes the orbits in the stable orbit of  $\gamma$ .

### Lemma 2.4.1

There is a bijection between  $\mathcal{D}_G(T)$  and  $\Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$ . Here  $\Omega(M(\mathbb{R}), T(\mathbb{R}))$  denotes the subgroup of  $\Omega(M, T)$  consisting of elements realized in  $M(\mathbb{R})$ .

Proof: See[S1, Theorem 2.1]. In summary: (i) every element of  $\mathcal{D}_G(T)$  has a representative in  $M$ , so that  $\mathcal{D}_G(T) = \mathcal{D}_M(T) \dots$  an analogue is true for groups over any field of characteristic zero ... and (ii) the representative from (i) can be chosen to normalize  $T$ , because a real group, in this case  $M(\mathbb{R})$ , has (at most) one conjugacy class of CSG's compact modulo the center. Finally, each element of  $\Omega(M, T)$  preserves  $T(\mathbb{R})$ .

### Definition 2.4.2

For  $f \in \mathcal{C}(G(\mathbb{R}))$ ,  $\gamma \in T(\mathbb{R})_{\text{reg}}$  and Haar measures  $dt$  on  $T(\mathbb{R})$  and  $dg$  on  $G(\mathbb{R})$  set

$$\phi_T^{\text{st}}(\gamma, f) = \phi_T^{\text{st}}(\gamma, f; dt, dg) = [\Omega(M(\mathbb{R}), T(\mathbb{R}))]^{-1} \sum \phi_T(\gamma^\omega, f; dt, dg),$$

where the summation is over  $\omega$  in  $\Omega(M, T)$ .

This, clearly, is the stable orbital integral of  $f$  in the sense

of Langlands [L5]. Note that by  $\gamma^\omega$  we mean  $w^{-1}\gamma w$ , where  $w \in G(\mathbb{C})$  realizes  $\omega$ . In view of Harish-Chandra's result for  $\phi_T(\gamma, f)$  we have immediately that  $\phi_T^{\text{st}}(\gamma, f)$  is well-defined and that the function  $\gamma \rightarrow (\prod_{\alpha} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi_T^{\text{st}}(\gamma, f)$ , where the product is over all  $\alpha \in R(G, T)$ , belongs to the Schwartz space of  $T(\mathbb{R})_{\text{reg}}$ .

We now write  $\phi^T(\gamma, dt, dg)$  in place of  $\phi_T^{\text{st}}(\gamma, f; dt, dg)$ .

Then:

$$(2.4.3) \quad \phi^T(\gamma, \alpha dt, \beta dg) = \beta/\alpha \phi^T(\gamma, dt, dg) \quad \text{for } \alpha, \beta > 0,$$

$$(2.4.4) \quad \phi^T(\gamma, dt, dg) = \phi^{T^w}(\gamma^w, (dt)^w, dg) \quad \text{for } w \in \mathcal{O}(T),$$

where  $T^w = w^{-1}T w$ , etc.,

(2.4.5)  $\gamma \rightarrow (\prod_{\alpha} |1 - \alpha(\gamma^{-1})|^{1/2}) \phi^T(\gamma, dt, dg)$  extends smoothly to (i.e. extends to a Schwartz function on)

$$T_{\text{reg}}^I(\mathbb{R}) = \{\gamma \in T(\mathbb{R}) : \alpha(\gamma) \neq 1, \alpha \in R(M, T)\} = T(\mathbb{R}) \cap M_{\text{reg}}.$$

For (2.4.3) and (2.4.4) the proof is immediate; (2.4.5) follows from the analogous result for  $\phi_T(\gamma, f)$  which is part of [HC, Theorem 17.1] and is proved, for example, by "descent to  $M$ " (see [W2, Section 8.5]).

The boundary of  $T_{\text{reg}}^I(\mathbb{R})$  is the union of the "imaginary walls" in  $T(\mathbb{R})$ , i.e. the union of the kernels in  $T(\mathbb{R})$  of the imaginary roots. We will need to know the behavior of  $\phi^T$  near points that lie on a single wall.

## 2.5 Semi-regular elements

Suppose that  $\gamma, \gamma' \in G(\mathbb{R})$  are semisimple, and that  $\gamma' = g^{-1}\gamma g$  for some  $g \in G(\mathbb{C})$ . Then  $\sigma_G(g)g^{-1}$  lies in  $G_{\gamma}(\mathbb{C})$ ,

the centralizer of  $\gamma$  in  $G(\mathbb{C})$ . Following Kottwitz's suggestion we call  $\gamma$  and  $\gamma'$  stably conjugate if  $g$  can be chosen so that  $\sigma_G(g)g^{-1}$  lies in  $G_Y^0(\mathbb{C})$ ,  $G_Y^0$  indicating the connected component of the identity in  $G_Y$ . The stable orbit, or stable conjugacy class, of  $\gamma$  is then the set of all elements stably conjugate to  $\gamma$ . This generalizes the notion for regular semisimple elements.

Lemma 2.5.1

Let  $\gamma \in G(\mathbb{R})$  be semisimple. Then:

(i) there exists a Cartan subgroup  $T(\mathbb{R})$  of  $G(\mathbb{R})$  such that  $\gamma \in T(\mathbb{R})$  and if  $\gamma \in T'(\mathbb{R})$  then  $T \leq T'$ . Then say that  $\gamma$  occurs fundamentally in  $T(\mathbb{R})$ .

(ii) If  $\gamma \in T'(\mathbb{R})$  then  $\{w^{-1}\gamma w : w \in \mathcal{A}(T')\}$  is contained in the stable orbit of  $\gamma$ .

(iii) If  $\gamma$  occurs fundamentally in  $T(\mathbb{R})$  then  $\{w^{-1}\gamma w : w \in \mathcal{A}(T)\}$  coincides with the stable orbit of  $\gamma$ .

Proof: For (i), suppose that  $\gamma \in T'(\mathbb{R})$ . Then  $T' \subset G_Y^0$  and  $T$  satisfies the conditions of (i) if and only if  $T$  is a fundamental maximal torus in  $G_Y^0$ . Thus (i) follows; (ii) is immediate from the definitions. For (iii) we have to show that  $\{w^{-1}\gamma w : w \in \mathcal{A}(T)\}$  contains the stable orbit of  $\gamma$ . Suppose that  $\gamma' = g^{-1}\gamma g$ , where  $\sigma_G(g)g^{-1} \in G_Y^0$ . Then  $\text{ad } g^{-1} : G_Y^0 \rightarrow G_Y^0$ , is an inner twist. [S1, Lemma 2.8] implies then that there is  $h$  in  $G_Y^0$ , such that  $\text{ad } h \circ \text{ad } g^{-1}|_T$  is defined over  $\mathbb{R}$ . Let  $w = gh^{-1}$ . Then  $w \in \mathcal{A}(T)$  and  $\gamma' = w^{-1}\gamma w$ . Thus the lemma is proved.

Returning to the notation of (2.4) we assume that  $\gamma_0$  is an element of  $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$ . Fix an imaginary root  $\alpha$  such that  $\alpha(\gamma_0) = 1$ . Assume that  $\gamma_0$  is semi-regular, i.e. that if  $\beta(\gamma_0) = 1$ ,  $\beta \in R(G, T)$ , then  $\beta = \pm \alpha$ , i.e. that  $G_{\gamma_0}$  is of type  $A_1$ . Note that  $\gamma_0$  occurs fundamentally in  $T(\mathbb{R})$ .

Call  $\gamma_0$  of totally compact type if  $\alpha$  is totally compact; this means that for  $\gamma'_0$  in the stable orbit of  $\gamma_0$  the group  $G_{\gamma'_0}$  is anisotropic modulo its center.

Lemma 2.5.2

For  $\gamma_0$  semi-regular in  $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$  we have that:

- (i) the stable orbit of  $\gamma_0$  meets only CSG's conjugate to  $T(\mathbb{R})$  if  $\gamma_0$  is of totally compact type,
- (ii) the stable orbit of  $\gamma_0$  meets CSG's conjugate to  $T(\mathbb{R})$  and exactly one conjugacy class of CSG's adjacent to  $T(\mathbb{R})$  if  $\gamma_0$  is not of totally compact type.

Proof: (i) is clear from the definitions. For (ii) "at most one" rather than "exactly one" is clear since for  $\gamma'_0$  in the stable orbit of  $\gamma_0$ ,  $G_{\gamma'_0}$  contains at most two conjugacy classes of CSG's and the union of the CSG's in the various  $G_{\gamma'_0}$  forms two conjugacy classes in  $G(\mathbb{R})$ . To produce the "exactly one" adjacent conjugacy class, choose a Cayley transform  $s: T \rightarrow T^S$  with respect to  $\alpha$  (recall (2.2.2)). Then  $\gamma_0^S$  is contained in  $T^S(\mathbb{R})$  and is stably conjugate to  $\gamma_0$  (see [S1, Section 2]). The conjugacy class of  $T^S(\mathbb{R})$  is the desired one.

## 2.6 Stable orbital integrals (continued)

Fix  $f \in \mathcal{C}(G(\mathbb{R}))$  and a CSG  $T(\mathbb{R})$ . For  $\gamma \in T$  we write  $|D(\gamma)|$  for  $\prod |1 - \alpha(\gamma^{-1})|$ , the product being taken over  $\alpha \in R(G, T)$ . Also we write  $\psi^T(\gamma) = \psi^T(\gamma, f; dt, dg)$  for the smooth extension of  $|D(\gamma)|^{1/2} \phi^T(\gamma) = |D(\gamma)|^{1/2} \phi_T^{\text{st}}(\gamma, f; dt, dg)$  to  $T_{\text{reg}}^I(\mathbb{R})$ , and  $\mathcal{A}_T(T)$  for the set of elements of  $\mathcal{A}(T)$  which normalize  $T$ .

### Lemma 2.6.1

Let  $\gamma_0$  be a semi-regular element in  $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$ . Then there is a neighborhood  $N(\gamma_0)$  of  $\gamma_0$  in  $T(\mathbb{R})$  invariant under  $\mathcal{A}_T(T)$ , and on  $N(\gamma_0)$   $C^\infty$ -functions  $\gamma \rightarrow \Lambda_i^T(\gamma_0, \gamma)$ ,  $i = 1, 2$ , (depending on  $f, dt, dg$  also and written  $\Lambda_i^T(\gamma_0, \gamma, f; dt, dg)$  when the occasion demands) such that:

$$(2.6.2) \quad \psi^T(\gamma) = \Lambda_1^T(\gamma_0, \gamma) |D(\gamma)|^{1/2} + \Lambda_0^T(\gamma_0, \gamma)$$

for all  $\gamma \in N(\gamma_0) \cap T_{\text{reg}}^I(\mathbb{R})$ .

Proof: See (2.7).

Let  $\alpha(\gamma_0) = 1$ ; recall that the root  $\alpha$  is unique up to sign. The Weyl reflection  $\omega_\alpha$  is realized in  $\mathcal{A}_T(T)$ . Thus if  $X$  is an invariant differential operator on  $T$  odd with respect to  $\omega_\alpha$  we have, with the obvious abuse of notation,

$$(2.6.3) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0) = 0.$$

Assume that  $X$  is fixed by  $\omega_\alpha$ . Set  $\gamma_\nu = \gamma_0 \exp \nu \alpha$ . Then for  $\nu$  in some deleted neighborhood of zero  $\gamma_\nu \in T_{\text{reg}}^I(\mathbb{R})$ . Applying  $X$  to (2.6.2) we obtain easily that:

$$(2.6.4) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0) = \lim_{\nu \rightarrow 0} (X\psi^T)(\gamma_\nu).$$

On the other hand,  $(X\Lambda_0^T)(\gamma_0, \gamma_0)$  is computed explicitly by, for example, reduction to  $SL_2$  (see references next page); the answer involves integrals along non-semisimple orbits. None of this is needed here. Nor do we need further information on  $(X\Lambda_1^T)(\gamma_0, \gamma_0)$ , although we will stop to mention  $\Lambda_1^T(\gamma_0, \gamma_0)$  as an illustration of Kottwitz's suggestion about stable orbital integrals for non-regular semisimple orbits [Ko] and as preamble to (2.9). Note that (2.6.2) gives:

$$(2.6.5) \quad \Lambda_1^T(\gamma_0, \gamma_0) = 1/2 \lim_{\nu \rightarrow 0} (H_\alpha(|1 - \alpha^{-1}| \phi^T))(\gamma_\nu),$$

where  $H_\alpha$  denotes the coroot of  $\alpha$  regarded as element of the Lie algebra of  $T$ .

There are two properties of  $X\Lambda_0^T$  which will be needed:

Lemma 2.6.6

Assume that  $X$  is fixed by  $\omega_\alpha$ .

(i) If  $\gamma_0$  is of totally compact type then  $(X\Lambda_0^T)(\gamma_0, \gamma_0) = 0$ .

(ii) If  $\gamma_0$  is not of totally compact type then  $(X\Lambda_0^T)(\gamma_0, \gamma_0)$

is independent of  $T$  in the following sense. Suppose that

$\gamma_0 \in T'$ . This implies that  $T'$  is either  $G(\mathbb{R})$ -conjugate to

or adjacent to  $T$ . Then  $T' = s^{-1}Ts$ , with  $s$  either  $\mathbb{R}$ -

rational or a Cayley transform with respect to the root  $\alpha$

which annihilates  $\gamma_0$  and:

$$(2.6.7) \quad (X\Lambda_0^T)(\gamma_0, \gamma_0, f; dt, dg) = (X^s \Lambda_0^{T^s})(\gamma^s, \gamma_0^s, f; (dt)^s, dg),$$

where in the case that  $s$  is a Cayley transform the right side

is to be interpreted as  $(X^s \Psi^{T^s})(\gamma_0^s)$ .

Proof: See (2.7).

Note that  $\gamma_0^s \in (T^s)_{\text{reg}}^I(\mathbb{R})$ , that  $(dt)^s$  is defined as in [S1]

and that the right side of (2.6.7) is independent of the choice of  $s$  (see [S1, Proposition 2.7]).

To compute  $\Lambda_1^T(\gamma_0, \gamma_0)$  we first assume that  $f \in C_c^\infty(G(\mathbb{R}))$ . Suppose that  $\gamma_0$  is of totally compact type. Recall that this means that  $G_\delta$  is anisotropic modulo center, for all  $\delta$  in the stable orbit of  $\gamma_0$ . Part (i) of the lemma above says simply that  $\phi^T$  itself extends smoothly to  $\gamma_0$ , so that  $\Lambda_1^T(\gamma_0, \gamma_0)$  is nothing but  $\lim_{\gamma \rightarrow \gamma_0} \phi^T(\gamma)$ ; i.e. the behavior is as if  $G(\mathbb{R})$  were compact. Reduction to  $SL_2/SU(2)$  (... in this case  $SU(2)$ ) via Harish-Chandra's compactness principle (see [W2, Section 8.5] or [S1, page 17], for example) then shows that  $\Lambda_1^T(\gamma_0, \gamma_0)$  is

the sum over (representatives  $\delta$  for the elements of)  $\mathcal{Q}(T)$  of

$$\text{vol}(T(\mathbb{R}) \backslash G_\delta^0(\mathbb{R})) \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where  $dh_\delta$  is a Haar measure on  $G_\delta^0(\mathbb{R})$  to be used also in the calculation of the volume (so that the choice is of no consequence).

If  $\delta = w\gamma_0$ , with  $w$  in  $\mathcal{A}(T)$  then  $\text{ad}w: G_\delta^0 \rightarrow G_{\gamma_0}^0$  is an inner twist (recall the proof of Lemma 2.5.1) and, in fact, an  $\mathbb{R}$ -isomorphism since both groups are anisotropic modulo center.

Fix a Haar measure  $dh$  on  $G_{\gamma_0}^0(\mathbb{R})$  and take  $dh_\delta$  to be the twist of  $dh$  by  $w$ . This allows us to rewrite the sum as

$$\text{vol}(T(\mathbb{R}) \backslash G_{\gamma_0}^0(\mathbb{R})) \sum_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where the summation remains over  $\mathcal{Q}(T)$ .

This, however, is misleading for a general definition of "stable orbital integral of  $f$  relative to  $\gamma_0$ ." Indeed,

suppose that  $\gamma_0$  is not of totally compact type. Then by reduction to  $SL_2/SU(2)$  we obtain that the right side of (2.6.5) equals

$$\text{vol}(\bar{T}(\mathbb{R}) \backslash \bar{G}_{\gamma_0}^0(\mathbb{R})) \sum (-1)^{q(G_\delta^0)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where the summation is now over  $\Omega(G_{\gamma_0}^0, T) \backslash \Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$  [ $\delta = \gamma_0^w$ ;  $w \in \mathcal{O}(T)$  realizes an element  $\omega$  of  $\Omega(M, T)$  and then  $\omega$  represents an element of the double quotient];  $\bar{\quad}$  indicates passage by an inner twist to the form anisotropic modulo center [L3, Chapter 6]. The volume is to be calculated using the twists of  $dh, dt$  to  $\bar{G}, \bar{T}$  respectively;  $dh_\delta$  is as before. As in [Ko],  $q(G_\delta^0)$  is one-half the dimension of the symmetric space attached to  $G_\delta^0$ , i.e. we take  $+1$  for  $(-1)^q$  if  $G_\delta^0$  is anisotropic modulo center, but  $-1$  otherwise.

For a Haar measure  $dh$  on  $G_{\gamma_0}^0(\mathbb{R})$  we set

$$O^{\text{st}}(\gamma_0, f) = O^{\text{st}}(\gamma_0, f; dh, dg) = \sum (-1)^{q(G_\delta^0)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

the summation being over  $\Omega(G_{\gamma_0}^0, T) \backslash \Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$ , as above. The conventions for measures remain the same also. Then  $O^{\text{st}}(\gamma_0, f)$  is well-defined for all  $f \in \mathcal{C}(G(\mathbb{R}))$  (see [W2, Section 9.3.1]) and  $f \mapsto O^{\text{st}}(\gamma_0, f)$  is a stable tempered distribution in the sense of [S1, Section 6] (see (2.9)).

Lemma 2.6.8

$\Lambda_1^T(\gamma_0, \gamma_0, f; dt, dg) = \text{vol}(\bar{T}(\mathbb{R}) \backslash \bar{G}_{\gamma_0}^0(\mathbb{R})) O^{\text{st}}(\gamma_0, f; dh, dg)$   
for all  $f \in \mathcal{C}(G(\mathbb{R}))$ .

Note that when  $\gamma_0$  is of totally compact type the double coset space in the summation coincides with  $\mathfrak{A}(T)$  itself. Thus our remarks above prove the lemma when  $f \in C_c^\infty(G(\mathbb{R}))$ . A simple continuity argument extends the formula to the Schwartz space.

## 2.7 Jump formulas

The results of Lemmas 2.6.1 and 2.6.6 can be expressed in terms of "jump" formulas analogous to those for  $'F_f$  given by Harish-Chandra. It is these formulas rather than the expansions (2.6.2) which guide us to the construction in (4.5) of the "transfer factors" for the main Matching Theorem.

First we will change  $\psi^T$  to a function  $\bar{\psi}^T$  by replacing  $|D(\gamma)|^{1/2}$  with the normalizing factor for  $'F_f$ . This factor will depend on our choice of a positive system for the imaginary roots of  $T$ ; care in that choice will greatly simplify the results.

For any choice of positive imaginary roots the function  $\gamma \rightarrow \prod_{\alpha \in \Sigma^+} (1 - \alpha(\gamma^{-1})) \prod_{\alpha \in \Sigma} |1 - \alpha(\gamma^{-1})|^{1/2} \phi^T(\gamma)$ , where the first product is over all positive imaginary roots and the second over all non-imaginary roots, is defined on  $T(\mathbb{R})_{\text{reg}}$  and equals  $\psi^T$  up to a factor which is bounded and has a  $C^\infty$ -extension to  $T_{\text{reg}}^I(\mathbb{R})$ . Thus it extends to a Schwartz function on  $T_{\text{reg}}^I(\mathbb{R})$ ; this function will be denoted  $\bar{\psi}^T$ , mention of the choice of positive imaginary roots being omitted in notation. In (2.7.2) any choice will do. For (2.7.3) we need some preparation.

Let  $\alpha$  be an imaginary root of  $T$ . Then a positive system for the imaginary roots is adapted to  $\alpha$  if all imaginary roots  $\beta$  for which  $\langle \alpha, \beta^\vee \rangle > 0$  are positive. If  $I^+$  is such a system and  $s$  is a Cayley transform with respect to  $\alpha$  then there is a unique positive system for the imaginary

roots of  $T^S$  transported into  $I^+$  by  $\text{ads}$ ; denote this system by  $I_S^+$ . In (2.7.3) we use a system  $I^+$  adapted to either one of the two roots annihilating  $\gamma_0$  and the related system  $I_S^+$  for  $T^S$ . Also, denote by  $X \rightarrow X'$  the automorphism of the algebra of invariant differential operators on  $T$  induced by the map  $H \rightarrow H + \nu(H)I$ , for  $H$  in the Lie algebra of  $T$ , where  $\nu$  is one-half the sum of the roots in  $I^+$ ; let  $X \rightarrow X''$  be the automorphism of the operators on  $T^S$  induced by the map  $H \rightarrow H + \nu_S(H)I$ , for  $H$  in the Lie algebra of  $T^S$ , with  $\nu_S$  one-half the sum of the roots in  $I_S^+$ .

Lemma 2.7.1

Suppose that  $\{\gamma \rightarrow \phi^T(\gamma) = \phi^T(\gamma, dt, dg)\}$  is a family of functions satisfying (2.4.3), (2.4.4) and (2.4.5). Form  $\psi^T, \bar{\psi}^T$  as above.

Then  $\{\psi^T\}$  satisfies the statements of Lemmas 2.6.1 and 2.6.6 for all semi-regular elements  $\gamma_0$  in  $T(\mathbb{R}) - T_{\text{reg}}^I(\mathbb{R})$  if and only if for all such  $\gamma_0$  we have:

(2.7.2)  $\lim_{\nu \rightarrow 0} (X\bar{\psi}^T)(\gamma_\nu) = 0$  if  $\gamma_0$  is of totally compact type  
and

(2.7.3)  $\lim_{\nu \rightarrow 0} (X'\bar{\psi}^T)(\gamma_\nu) = i ((X^S)''\bar{\psi}^{T^S})(\gamma_0^S)$  if  $\gamma_0$  is not of  
totally compact type,

for all invariant differential operators  $X$  on  $T$ .

Proof: Note that for  $X$  odd with respect to the Weyl reflection  $\omega_\alpha$ ,  $\alpha$  a root annihilating  $\gamma_0$ , (2.7.2) and (2.7.3) are automatically satisfied. Thus in the statement of the lemma we may as well take  $X$  fixed by  $\omega_\alpha$ . Then (2.7.2) follows from (2.6.4) and (i) in Lemma 2.6.6; (2.7.3) follows from (2.6.4) and (ii)

in Lemma 2.6.6, although now some (elementary) calculations are needed. We omit the details (see [S1,page 27] regarding a crucial property of  $I^+$ ).

For the converse, assume (2.7.2) and (2.7.3). We have just to verify the existence of expansions (2.6.2) for then (2.6.4) is true and reversing the arguments of the first part of the proof yields Lemma 2.6.6. Fix  $T$  and  $\gamma_0$ . If  $\gamma_0$  is of totally compact type then (2.7.2) implies that  $\bar{\psi}^T$  is  $C^\infty$  around  $\gamma_0$ . It then follows easily that  $\phi^T$  itself is  $C^\infty$  around  $\gamma_0$  also, and (2.6.2) is true.

Now assume that  $\gamma_0$  is not of totally compact type. We will give just an outline of the steps involved in verifying an expansion (2.6.2). We may assume that  $G$  is simply-connected and semisimple, first by a routine reduction to  $M = M_T$  and secondly by reduction to the simply-connected covering of the derived group of  $M$ . This second reduction rests on the definition of stable conjugacy. Next we define  $\Lambda_0^T$ . Fix a Cayley transform  $s$  with respect to a root  $\alpha$  annihilating  $\gamma_0$ . Then the restriction of  $\bar{\psi}^{T^S}$  to a suitable neighborhood of  $\gamma_0^S$  in  $T^S(\mathbb{R})$  can be extended to an  $\Omega(G,T)$ -invariant  $C^\infty$ -function on a neighborhood of  $\gamma_0^S$  in  $T^S(\mathbb{C})$ . Transport this function to  $T(\mathbb{C})$  by  $s$  and restrict to (a neighborhood of  $\gamma_0$  in)  $T(\mathbb{R})$ . This defines our  $\Lambda_0^T$ . Note that (2.6.7) is then automatic. Finally we use (2.7.3) to show that  $\psi^T$  is  $C^\infty$ -divisible by  $|D(\gamma)|^{1/2}$  around  $\gamma_0$ . See [L3,Chapter 6] for a similar style of argument.

It remains then to recall the proofs of Lemmas 2.6.1 and 2.6.6 and the formulas (2.7.2) and (2.7.3) for  $\phi^T(\gamma) = \phi_T^{\text{st}}(\gamma, f)$ . The formulas are proved in Section 4 of [S1]. The arguments combine those of Harish-Chandra for  $'F_f$  (involving reduction to  $SL_2/SU(2)$ ) with facts about the orbits in a stable orbit. For example, the left side of (2.7.3) involves  $[\mathfrak{L}(T)]$  orbits and the right side  $[\mathfrak{L}(T^S)]$  orbits, but only either  $[\mathfrak{L}(T^S)]$  or  $2[\mathfrak{L}(T^S)]$  orbits on the left contribute to the limit. For the proofs of Lemmas 2.6.1 and 2.6.6 we can invoke the last lemma, or argue directly by reduction to  $SL_2/SU(2)$ , recalling again [L3, Chapter 6] and [S1, Section 4].

## 2.8 Characterization of stable orbital integrals

### Theorem 2.8.1

Suppose that  $\{\gamma \rightarrow \phi^T(\gamma, dt, dg)\}$  is a family satisfying (2.4.3), (2.4.4), (2.4.5), (2.7.2) and (2.7.3). Then there exists  $f \in \mathcal{C}(G(\mathbb{R}))$  such that:

$$\phi^T(\gamma, dt, dg) = \phi_T^{\text{st}}(\gamma, f; dt, dg), \quad \gamma \in T(\mathbb{R})_{\text{reg}},$$

for all  $T$ ,  $dt$  and  $dg$ .

Proof: See [S1, Theorem 4.7]. Suppose that  $\phi^{T'} = 0$  whenever  $T' \not\geq T$  and  $T'$  is not  $G(\mathbb{R})$ -conjugate to  $T$ . Then  $\bar{\psi}^T$  extends smoothly to the union of the regular and the semi-regular elements in  $T(\mathbb{R})$  and thence, by a well-known principle of Harish-Chandra, smoothly to all of  $T(\mathbb{R})$ , i.e.  $\bar{\psi}^T$  extends to a Schwartz function on  $T(\mathbb{R})$ . An inductive argument then

shows that to prove the theorem we need only verify the following for each Cartan subgroup  $T(\mathbb{R})$ :

(2.8.2) if  $\bar{\psi}^T$  extends to a Schwartz function on  $T(\mathbb{R})$  then there exists  $f \in \mathcal{C}(G(\mathbb{R}))$  such that  $\phi_T^{\text{st}}(\cdot, f) \equiv \phi^T$  and  $\phi_{T'}^{\text{st}}(\cdot, f) \equiv 0$  unless  $T' \leq T$ .

In [S1] we construct  $f$  as a sum of (scalar projections of) wave-packets. In the case that  $T(\mathbb{R})$  is compact the constructed  $f$  is a sum of  $K$ -finite matrix coefficients of discrete series representations, with some constraints on the  $K$ -types involved.

## 2.9 A limit formula

The following is a short exercise based on our discussion so far (and some fundamental results of Harish-Chandra for which we refer to [W2]), and concerns one of the simplest problems associated with applications of the trace formula ([L4], [L6]).

Suppose that  $G(\mathbb{R})$  has Cartan subgroups compact modulo the center of  $G(\mathbb{R})$  (CCSG's). We consider a function  $f$  with the following property:

(2.9.1)  $\phi_{T'}^{\text{st}}(\cdot, f) \equiv 0$  unless  $T'(\mathbb{R})$  is a CCSG.

In [L4] (see also [L6]) the constraints on  $f$  are greater; moreover,  $\phi_T^{\text{st}}(\cdot, f)$  is specified for  $T(\mathbb{R})$  a CCSG, and  $f$  is to be  $C_c^\infty$ . This will not concern us here.

Recall that (2.9.1) implies that  $\bar{\psi}^T$  extends to a Schwartz function on  $T(\mathbb{R})$ , for  $T(\mathbb{R})$  a CCSG.

Lemma 2.9.2

Assume (2.9.1). Then  $\gamma \mapsto \Phi_T^{\text{St}}(\gamma, f)$  extends smoothly to  $T(\mathbb{R})$ , for each CCSG  $T(\mathbb{R})$ .

Proof: After a straightforward reduction to the simply-connected covering of the derived group of  $G$ , we find that it is sufficient to show the following. Suppose that  $G$  is simply connected and semisimple and that  $T(\mathbb{R})$  is a CCSG. Then a  $C^\infty$ -function  $F$  on  $T(\mathbb{R})$  satisfying  $F(\gamma^\omega) = \det \omega F(\gamma)$ ,  $\omega \in \Omega(G, T)$ , is  $C^\infty$ -divisible by  $\Delta$ , where, as usual,

$$\Delta(\gamma) = \prod_{\alpha \in \Sigma^+} (1 - \alpha(\gamma^{-1})),$$

$\alpha$  denoting one-half the sum of the roots in some positive system for  $R(G, T)$  and the product being taken over these roots. But this follows easily from rearrangement of the Fourier series for  $F$ .

For any  $\gamma_0 \in T(\mathbb{R})$  we define  $O^{\text{St}}(\gamma_0, f; dh, dg)$  by the formula of (2.6). Thus:

$$O^{\text{St}}(\gamma_0, f; dh, dg) = \sum (-1)^{q(\delta)} \int_{G_\delta^0(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\delta g) \frac{dg}{dh_\delta},$$

where  $q(\delta) = q(G_\delta^0)$  is well-defined since  $G(\mathbb{R})$  has CCSG's,  $\delta$  is of the form  $\gamma_0^w$ , with  $w$  in  $\mathcal{U}(T)$ ,  $dh_\delta$  is the twist of  $dh$  (a Haar measure on  $G_{\gamma_0}^0$ ) by  $w$ , and the summation is over  $w$  in a set of representatives for

$$\Omega(G_{\gamma_0}^0, T) \backslash \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R}));$$

more precisely, we choose representatives  $\underline{w}$  in  $\Omega(G, T)$  for these double cosets and then for each  $\underline{w}$  an element  $w$

in  $G$  (i.e. in  $\tilde{\mathcal{A}}_T(T)$ ) realizing  $\underline{w}$ . Note that relative to the natural projection of  $\mathcal{D}(T) = \Omega(G,T)/\Omega(G(\mathbb{R}),T(\mathbb{R}))$  onto  $\Omega(G_{\gamma_0}^0, T) \setminus \Omega(G,T) / \Omega(G(\mathbb{R}),T(\mathbb{R}))$  the preimage of the double coset of  $\underline{w} \in \Omega(G,T)$  contains  $[\Omega(G_\delta^0, T) / \Omega(G_\delta^0(\mathbb{R}), T(\mathbb{R}))]$  =  $[\mathcal{D}_{G_\delta}^0(T)]$  elements, where  $\delta = (\gamma_0)^{\underline{w}}$ .

Lemma 2.9.3

Assume (2.9.1). Then:

$$\lim_{\gamma \rightarrow \gamma_0} \phi_T^{\text{st}}(\gamma, f; dt, dg) = \text{vol}(\bar{T}(\mathbb{R}) \setminus \bar{G}_{\gamma_0}^0(\mathbb{R})) O^{\text{st}}(\gamma_0, f; dh, dg)$$

for each CCSG  $T(\mathbb{R})$ .

Here, as in (2.6),  $\bar{\quad}$  indicates passage to the inner form anisotropic modulo center, and the volume is to be calculated relative to the twists of  $dh, dt$  (so that the right side of our formula, like the left side, is independent of the choice for  $dh$ ).

We will include a proof of the lemma just for the case  $G$  simply-connected and semisimple, where a CCSG  $T(\mathbb{R})$  is compact and we have at our disposal the normalizing factor  $\Delta$  from the proof of Lemma 2.9.2. The general case involves only minor modification of the arguments, but considerably more notation.

For the rest of this section, then,  $G$  will be simply connected and semisimple and  $T(\mathbb{R})$  compact. If  $\delta = \gamma_0^{\underline{w}}$ , with  $\underline{w} \in \tilde{\mathcal{A}}_T(T)$ , then we write  $H_{\underline{w}}$  for  $G_\delta^0$ ; in particular,  $H_1 = G_{\gamma_0}^0$ . The lemma has only to be proved for one choice of  $dg, dh$  and  $dt$ . We fix maximal compact subgroups

$K, K_w$  for  $G(\mathbb{R}), H_w(\mathbb{R})$  containing  $T(\mathbb{R})$  and denote by  $dg, dh_w, dt$  the standard Haar measures on  $G(\mathbb{R}), H_w(\mathbb{R}), T(\mathbb{R})$ , all as in [HC, Sections 3,7]. Note that  $\text{vol}(\overline{T}(\mathbb{R}))$  is now 1. Fix a positive system for  $R(G,T)$  and define the factor  $\Delta$  relative to this system. Also set  $\Delta^*(\gamma_0) = \prod_{\alpha} (1 - \alpha(\gamma_0^{-1}))$ , where the product is over positive roots not in  $R(H_1, T)$ . The positive system for  $R(H_1, T)$  will consist of the roots positive for  $R(G, T)$ , and the system for  $R(H_w, T)$  will be the twist of this one by  $w$ . We define  $\nu_1, \nu_w$  and the operators  $\varpi_1, \varpi_w$  (see [HC, Section 17]) accordingly.

Fix  $w \in \mathcal{O}_T(T)$  and let  $\delta = \gamma_0^w$ . From [HC, Section 37] we have that  $dh$ , the twist of  $dh$  by  $w$ , is  $v(K_w)/v(K_1) dh_w$ , where  $v$  is as defined in the reference. Thus, for any Schwartz function  $f$  we have:

$$\begin{aligned}
 O(\delta, f; dh_\delta, dg) &= C_w \lim_{\gamma \rightarrow \gamma_0} (\varpi_w(\Delta^w \phi_T(\cdot, f; dt, dg)))(\gamma^w), \text{ where } C_w \\
 &= (\Delta^*(\gamma_0))^{-1} (-1)^{q(w)} v(K_1) (v(T(\mathbb{R})))^{-1} [\Omega(H_w(\mathbb{R}), T(\mathbb{R}))]^{-1} (2\pi)^{-r(w)}, \\
 q(w) &\text{ denoting } q(H_w) \text{ and } r(w) = r(1) \text{ one-half the dimension} \\
 &\text{of } H_w/T \text{ (see [W2, Section 9.3.1, Theorem 8.5.1.6, Section 8.1.3]} \\
 &\text{and [HC, Lemma 37.4])}. \text{ To form } O^{\text{st}}(\gamma_0, f) \text{ we have to sum} \\
 &\text{over } w \text{ for which } \underline{w} = wT(\mathbb{C}) \text{ form a set of representatives} \\
 &\text{for } \Omega(H_1, T) \setminus \Omega(G, T) / \Omega(G(\mathbb{R}), T(\mathbb{R})). \text{ If we replace } w \text{ by } w' \\
 &\text{such that } \underline{w}' \text{ lies in the preimage in } \mathcal{O}(T) \text{ of the double coset} \\
 &\text{of } \underline{w} \text{ then the expression above does not change. Thus we have} \\
 &\text{that } O^{\text{st}}(\gamma_0, f; dh, dg) =
 \end{aligned}$$

$$C \lim_{\gamma \rightarrow \gamma_0} (\alpha_1(\Delta \phi_T^{st}(\cdot, f; dt, dg)))(\gamma),$$

where  $C = (\Delta^*(\gamma_0))^{-1} v(K_1)(v(T(\mathbb{R}))^{-1} [\Omega(H_1, T)]^{-1} (2\pi)^{-r(1)})$ .

Note that this implies that the tempered distribution  $f + O^{st}(\gamma_0, f)$  is stable in the sense of [S1, Section 6]. We may therefore argue that to prove the lemma for a given  $f$  satisfying (2.9.1) we may replace  $f$  by any Schwartz function  $f'$  with same stable orbital integrals, i.e.  $\phi_{T'}^{st}(\cdot, f) \equiv \phi_{T'}^{st}(\cdot, f')$  for all CSG's  $T'(\mathbb{R})$ . We choose  $f'$  to be the function constructed in our proof of Theorem 2.8.1 (see [S1, Section 4]). It then follows that we have only to verify the formula in the statement of the lemma for the case that  $f$  is a  $K$ -finite matrix coefficient of a discrete series representation of  $G(\mathbb{R})$ .

Let  $\Lambda$  be a regular element in the positive chamber for  $X^*(T)$  and  $\pi(\Lambda)$  be the discrete series (class of) representation(s) attached to  $\Lambda$ . Let  $\theta_\Lambda^* = \sum \det w \Lambda / \sum \det w w_1$ , so that the stabilized character of  $\pi(\Lambda)$  is  $(-1)^{q(G)} \theta_\Lambda^*$ . If  $f$  is a  $K$ -finite matrix coefficient for  $\pi(\Lambda)$  then  $\phi_T^{st}(\gamma, f; dt, dg) = (-1)^{q(G)} f(1) d_\Lambda^{-1} \theta_\Lambda^*(\gamma)$ , where  $d_\Lambda$  is the formal degree of  $\pi(\Lambda)$  relative to  $dg$  (see [W2]).

Thus  $\lim_{\gamma \rightarrow \gamma_0} \phi_T^{st}(\gamma, f; dt, dg) =$

$$(-1)^{q(G)} f(1) d_\Lambda^{-1} \alpha_1(\sum \det w \Lambda (\Delta^*(\gamma_0) \alpha_1(1_1) [\Omega(H_1, T)]))^{-1}$$

(with the usual abuse of notation). This equals  $O^{st}(\gamma_0, f; dh, dg)$

up to a constant which Lemma 37.4 of [HC] shows to be

$\text{vol}(\bar{T}(\mathbb{R}) \setminus \bar{H}_1(\mathbb{R}))$ . Hence Lemma 2.9.3 is proved.

Similar arguments show that  $d_\Lambda = \theta_\Lambda^*(1) / \text{vol}(\bar{G}(\mathbb{R}))$ , where the volume is to be calculated relative to the twist of the

measure used to define  $d_\Lambda$ , i.e. formal degree is "invariant under inner twisting." (See [W2, Theorem 10.2.4.1] for Harish-Chandra's formula for  $d_\Lambda$ ).

## 2.10 $\mathfrak{D}(T)$ , $\mathfrak{E}(T)$ and $\kappa$ .

This section is preparation for the definition of "unstable", i.e. "k-", orbital integrals. Recall  $\mathfrak{D}(T)$  as  $T(\mathbb{C}) \backslash \mathcal{O}(T) / G(\mathbb{R})$ . If  $w \in \mathcal{O}(T)$  then  $\sigma_G(w)w^{-1} \in T(\mathbb{C})$  and the map  $w \rightarrow \{1 \rightarrow 1, \sigma \rightarrow \sigma_G(w)w^{-1}\}$  induces an embedding of  $\mathfrak{D}(T)$  in  $H^1(T) = H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), T(\mathbb{C}))$ . The image of  $\mathfrak{D}(T)$  is contained in the subgroup  $\mathfrak{E}(T)$  of [L2]. In fact,  $\mathfrak{E}(T)$  is the smallest subgroup of  $H^1(T)$  containing  $\mathfrak{D}(T)$  and "invariant under inner twisting" in the obvious sense (we leave a proof of this to the reader). Following [L2], we regard  $\mathfrak{E}(T)$  as a quotient of the span of the imaginary coroots; more precisely, as  $\mathbb{Z}[R^\vee(M, T)] / \mathbb{Z}[R^\vee(M, T)] \cap \{\lambda^\vee - \sigma_T \lambda^\vee : \lambda^\vee \in X_*(T)\}$ . To see how  $\mathfrak{D}(T)$  appears in this realization we return to  $\mathfrak{D}(T)$  as  $\Omega(M, T) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$ . Consider the coset of  $\omega_\alpha$ , for  $\alpha$  an imaginary root. If  $\alpha$  is compact then  $\omega_\alpha$  is realized in  $G(\mathbb{R})$  so that the coset of  $\omega_\alpha$  is trivial and the corresponding element of  $\mathfrak{E}(T)$  is trivial also. If  $\alpha$  is noncompact then the coset of  $\omega_\alpha$  corresponds to the coset of  $\alpha^\vee$ . More generally, there is the following (unpublished) result of Langlands. First, an inductive argument shows that there exists  $\Lambda^\vee \in \mathbb{Z}[R^\vee(M, T)]$  such that  $\langle \Lambda^\vee, \alpha \rangle$  is even if  $\alpha$  is compact and  $\langle \Lambda^\vee, \alpha \rangle$  is odd if  $\alpha$  is noncompact.

Then  $\omega \in \Omega(M(\mathbb{R}), T(\mathbb{R})) \in \mathfrak{D}(T)$  is identified with the coset of  $\lambda^\vee - \omega \lambda^\vee$ .

A character  $\kappa$  on  $\mathfrak{E}(T)$  has order two and so defines by restriction a function on  $\mathfrak{D}(T)$  assuming the values  $\pm 1$ . Recall that the source of  $\kappa$ -orbital integrals is the trace formula [L5, Chapter 8]. There  $\kappa$  appears as the restriction to  $\mathfrak{D}(T)$  of a character on all coroots of  $(G, T)$ . Thus, following Langlands, we will assume that  $\kappa$  is a quasicharacter on  $\mathbb{Z}[R^\vee(G, T)] / \mathbb{Z}[R^\vee(G, T)] \cap \{\lambda^\vee - \sigma_T \lambda^\vee : \lambda^\vee \in X_*(T)\}$ , i.e. that  $\kappa \in \bar{K}(T/\mathbb{R})$  as in [L5]. If  $T$  is anisotropic modulo center then this quotient is  $\mathfrak{E}(T)$ , so there is no change, but if  $T$  is split over  $\mathbb{R}$  then we will be allowing any quasicharacter on  $\mathbb{Z}[R^\vee(G, T)]$ .

An element  $w$  of  $\mathfrak{A}(T)$  defines a map  $\kappa \rightarrow \kappa^w$  from  $\bar{K}(T) = \bar{K}(T/\mathbb{R})$  to  $\bar{K}(T^w)$  in the obvious way. Let  $\Omega_0(G, T)$  be the subgroup of  $\Omega(G, T)$  consisting of the elements realized in  $\mathfrak{A}(T)$ , i.e.  $\Omega_0(G, T) = \{\omega \in \Omega(G, T) : \omega \sigma_T = \sigma_T \omega\}$ . Then  $\Omega_0(G, T)$  acts on  $\bar{K}(T)$ ; if  $w$  realizes  $\omega$  then  $\kappa^\omega = \kappa^w$ . Also we write  $\kappa(\omega) = \kappa(w)$  for the value of  $\kappa$  on the element of  $\mathfrak{D}(T)$  determined by  $\omega$  and  $w$ . Then, regarding the (Weyl group-) product in  $\Omega_0(G, T)$ , we have

$$(2.10.1) \quad \kappa(\omega \omega') = \kappa(\omega) \kappa^\omega(\omega')$$

for all  $\omega, \omega' \in \Omega_0(G, T)$ .

Suppose that  $s$  is a Cayley transform with respect to the imaginary root  $\alpha$  of  $T$ . Suppose also that  $\kappa(\alpha^\vee) = 1$ . Then a calculation shows that the map  $\mathbb{Z}[R^\vee(G, T)] \rightarrow \mathbb{Z}[R^\vee(G, T^S)]$

determined by  $\text{ad } s^{-1}$  induces a map  $\kappa \rightarrow \kappa^S$  of  $\bar{K}(T)$  to  $\bar{K}(T^S)$ , i.e. we may propagate a given  $\kappa$  to certain adjacent CSG's. (See [S2, Section 3] for this and some related observations. Note that there is an error in the statement of part (iii) of Proposition 3.3: " $\omega \in \mathcal{A}(T_S)$  which normalizes  $T_S$ " should read " $\omega$  in the imaginary Weyl group of  $T_S$ ." The error is of no consequence for the rest of the paper.)

Fix  $\kappa \in \bar{K}(T)$ . Suppose that  $H$  is a group (connected, reductive, defined over  $\mathbb{R}$ ) such that:

- (i)  $T$  is contained in  $H$ ,
- (ii)  $R^\vee(H, T) = \{\alpha^\vee \in R^\vee(G, T) : \kappa(\alpha^\vee) = 1\}$ .

Then a review of the axioms for root systems shows:

(2.10.2)  $\Omega(H, T)$  is naturally embedded in  $\Omega(G, T)$  as the subgroup generated by the reflections  $\omega_\alpha$ , where  $\alpha^\vee \in R^\vee(H, T)$ , and

(2.10.3) on fixing a nondegenerate symmetric bilinear form on  $X_*(T)$  invariant under  $\Omega(G, T)$ , we may identify  $R(H, T) = (R^\vee(H, T))^\vee$  with a subset (but, in general, not a subsystem) of  $R(G, T) = (R^\vee(G, T))^\vee$ .

Note that in both (2.10.2) and (2.10.3) the Galois action is respected.

If  $\omega \in \Omega(G, T)$  lies in the embedded  $\Omega(H, T)$  then we say that  $\omega$  is "from  $H$ ." Similarly, a root of  $\Omega(G, T)$  is "from  $H$ " if it is in the embedded  $R(H, T)$  (some suitable bilinear form will be assumed fixed). These notions are compatible

in the sense that a reflection from  $H$  is with respect to a root from  $H$ ; also, imaginary roots come only from imaginary roots of  $H$  and elements of  $\Omega_0(G, T)$  only from  $\Omega_0(H, T)$ .

Suppose that  $\omega \in \Omega_0(G, T)$  is from  $H$ . Then for  $\lambda^V$  in  $\mathbb{Z}[R^V(G, T)]$  we have that  $\lambda^V - \omega^{-1}\lambda^V$  lies in  $\mathbb{Z}[R^V(H, T)]$  and so is annihilated by  $\kappa$ . We conclude then that:

$$(2.10.4) \quad \kappa^\omega = \kappa.$$

Note that (2.10.4) does not characterize the elements from  $\Omega_0(H, T)$ .

Lemma 2.10.5

$\kappa(\omega) = 1$  provided that  $\omega$  is from the imaginary Weyl group of  $T$  in  $H$ .

Proof: See [S2, Proposition 7.4]. This is a simple exercise: if  $\alpha$  is a compact root then  $\kappa(\omega_\alpha) = 1$ , and if  $\alpha$  is noncompact then  $\kappa(\omega_\alpha) = \kappa(\alpha^V)$  so that  $\kappa(\omega_\alpha) = 1$  if  $\omega$  is from  $H$ . Thus the lemma is true for reflections. Next, (2.10.4) implies that  $\kappa$  is multiplicative on the elements from  $\Omega_0(H, T)$  and the proof is completed by an inductive argument.

For a general element of  $\Omega_0(G, T)$  from  $H$  Lemma 2.10.5 is false. Such an element  $\omega$  has two signatures,  $\text{sgn}_H(\omega)$  defined with respect to the imaginary roots from  $H$  and  $\text{sgn}_G(\omega)$  defined with respect to all imaginary roots of  $T$  in  $G$ .

Lemma 2.10.6

$$\kappa(\omega) = \text{sgn}_G(\omega) / \text{sgn}_H(\omega).$$

Proof: See [S2, Section 7 and Lemma 8.2]. The proof is quite long as we need to consider generators for  $\Omega_0(G, T)$ . In view of the proof of Lemma 2.4.1 there is an exact sequence:

$$1 \rightarrow \Omega(M, T) \rightarrow \Omega_0(G, T) \rightarrow W_T \rightarrow 1,$$

where  $W_T$  is the relative Weyl group of  $T$ , i.e.  $W_T = \text{Norm}(M(\mathbb{R}), G(\mathbb{R})) / M(\mathbb{R})$ . The results of [Kn] give generators for  $W_T$ . We then form generators for  $\Omega_0(G, T)$  in the obvious way. The proof of the lemma consists of reduction to and examination of certain "types" of generators. Details are given in the reference.

## 2.11 $\kappa$ -orbital integrals

Let  $f$  be a Schwartz function on  $G(\mathbb{R})$ ,  $T(\mathbb{R})$  be a CSG and  $\kappa$  an element of  $\mathcal{K}(T)$ . Then, as in [L5], we set:

$$\phi_T^K(\gamma, f) = \phi_T^K(\gamma, f; dt, dg) = \sum \kappa(w) \phi_{T^w}^K(\gamma^w, f; (dt)^w, dg),$$

$\gamma \in T(\mathbb{R})_{\text{reg}}$ , where the summation is over representatives  $w$  for the elements of  $\mathcal{Q}(T)$ . Then:

$$(2.11.1) \quad \phi_T^K(\gamma, f; \alpha dt, \beta dg) = \beta/\alpha \phi_T^K(\gamma, f; dt, dg),$$

$$(2.11.2) \quad \kappa(w) \phi_{T^w}^K(\gamma^w, f; (dt)^w, dg) = \phi_T^K(\gamma, f; dt, dg)$$

for  $w \in \mathcal{A}(T)$ .

We normalize  $\phi_T^K$  by the factor from (2.7), following the same conventions for positive imaginary roots in (2.11.5) below. Let  $\bar{\Psi}^K$  be the function so obtained (so that  $\bar{\Psi}^1 = \bar{\Psi}^T$ ). Then:

$$(2.11.3) \quad \bar{\Psi}^K \text{ extends to a Schwartz function on}$$

$$T_{\text{reg}}^{I, K}(\mathbb{R}) = \{\gamma \in T(\mathbb{R}) : \alpha(\gamma) \neq 1 \text{ for imaginary roots } \alpha \text{ such}$$

that  $\kappa(\alpha^V) = 1$ ).

This follows from the results for ordinary orbital integrals and the observation that (anti-)symmetry forces  $\bar{\psi}^K$  to be smooth across the semi-regular elements on walls  $\alpha = 1$  where  $\alpha$  is not totally compact and  $\kappa(\alpha^V) = -1$ . (See [S2, Lemma 4.3]).

We now follow the notation of Lemma 2.7.1. Suppose that  $\alpha$  is an imaginary root for which  $\kappa(\alpha^V) = 1$ . Then:

$$(2.11.4) \quad \lim_{v \rightarrow 0} (X \bar{\psi}^K)(\gamma_v) = 0 \quad \text{for all semi-regular}$$

elements  $\gamma_0$  annihilated by  $\alpha$ , if  $\alpha$  is totally compact, and

$$(2.11.5) \quad \lim_{v \rightarrow 0} (X \bar{\psi}^K)(\gamma_v) - \lim_{v \rightarrow 0} (X \bar{\psi}^K)(\gamma_v) =$$

$$2i \varepsilon_\kappa(s) ((X^S) \bar{\psi}^{K^S})(\gamma_0^S) \quad \text{for all semi-regular}$$

elements  $\gamma_0$  annihilated by  $\alpha$ , if  $\alpha$  is not totally compact.

The new term  $\varepsilon_\kappa(s)$  is the  $\kappa$ -signature of  $s$  defined in Section 4 of [S2]; it assumes the values  $\pm 1$  ... this is all that concerns us for the present. For a proof of (2.11.4) and (2.11.5) see [S2, Section 4].

In Part 4 we will take up one of the themes of endoscopic groups, that  $\kappa$ -orbital integrals are "stable orbital integrals on lower-dimensional groups." Note that if  $H$  is as in (2.10) then:

$$(i) \quad T_{\text{reg}}^{I, \kappa}(\mathbb{R}) \quad \text{is} \quad T_{\text{reg}}^I(\mathbb{R}) \quad \text{relative to } H, \quad \text{and}$$

$$(ii) \quad \phi_T^K(\gamma^\omega, f) = \text{sgn}_H(\omega) / \text{sgn}_G(\omega) \phi_T^K(\gamma, f), \quad \gamma \in T(\mathbb{R})_{\text{reg}},$$

for all  $\omega \in \Omega_0(G, T)$  which are "from  $H$ ."

### 3. TEMPERED SPECTRUM AND L-GROUPS

#### 3.1 Notation

For L-group data we will follow the notation of [S4] which is essentially that of [L5]. Thus  $G$  remains a connected, reductive group over  $\mathbb{R}$ ;  $\psi: G \rightarrow G^*$  is an inner twist from  $G$  to a quasi-split form  $G^*$ , and  $L_G = L_G^0 \rtimes W$  is the L-group of  $G^*$  and of  $G$ . We realize the Weil group  $W$  as  $\{z \times \tau: z \in \mathbb{C}^\times, \tau \in \text{Gal}(\mathbb{C}/\mathbb{R})\}$ , with  $(z \times \tau)(z' \times \tau') = z\tau(z')a_{\tau, \tau'} \times \tau\tau'$ , where  $a_{\tau, \tau'} = 1$  unless  $\tau = \tau' = \sigma$ , and  $a_{\sigma, \sigma} = -1$ . Also included in the "L-group data" is a Borel subgroup  $B^*$  over  $\mathbb{R}$  in  $G^*$ ;  $B^*$  contains the maximal torus  $T^*$  over  $\mathbb{R}$ ;  $L_G^0$ ,  $L_B^0$  and  $L_T^0$  are the "dual" complex groups. The group  $W$  acts on  $L_G^0$  via a homomorphism  $\rho_G: W \xrightarrow{\text{proj}} \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow \text{Aut}(L_G^0, L_B^0, L_T^0, \{X_{\alpha^v}\})$ . The action is completely specified by the automorphism  $\sigma_G = \rho_G(1 \times \sigma)$  which is the "algebraic dual" of the Galois automorphism of  $G^*$ . Recall that  $X_{\alpha^v}$  is a root vector for the simple root  $\alpha^v$  of  $L_T^0$  in  $L_B^0$ .

#### 3.2 L-groups for maximal tori and Levi groups

Let  $T$  be a maximal torus over  $\mathbb{R}$  in  $G^*$ . We will call  $T$  standard if  $S_T$  is contained in  $T^*$  (i.e. in  $S_{T^*}$ ). Every maximal torus over  $\mathbb{R}$  in  $G^*$  is  $G^*(\mathbb{R})$ -conjugate to a standard

one. Suppose then that  $T$  is standard. We may choose  $m \in M_T$  so that  $m^{-1}Tm = T^*$ . The automorphism  $\text{adm}^{-1} \circ \sigma_T \circ \text{adm}$  of  $T^*$  (and hence of  $L_{T^0}$ ) is independent of the choice for  $m$ ; we denote the automorphism by  $\sigma_T$  again. For  $L_T$  we take  $L_{T^0} \rtimes W$ , with  $W$ -action given by  $\rho_T(1 \times \sigma) = \sigma_T$ . Moreover for  $L_M = L_{(M_T)}$  we may take  $L_{M^0} \rtimes W$ , where  $L_{M^0}$  contains  $L_{T^0}$ ,  $R(L_{M^0}, L_{T^0}) = \{\alpha^v \in R(L_G^0, L_{T^0}) : \sigma_T \alpha^v = -\alpha^v\}$ , and  $W$  acts by restriction of  $\rho_G$  to  $L_{M^0}$ , i.e.  $\sigma_M$  is the restriction of  $\sigma_G$  to  $L_{M^0}$ .

Suppose now that  $T$  is a maximal torus over  $\mathbb{R}$  in  $G$ . Fix a standard torus  $T^{**}$  in  $G^*$  and  $x \in G^*$  such that  $\psi_x = \text{ad } x^{-1} \circ \psi$  maps  $T$  to  $T^{**}$  over  $\mathbb{R}$  (see [L1]). Fix  $m \in M_{T^{**}}$  such that  $m^{-1}(T^{**})m = T^*$ . We write  $\eta$  for the pair  $(\psi_x, m)$  and call it a pseudodiagonalization (p-d.) of  $T$ . Via  $\text{adm}^{-1} \circ \psi_x$ , or more informally "via  $\eta$ ", we transport  $\sigma_T$  to an automorphism  $\sigma_{(T, \eta)}$  of  $T^*$  and hence of  $L_{T^0}$ . Clearly  $\sigma_{(T, \eta)} = \sigma_{T^{**}}$ . We have then a realization of the L-group for  $T$ , namely  $L_T = L_{(T^{**})}$ . Since  $\psi_x$  is an inner twist from  $M_T$  to  $M_{T^{**}}$ , we have also the realization of  $L_M = L_{(M_T)}$  as  $L_{(M_{T^{**}})}$ . Whenever  $\eta$  has been given we will assume  $L_T$  and  $L_M$  chosen in this way. Note that if  $T$  is anisotropic modulo the center of  $G$ , so that  $L_M = L_G$ , then  $L_T$  is independent of the choice for  $\eta$ .

Remark: We make this definition of p-d. for use in parts 3 and 4 only, and only for the purpose of fixing L-groups. It will simplify our discussion of some constructions. Later (see (4.4)) we will allow any map from  $T$  to  $T^*$  of the form  $\text{ad } y^{-1} \circ \psi$ , with  $y$  in  $G^*$ , as is appropriate for Langlands' notion of "diagram" [L5].

### 3.3 Embedding $L_T$ in $L_G$

We recall some material from [S3, Section 1.3]. Fix a maximal torus  $T$  over  $\mathbb{R}$  in  $G$  and  $p$ -d.  $\eta$  of  $T$ . We have  $L_M$  naturally embedded in  $L_G$ . To embed  $L_T$  in  $L_G$  it is then sufficient to embed  $L_T$  in  $L_M$ . We therefore assume that  $T$  is anisotropic modulo the center of  $G$ . An admissible embedding  $\xi: L_T \rightarrow L_G$  will be a homomorphism  $\xi(t \times w) = t\xi(w)$ ,  $t \in L_T^0$ ,  $w \in W$ , where  $\xi(w)$  is of the form  $\xi_0(w) \times w \in L_G^0 \times w$ ,  $w \in W$ . Since  $\xi(\mathbb{C}^\times \times 1)$  must act (by conjugation) trivially on  $L_T^0$  we have the restriction of  $\xi_0$  to  $\mathbb{C}^\times \times 1$  is a homomorphism into  $L_T^0$ . On the other hand,  $\xi(1 \times \sigma)$  must act on  $L_T^0$  as  $\sigma_{(T, \eta)}$ ;  $1 \times (1 \times \sigma)$  preserves the positive roots of  $L_T^0$  (i.e. the roots of  $L_T^0$  in  $L_B^0$ ) and  $\sigma_{(T, \eta)}$  acts as  $-1$  on all roots. Thus we conclude that  $\xi_0(1 \times \sigma)$  must be an element of  $\text{Norm}(L_T^0, L_G^0)$  which takes the positive roots to negative roots. Finally,  $(1 \times \sigma)^2 = -1 \times 1$  requires that  $\xi_0(1 \times \sigma) \sigma_G(\xi_0(1 \times \sigma)) = \xi_0(-1 \times 1)$ .

Lemma 3.2 of [L1] shows how to construct  $\xi$ . (The proof of this lemma is reproduced in an appendix to [S3].) In [L1] the lemma is the crucial step in constructing parameters for discrete series representations. Here we construct embeddings of  $L_T$  and the parameters will appear simply as "lifts" from  $T$ .

Let  $\lambda$  be one-half the sum of the coroots for the roots of  $L_T^0$  in  $L_B^0$ . Define  $\xi_0$  on  $\mathbb{C}^\times \times 1$  by  $\lambda^v(\xi_0(z \times 1)) = (z/\bar{z})^{\langle 1, \lambda^v \rangle}$ ,  $\lambda^v \in X^*(L_T^0)$ . For  $\xi_0(1 \times \sigma)$  take any element in the normalizer of  $L_T^0$  in the derived group of  $L_G^0$  which maps the positive roots to negative roots (such elements do

exist [L1], see [S3, Appendix]). Then the "Lemma 3.2" implies that we may reverse the steps above and extend  $\xi_0$  to an admissible embedding  $\xi$  of  $L_T$  in  $L_G$ . If  $\xi'$  is another admissible embedding of  $L_T$  in  $L_G$  then  $\xi'(t \times w) = a(w)\xi(t \times w)$ ,  $t \in L_T^0$ ,  $w \in W$ , where  $a(w)$  is a 1-cocycle of  $W$  in  $L_T^0$  with the action of  $W$  determined by  $\sigma_{(T,\eta)}$ . In particular, the image of  $L_T$  under  $\xi'$  is the same as that under  $\xi$ . Also it follows that in replacing  $\xi$  by  $\xi'$  we replace  $\tau$  by, roughly speaking,  $\tau$  plus the data for a quasi-character on  $T(\mathbb{R})$ , i.e. there is an essential "twist" by  $\tau$  when we embed  $L_T$  in  $L_G$  ... we could state this more precisely following [S3, Section 1.3] and the discussion of the next section.

The existence of an admissible embedding of  $L_T$  in  $L_G$ , for arbitrary  $T, \eta$  can also be verified directly from [L2, Lemma 4].

### 3.4 Langlands parameters

Recall that an admissible homomorphism  $\phi: W \rightarrow L_G$  is of the form  $\phi(w) = \phi_0(w) \times w$ ,  $w \in W$ , where  $\phi_0(w)$  is a semi-simple element of  $L_G^0$ ;  $\phi$  is equivalent to  $\phi'$  if  $\phi' = \text{ad}_g \circ \phi$ , for some  $g$  in  $L_G^0$ . When it is necessary to distinguish in notation between a homomorphism and its equivalence class we will use  $\{\phi\}$  for the class of  $\phi$ . The set  $\Phi(G^*)$  is the collection of all equivalence classes of admissible  $\phi: W \rightarrow L_G$ , and  $\Phi(G)$  consists of those classes "relevant to  $G$ ." See [B].

Let  $\phi:W \rightarrow L_G$  be admissible. If  $\phi_0(\mathbb{C}^\times \times 1)$  is contained in  $L_T^0$  and  $\phi(1 \times \sigma)$  normalizes  $L_T^0$  and acts on  $L_T^0$  as  $c_{T^{**}}$ , for some standard torus  $T^{**}$  in  $G^*$ , then  $\phi(W)$  is contained in the image of  $L_{T^{**}}$  under the admissible embeddings of  $L_{T^{**}}$  in  $L_{M_{T^{**}}} \subseteq L_G$ . Fix such an embedding  $\xi$ . Then clearly  $\phi = \xi \circ \phi'$ , for some admissible homomorphism  $\phi':W \rightarrow L_{T^{**}}$ , i.e.  $\{\phi\}$  is contained in the image of the map from  $\phi(T^{**})$  to  $\phi(G^*)$  induced by  $\xi$ . Since the image of  $\phi(T^{**})$  is independent of the choice of  $\xi$  we will say simply that  $\{\phi\}$  "factors through  $\phi(T^{**})$ ." Suppose now that  $T$  is a maximal torus over  $\mathbb{R}$  in  $G$ . Fix a p-d.  $\eta = (\psi_x, m)$  of  $T$ . Let  $T^{**} = \psi_x(T)$ . Then  $\phi(T) = \phi(T^{**})$  and from [L1] (see [B] also) we may check that  $\text{Image}(\phi(T)) = \text{Image}(\phi(T^{**}))$  is independent of the choice for  $\eta$  and:

$$(3.4.1) \quad \phi(G) = \bigcup_T \text{Image}(\phi(T)) \cap \phi(G).$$

Note that in our terms "relevant to  $G$ ", i.e. " $\in \phi(G)$ ", means simply  $\{\phi\}$  factors through only those  $\phi(T^{**})$  for which  $T^{**}$  "originates in  $G$ " [S1, Section 2], i.e.  $T^{**}$  is of the form  $\psi_x(T)$ .

We will consider only bounded parameters, i.e. classes with representatives  $\phi$  such that  $\phi_0(W)$  is bounded. Let  $\phi_0(G^*)$  be the set of all bounded parameters and  $\phi_0(G) = \phi_0(G^*) \cap \phi(G)$ . As long as we allow only embeddings  $\xi$  of  $L_T$  for which  $\xi_0(W)$  is bounded we have:

$$(3.4.2) \quad \phi_0(G) = \bigcup_T \text{Image}(\phi_0(T)) \cap \phi(G).$$

The discrete parameters in  $\phi_0(G)$  are those which factor through only the  $\phi_0(T)$  for  $T$  anisotropic modulo the center

of  $G$ . There are other parameters in the image of  $\phi_0(T)$ , for such  $T$ . They are not necessarily relevant to  $G$ ; we call them "limits of discrete parameters." A more concrete description is given as follows.

Fix a torus  $T$  anisotropic modulo the center of  $G$ , and a  $p$ -d.  $\eta$  of  $T$ . We embed  $L_T$  in  $L_G$  via the homomorphism  $\xi$  constructed in (3.3). A class factoring through  $\phi_0(T)$  has a representative  $\phi$  such that  $\phi_0(\mathbb{C}^\times \times 1)$  is contained in  $L_T^0$  and  $\phi(1 \times \sigma) = n \times (1 \times \sigma)$ , where  $n \in L_G^0$  normalizes  $L_T^0$  and maps the positive roots to negative roots. We write  $\phi = \phi(\mu, \lambda)$  if  $\lambda^V(\xi_0(z \times 1)) = z^{\langle \mu, \lambda^V \rangle} \bar{z}^{\langle \bar{\sigma}\mu, \lambda^V \rangle}$ ,  $\lambda^V \in X^*(L_T^0)$ , where  $\bar{\sigma}$  abbreviates  $\sigma_{(T, \eta)}$ , and  $\lambda^V(n) = e^{2\pi i \langle \lambda, \lambda^V \rangle}$ ,  $\lambda^V \in X^*(L_G^0)$ . Both  $\mu$  and  $\lambda$  are elements of  $X_*(L_T^0) \otimes \mathbb{C}$  which we can identify with  $X^*(T) \otimes \mathbb{C}$  via  $\eta$ . While  $\mu$  is uniquely determined by  $\phi$ , there is ambiguity in  $\lambda$  (which we will ignore); see [S4, Section 4.3] for details. Note that  $\phi(\mu, \lambda)$  is equivalent to  $\phi(\mu', \lambda')$  if and only if  $\mu'$  is in the orbit of  $\mu$  under  $\Omega(L_G^0, L_T^0)$  and  $\lambda$  equals  $\lambda'$  up to the allowed ambiguity [S4, Lemma 4.3.1]. The class of  $\phi(\mu, \lambda)$  forms a discrete parameter if and only if  $\langle \mu, \alpha^V \rangle \neq 0$  for all roots  $\alpha^V$  of  $L_T^0$  ([L1], see also [B]). In any case we can choose for a class  $\{\phi\}$  a representative  $\phi(\mu, \lambda)$  such that  $\langle \mu, \alpha^V \rangle \geq 0$  for all roots  $\alpha^V$  of  $L_T^0$  in  $L_B^0$ , i.e. with  $\mu$  dominant. Then  $\mu$  is uniquely determined by  $\{\phi\}$ .

Clearly we can write any  $\phi(\mu, \lambda)$  as  $\xi \circ \phi'$ , where  $\phi'$  has parameters  $(\mu - \nu, \lambda)$  relative to  $T$ . By the Langlands correspondence for tori we have then that  $(\mu - \nu, \lambda)$  is datum for a character on  $T(\mathbb{R})$  (see [S3, Section 4.1] for a review of the correspondence); so also is  $(\omega\mu - \omega\nu, \lambda)$ ,  $\omega \in \Omega(G, T)$ .

### 3.5 L-packets

We recall now the Langlands correspondence for tempered representations of  $G$ , i.e. the assignment to an element of  $\Phi_0(G)$  of an L-packet of (equivalence classes of) tempered representations of  $G(\mathbb{R})$ .

Suppose that  $\{\phi\}$  is discrete. Choose a maximal torus  $T$  compact modulo center and a p-d.  $\eta$  of  $T$ . Let  $\phi = \phi(\mu, \lambda)$ , with  $\mu$  dominant, represent  $\{\phi\}$ . Transport  $\mu, \lambda$  to  $T$  via  $\eta$ . Then for each  $\omega \in \Omega(G, T)$  the pair  $(\omega\mu, \lambda)$  is datum for a unique (equivalence class of) discrete series representation(s); note that  $\omega\mu$  is strictly dominant with respect to the ordering obtained from  $(L_B^0, L_T^0)$  by transport by  $\eta$  and then  $\omega$ . We denote this representation, and its character, by  $\theta_\omega(\mu, \lambda)$ . The L-packet attached to  $\{\phi\}$  is  $\{\theta_\omega(\mu, \lambda) : \omega \in \Omega(G, T)\}$ . It is independent of the choice for  $T$  and  $\eta$ .

In general, we can find a pair  $(T, \eta)$  so that  $\{\phi\}$  has a representative  $\phi$  where  $\phi(W)$  is contained in  $L_M$  and the class of  $\phi$  in  $\Phi(M)$  is discrete. The L-packet attached to  $\{\phi\}$  is the set of constituents of the unitary principal

series representations obtained from the representations in the L-packet of discrete series representations of  $M(\mathbb{R})$  attached to the class of  $\phi$  in  $\phi(M)$ . It is independent of the choice of  $T$ ,  $\eta$  and  $\phi$  [L1].

We will need to identify L-packets in terms of "limits of discrete series" and "induced limits of discrete series" characters, i.e. in terms of the "basic" characters of [KZ].

Fix  $(T, \eta)$  with  $T$  anisotropic modulo the center of  $G$ . Let  $\{\phi\}$  factor through  $\phi(T)$ . Suppose that  $\phi = \phi(\mu, \lambda)$ , with  $\mu$  dominant, represents this class. Transfer  $\mu, \lambda$  to  $T$  via  $\eta$ . Then for each  $\omega \in \Omega(G, T)$  the pair  $(\omega\mu, \lambda)$  together with  $\{\alpha^v \in R^v(G, T) : \omega^{-1}\alpha^v \text{ is the transfer to } T \text{ via } \eta \text{ of a root in } R(L_B^0, L_T^0)\}$ , with respect to which  $\omega\mu$  is dominant, are data for a limit of discrete series character which we will denote by  $\Theta_\omega(\mu, \lambda)$ . Recall that this character is either irreducible or zero [KZ]. Clearly  $\{\Theta_\omega(\mu, \lambda) : \omega \in \Omega(G, T)\}$  depends only on  $\{\phi\}$ , i.e. is independent of the choice of  $(T, \eta)$  (and the ambiguity in  $\lambda$ ). Note that if  $\Theta_\omega(\mu, \lambda)$  and  $\Theta_{\omega'}(\mu', \lambda')$  are nonzero and equal then we must have  $\omega'\mu' = \omega\bar{\omega}\mu$ , for some  $\bar{\omega} \in \Omega(G(\mathbb{R}), T(\mathbb{R}))$ , and  $\lambda'$  equal to  $\lambda$  up to the allowed ambiguity (see [V] or [KZ]).

### Theorem 3.5.1

- (i) If  $\{\phi\}$  is not relevant to  $G$  then each  $\Theta_\omega(\mu, \lambda)$ ,  $\omega \in \Omega(G, T)$ , is zero.
- (ii) If  $\{\phi\}$  is relevant to  $G$  then the nonzero characters in  $\{\Theta_\omega(\mu, \lambda) : \omega \in \Omega(G, T)\}$  are exactly the characters of the representations in the L-packet attached to  $\{\phi\}$ .

Proof: See [S4, Theorem 4.3.2]. The argument given is as follows. We may verify the theorem directly in a special case (recalled below). This case is sufficient for us to prove functoriality of the lifting of tempered characters from an endoscopic group (see (4.4)). Via that functoriality we may use induction on the dimension of  $G$  to prove the theorem in the general case.

The special case is that where  $R_\mu^\vee = \{\alpha^\vee \in R^\vee(G, T) : \langle \mu, \alpha^\vee \rangle = 0\}$  consists of superorthogonal roots, i.e. is of type  $(A_1)^\mathbb{N}$ . We then use in  $L_{G^0}$  the analogue of a sequence of Cayley transforms (with respect to roots in  $R_\mu^\vee$ ) to find a standard maximal torus  $T^{**}$  in  $G^*$ , and parameter  $\phi_1$  equivalent to  $\phi$  and discrete relative to  $M_{T^{**}}$ . See the proof of Lemma 4.3.5 in [S4]. If  $\phi$  is not relevant to  $G$  then  $T^{**}$  does not "originate in  $G$ ." An argument with Cayley transforms in  $G$  then shows that for each  $\omega \in \Omega(G, T)$  some root in  $\{\omega\alpha : \alpha^\vee \in R_\mu^\vee\}$  is compact. This implies that each  $\Theta_\omega(\mu, \lambda)$  is zero ([V] or [KZ]). If  $\phi$  is relevant to  $G$  we have that the associated L-packet consists of the constituents of the principal series representations defined by  $\phi_1$ . It remains then to decompose these representations using the Schmid identities, to observe which of the  $\Theta_\omega(\mu, \lambda)$  their constituents are and to verify that the remaining  $\Theta_\omega(\mu, \lambda)$  are zero. See the proof of Lemma 4.3.7 in [S4] (our discussion in the next section will also be relevant). One point we will note here is the L-group formulation of the condition for a character to appear as the right side of a Schmid identity.

Fix a maximal torus  $T$  and  $p$ -d.  $\eta$  of  $T$ . Let  $\{\phi\} \in \Phi(G)$  factor through  $\Phi(T)$  and  $\phi = \phi(\mu, \lambda)$ , with  $\mu$  dominant relative to the roots of  $L_B^0 \cap L_M^0$ , represent  $\{\phi\}$ .

Let  $\Theta_\omega^M(\mu, \lambda)$  be an associated limit of discrete series character of  $M(\mathbb{R})$ . Let  $\alpha$  be a real root of  $T$ . Fix  $(T_1, \eta_1)$  so that  $T_1$  is adjacent to  $T$ ,  $S_{T_1}$  is contained in  $S_T$ , and  $\alpha$  (more precisely, the transfer of  $\alpha$  by  $\eta$  then  $\eta_1^{-1}$ ) is imaginary for  $T_1$ . Let  $M_1 = M_{T_1}$  and  $P^1 = MN^1$  be a parabolic subgroup over  $\mathbb{R}$  in  $M_1$  with  $M$  as Levi subgroup. Set  $\Theta_\omega^{M_1}(\mu, \lambda)$  equal to  $\text{Ind}(\Theta_\omega^M(\mu, \lambda) \otimes I_{N^1}; P^1(\mathbb{R}), M_1(\mathbb{R}))$ .

The element  $\lambda$  belongs to  $X^*(T) \otimes \mathbb{C}$  (after transfer via  $\eta$ ). It determines a character on the finite subgroup  $\{\exp i\pi\lambda^V : \lambda^V \in X_*(T), \sigma_T \lambda^V = \lambda^V\}$  of  $T(\mathbb{R})$ , namely  $\exp i\pi\lambda^V \rightarrow \exp(2\pi i \langle \lambda, \lambda^V \rangle)$ . Thus the necessary and sufficient condition for  $\Theta_\omega^{M_1}(\mu, \lambda)$  to be the right side of a Schmid identity ([KZ]) becomes  $\langle \omega\mu, \alpha^V \rangle = 0$  and  $\exp(2\pi i \langle \lambda, \alpha^V \rangle) = (-1)^{\langle \rho_\alpha, \alpha^V \rangle}$ , where  $\rho_\alpha$  has its usual meaning (one-half the sum of the roots  $\beta$  of  $(G, T)$  such that the restriction of  $\beta$  to  $S$  is a positive multiple of  $\alpha$ ). Note that  $\langle \omega\mu, \alpha^V \rangle = \langle \mu, \alpha^V \rangle$ .

On the other hand,  $\lambda^V(\phi(1 \times \sigma)) = \exp(2\pi i \langle \lambda, \lambda^V \rangle)$ ,  $\lambda^V \in X^*(L_M^0)$ . According to a result of Langlands (see [A]) this implies that  $\phi(1 \times \sigma) X_{\alpha^V} = -(-1)^{\langle 2\lambda + \rho_\alpha, \alpha^V \rangle} X_{\alpha^V}$  for any root vector  $X_{\alpha^V}$  for  $\alpha^V$ . We conclude then that  $\Theta_\omega^{M_1}(\mu, \lambda)$  is the right side of a Schmid identity if and only if  $\Theta_1^{M_1}(\mu, \lambda)$  is, and:

(3.5.2)  $\Theta_1^{M_1}(\mu, \lambda)$  is the right side of a Schmid identity if and only if  $\langle \mu, \alpha^V \rangle = 0$  and  $\phi(1 \times \sigma) X_{\alpha^V} = -X_{\alpha^V}$ .

Using "induction in stages" we deduce from Theorem 3.5.1 that the constituents of the representations in any collection  $\{\Theta_{\omega}^G(\mu, \lambda) = \text{Ind}(\Theta_{\omega}^M(\mu, \lambda) \otimes I_N ; P(\mathbb{R}), G(\mathbb{R})) : \omega \in \Omega(M, T)\}$ , where  $P = MN$  is a parabolic subgroup over  $\mathbb{R}$  containing given  $M = M_T$  as Levi subgroup, either are all zero or form an L-packet of tempered representations.

### 3.6 Internal structure of L-packets

We fix a bounded parameter  $\{\phi\}$  in  $\phi(G^*)$ . If  $\{\phi\}$  is relevant to  $G$  then  $\mathbb{I}$  will denote the associated L-packet of representations of  $G(\mathbb{R})$  (or their characters); otherwise  $\mathbb{I}$  will consist of the zero character.

In presenting  $\mathbb{I}$  as  $\{\Theta_{\omega}^G(\mu, \lambda) : \omega \in \Omega(M, T)\}$  we have various choices for  $(T, \eta)$  and  $\phi = \phi(\mu, \lambda)$ .

#### Lemma 3.6.1

The orbit of  $\mu$  under  $\Omega(L_G^0, L_T^0)$  is uniquely determined by  $\{\phi\}$ .

Proof: A replacement  $\phi'$  for  $\phi$  must satisfy:  $\phi'_0(\mathbb{C}^{\times} \times 1)$  is contained in  $L_T^0$  and  $\phi'(1 \times \sigma)$  normalizes  $L_T^0$ . Also there must be  $g \in L_G^0$  such that  $\phi' = \text{ad}_g \circ \phi$ . Multiplying  $g$  by an element of the connected reductive group  $\text{Cent}(\phi_0(\mathbb{C}^{\times} \times 1), L_G^0)$  if necessary, we may assume that  $g$  normalizes  $L_T^0$ . Then the lemma follows.

The element  $\mu$  lies in  $X_*(L_T^0) \otimes \mathbb{C}$  which can be identified with the dual of the Lie algebra of  $T(\mathbb{C})$ ; the  $\Omega(L_G^0, L_T^0)$ -orbit of  $\mu$  then is canonically identified with an orbit of  $\Omega(G, T)$  in this dual. Here  $T$  is any maximal torus in  $G$ . In the case that  $\{\phi\}$  is relevant to  $G$  the latter orbit is identified

in the usual way with the infinitesimal character of the representations in  $\mathbb{H}$ .

Note that in fixing  $\mu$ , or its orbit, but varying  $(T, \eta)$  and  $\phi(1 \times \sigma)$  which acts as  $\sigma_{(T, \eta)}$ , we may easily pass to a unbounded parameter (see [L7, Section 7] and the example of the appendix to these notes).

Our concern now is to present  $\mathbb{H}$  with a "best" choice of data. The notion of "nondegenerate data" is introduced in [KZ]. Fix  $(T, \eta)$  and associated  $\phi = \phi(\mu, \lambda)$  representing  $\{\phi\}$ . Let  $R_\mu^V = \{\alpha^V \in R(L_M^0, L_T^0) : \langle \mu, \alpha^V \rangle = 0\}$ . We transfer data to  $T$  via  $\eta$  without change in notation. Assume:

(3.6.2)  $R_\mu^V$  is of type  $(A_1)^n$  and

(3.6.3) no element of  $R_\mu^V$  lies in  $(I - \sigma_T)X_*(T)$ , i.e.  
each element of  $R_\mu^V$  defines a nontrivial  $(-1)$ -cocycle of  
Gal $(\mathbb{C}/\mathbb{R})$  in  $X_*(T)$ .

Lemma 3.6.4

(i) Each  $\theta_\omega^G(\mu, \lambda)$  is either zero or presented with non-  
degenerate data,

(ii)  $\theta_\omega^G(\mu, \lambda)$  is nonzero if and only if  $\{\omega\alpha : \alpha^V \in R_\mu^V\}$  contains  
no compact roots.

(iii) Suppose that  $\theta_1^G(\mu, \lambda) \neq 0$ . Then  $\theta_\omega^G(\mu, \lambda) \neq 0$  if and  
only if  $\omega$  belongs to  $\Omega_\mu(M, T)$ , the subgroup of  $\Omega(M, T)$   
generated by the reflections with respect to the roots in

$R_\mu = (R_\mu^V)^V$  and  $(R_\mu^V)^\perp = \{\alpha \in R(M, T) : \langle \alpha, \beta^V \rangle = 0, \beta \in R_\mu^V\}$ .

Proof: Each positive root in  $R_\mu^V$  is simple (for  $L_B^0 \cap L_M^0$ , as usual).

Thus  $\Theta_{\omega}(\mu, \lambda)$ , and hence  $\Theta_{\omega}^G(\mu, \lambda)$ , is nonzero if and only if  $\{\omega\alpha: \alpha^{\vee} \in R_{\mu}^{\vee}\}$  contains a compact root ([KZ]). The requirement (3.6.3) implies that if  $\alpha$  is noncompact then the Weyl reflection with respect to  $\alpha$  is not realized in  $G(\mathbb{R})$  ... see (2.10). Clearly (3.6.3) holds for  $\omega R_{\mu}^{\vee}$  also. Thus (i) and (ii) follow. For (iii) we have just to determine for which  $\omega$  the set  $\{\omega\alpha: \alpha^{\vee} \in R_{\mu}^{\vee}\}$  contains only noncompact roots. See the argument for Lemma 4.3.7 of [S4]. The crucial point, already used in the jump formulas for stable orbital integrals, is that if  $\alpha$  is noncompact then the only other noncompact roots in the  $\Omega(M, T)$ -orbit of  $\alpha$  are  $\pm\omega_0\alpha$ , for  $\omega_0 \in \Omega(M(\mathbb{R}), T(\mathbb{R}))$  [S1, Lemma 4.2].

The Knapp-Zuckerman theory will show that if  $T$  is minimal with respect to (3.6.2) and (3.6.3) (i.e. if  $T' \leq T$  (see (2.2)) and  $(T', \eta')$  and some  $\phi' = \phi(\mu', \lambda')$  also satisfy (3.6.2) and (3.6.3) then  $T'$  is  $G(\mathbb{R})$ -conjugate to  $T$ ) then the nonzero characters among the  $\Theta_{\omega}^G(\mu, \lambda)$  are irreducible. Also, nonzero  $\Theta_{\omega}^G(\mu, \lambda)$  and  $\Theta_{\omega'}^G(\mu, \lambda)$  are distinct unless  $\omega' \in \omega\Omega(M(\mathbb{R}), T(\mathbb{R}))$  (this requires Lemma 3.2 of [S1]). Finally,  $T$ , etc., is unique up to  $G(\mathbb{R})$ -conjugacy, i.e. if  $(T', \eta')$  and  $\phi(\mu', \lambda')$  are also minimal then there exists  $g \in G(\mathbb{R})$  such that  $T' = g^{-1}Tg$ ,  $\mu' = \omega\bar{g}\mu$ ,  $\lambda' = \bar{g}\lambda$  (up to the allowed ambiguity), and  $\omega\bar{g}$  preserves  $R(L_M^0 \cap L_B^0, L_T^0)$ , where  $\omega$  is some element of  $\Omega(L_M^0, L_T^0)$  and  $\bar{g}$  denotes the automorphism  $\eta' \circ \text{ad } g^{-1} \circ \eta^{-1}$  of  $L_T^0$ . This uniqueness, of course, presumes  $\{\phi\}$  relevant to  $G$ . We take all this for granted until the next section.

Assume now that  $\{\phi\}$  is relevant to  $G$ . Fix  $(T, \eta)$  and some corresponding  $\phi = \phi(\mu, \lambda)$  satisfying (3.6.2) and (3.6.3). Assume that  $T$  is minimal. Fix also  $\omega_* \in \Omega(M, T)$  so that  $\Theta_{\omega_*}^G(\mu, \lambda)$  is nonzero, i.e. so that  $\omega_* R_\mu$  contains only non-compact roots. For  $\omega \in \Omega(M, T)$  set  $\Theta(\omega) = \Theta_{\omega_* \omega}^G(\mu, \lambda)$ . (\*) Then  $\mathbb{H} = \{\Theta(\omega) : \omega \in \Omega_{\omega_* \mu}(M, T)\}$ , where  $\Omega_{\omega_* \mu}(M, T)$  is the subgroup of  $\Omega(M, T)$  generated by the reflections with respect to the roots in  $\omega_* R_\mu$  and  $\omega_*(R_\mu^\vee)^\perp$ . Thus  $\mathbb{H}$  is in 1-1 correspondence with  $\Omega_{\omega_* \mu}(M, T) / \Omega_{\omega_* \mu}(M, T) \cap \Omega(M(\mathbb{R}), T(\mathbb{R})) = \Omega_{\omega_* \mu}(M, T) \Omega(M(\mathbb{R}), T(\mathbb{R})) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$ , a subset of  $\mathfrak{S}(T)$  and hence of  $\mathfrak{E}(T)$ .

For the moment we ignore this special choice of  $\omega_*$  and pair  $\mathbb{H}$  with  $\mathfrak{E}(T)^\vee$ , the character group of  $\mathfrak{E}(T)$ , in the obvious way(s). Fix any  $\omega_* \in \Omega(M, T)$ . Then for  $\kappa \in \mathfrak{E}(T)^\vee$  and  $\pi \in \mathbb{H}$  define  $\langle \kappa, \pi \rangle_{\omega_*} = \kappa(\omega)$  if  $\pi = \Theta_{\omega_* \omega}^G(\mu, \lambda)$ . Then:

$$(3.6.5) \quad \langle \kappa_1 \kappa_2, \pi \rangle_{\omega_*} = \langle \kappa_1, \pi \rangle_{\omega_*} \langle \kappa_2, \pi \rangle_{\omega_*} \quad \text{for all } \kappa_1, \kappa_2, \pi,$$

$$(3.6.6) \quad \langle \kappa, \pi \rangle_{\omega_*} = \langle \kappa, \pi' \rangle_{\omega_*} \quad \text{for all } \kappa \quad \text{if and only if} \\ \pi = \pi',$$

$$(3.6.7) \quad \langle \kappa, \pi \rangle_{\omega_*} = \kappa(\omega_*) \langle \kappa^{\omega_*^{-1}}, \pi \rangle_1 \quad \text{for all } \kappa, \pi.$$

The proof of (3.6.5) and (3.6.6) is immediate. For (3.6.7) see (2.10.1).

Our real purpose is to pair  $\mathbb{H}$  with the group  $\mathbb{S}$  defined as follows. Let  $\phi$  be any representative for  $\{\phi\}$ . Let  $S_\phi$  be the centralizer of  $\phi(W)$  in  $L_G^0$ , i.e. the group of fixed points of the automorphism of the group  $\text{Cent}(\phi_0(\mathbb{T}^\times \times 1), L_G^0)$

---

(\*) Recall that we have defined the action of  $\Omega(G, T)$  on  $T$  by  $\omega(t) = t^\omega = w^{-1}tw$ ,  $t \in T$ , if  $w \in G$  realizes  $\omega \in \Omega(G, T)$ . Thus  $\omega(\omega_* \mu) = (\omega_* \omega) \mu$ .

determined by conjugation by  $\phi(1 \times \sigma)$ . Let  $S_\phi^0$  be the connected component of the identity in  $S_\phi$ , and  $Z_G^W$  be the group of  $W$ -invariants in the center of  $L_G^0$ . Then  $S = S_\phi / Z_G^W S_\phi^0$ . Since  $S$  is abelian (Langlands, see also below), it is replaced by a canonically isomorphic copy when we replace  $\phi$  by another representative for  $\{\phi\}$ .

Recall the description of  $\mathfrak{E}(T)$  as  $\mathbb{Z}[R^\vee(M, T)] / \mathbb{Z}[R^\vee(M, T)] \cap \{v^\vee - \sigma_T v^\vee : v^\vee \in X_*(T)\}$ . A character on this quotient can be extended to a  $\sigma_T$ -invariant character on  $X_*(T)$ . Any such extension, after transfer by the fixed  $\eta$ , can be regarded as an element of  $L_T^0$  fixed by  $\sigma_{(T, \eta)}$ . The coset of this element in  $L_T^0 / Z_M^W$  is independent of the extension chosen. On the other hand, any of the representatives  $\phi(\omega\mu, \lambda)$  acts on  $L_T^0$  as  $\sigma_{(T, \eta)}$ . Thus  $\mathfrak{E}(T)^\vee$  is identified with  $S_\phi \cap L_T^0 / Z_M^W$ , for each  $\phi = \phi(\omega\mu, \lambda)$ ,  $\omega \in \Omega(M, T)$ .

Theorem 3.6.8

Let  $\phi = \phi(\omega\mu, \lambda)$  as above. Then:

- (i)  $L_T^0$  meets every component of  $S_\phi$ ,
- (ii)  $Z_W^G(S_\phi^0 \cap L_T^0)$  consists of the elements in  $S_\phi \cap L_T^0$  fixed by  $\omega R_\mu^\vee \cup \omega R_\mu^\perp$ .

Proof: See next section ... this result is a main step in our  $L$ -group version of some of the Knapp-Zuckerman results... and [S4, Theorem 5.4.4].

In view of the theorem we may identify  $S_\phi / Z_G^W S_\phi^0$  with  $S_\phi \cap L_T^0 / Z_G^W(S_\phi^0 \cap L_T^0)$ . We obtain then, for each  $\omega \in \Omega(M, T)$ , an exact sequence:

$$1 \rightarrow S_\phi^0 \cap L_T^0 / Z_M^W \rightarrow \mathfrak{E}(T)^\vee \rightarrow \mathbf{S} \rightarrow 1,$$

where  $\mathbf{S}$  is realized as  $S_\phi \cap L_T^0 / Z_G^W(S_\phi^0 \cap L_T^0)$ , with  $\phi = \phi(\omega\mu, \lambda)$ . In particular,  $\mathbf{S}$  is an abelian 2-group.

Lemma 3.6.9

If  $\Theta_{\omega_*}^G(\mu, \lambda) \neq 0$  then  $\langle \cdot, \cdot \rangle_{\omega_*}$  and the above exact sequence for  $\phi = \phi(\omega_*\mu, \lambda)$  determine a pairing between  $\mathbf{S}$  and  $\mathbb{I}$ .

Proof: We have just to show that if  $\kappa \in \mathfrak{E}(T)^\vee$  is in the image of  $S_\phi^0 \cap L_T^0 / Z_M^W$  then  $\kappa(\omega) = 1$  if  $\Theta_{\omega_*\omega}^G(\mu, \lambda) \neq 0$ . This follows from (ii) of Theorem 3.6.8 and our description of  $\mathbb{I}$  ... note that under the Tate-Nakayama isomorphism an element of  $\Omega_{\omega_*\mu}(M, T) \cong \Omega(M(\mathbb{R}), T(\mathbb{R})) / \Omega(M(\mathbb{R}), T(\mathbb{R}))$  corresponds to the coset of an element in  $\mathbb{Z}[\omega_*R_\mu^\vee \cup \omega_*R_\mu^\perp]$  (see (2.10).

Note that if  $\kappa \in \mathfrak{E}(T)^\vee$  is in the image of  $S_\phi^0 \cap L_T^0 / Z_M^W$  for  $\phi = \phi(\mu, \lambda)$  then  $\kappa^{\omega_*}$  is in the image of  $S_\phi^0 \cap L_T^0 / Z_M^W$  for  $\phi = \phi(\omega_*\mu, \lambda)$ . Thus if  $\omega_*$  is as in the lemma we have by (3.6.7) that  $\langle \kappa, \pi \rangle_1 = \langle \kappa^{\omega_*}, \pi \rangle_{\omega_*} (\kappa(\omega_*))^{-1} = \kappa(\omega_*)$ ,  $\pi \in \mathbb{I}$ , and  $\langle \cdot, \cdot \rangle_1$  need not factor through  $\mathbf{S}$ .

In conclusion, we will fix a pairing

$$\langle \cdot, \cdot \rangle : \mathbf{S} \times \mathbb{I} \longrightarrow \{\pm 1\}$$

for once and for all. First choose  $(T, \eta)$  so that  $\{\phi\}$  factors through  $\phi(T)$ . Assume that  $\phi(\mu, \lambda)$  (data defined relative to  $(T, \eta)$ ,  $\mu \in L_B^0 \cap L_M^0$ -dominant) represents  $\{\phi\}$ .

Assume further that  $(T, \eta)$  and  $\phi(\mu, \lambda)$  satisfy (3.6.2) and (3.6.3) and that  $T$  is minimal with respect to this property. Choose  $\omega_*$  such that  $\omega_*R_\mu$  contains only noncompact roots.

Then we define  $\langle \cdot, \cdot \rangle$  to be  $\langle \cdot, \cdot \rangle_{\omega_*}$ . Recall that  $T$  is unique up to  $G(\mathbb{R})$ -conjugacy but that  $\eta$  is arbitrary. Once  $(T, \eta)$  is fixed  $\phi(\mu, \lambda)$  can be replaced only by  $\phi(\mu', \lambda')$ , where  $\mu' = \omega \bar{g} \mu$ ,  $\lambda' = \bar{g} \lambda$  (up to the allowed ambiguity) and  $\omega \bar{g}$  preserves the roots of  $L_B^0 \cap L_M^0$ ; here  $\omega \in \Omega(M, T)$  and  $\bar{g}$  is the transfer to  $L_T^0$  via  $\eta$  of  $g \in \Omega(G(\mathbb{R}), T(\mathbb{R}))$ . From the proof of Lemma 3.2 in [S1] we conclude that  $\omega \in \Omega(M(\mathbb{R}), T(\mathbb{R}))$ . Thus we may and will assume that  $\omega = 1$ . Then  $\phi' = \phi(\mu', \lambda') = \phi(\bar{g} \mu, \bar{g} \lambda)$  and  $R_{\mu'}^V = \bar{g} R_{\mu}^V$ ; for  $\omega_*^!$  such that the roots of  $\omega_*^! R_{\mu}$  are noncompact we may take  $\bar{g}^{-1} \omega_*^! \bar{g}$ . There is a canonical isomorphism  $\mathbf{x} \rightarrow \mathbf{x}'$  from the realization of  $\mathbf{S}$  in terms of  $\phi$  to that in terms of  $\phi'$ . Since  $g \in \Omega(G(\mathbb{R}), T(\mathbb{R}))$  we have  $\kappa^{\mathbb{E}}(g^{-1} \omega g) = \kappa(\omega)$ ,  $\omega \in \Omega(M, T)$ ,  $\kappa \in \mathbb{E}(T)^V$ . Hence  $\langle \mathbf{x}', \pi \rangle_{\omega_*^!} = \langle \mathbf{x}, \pi \rangle_{\omega_*}$  for all  $\pi$  and  $\mathbf{x}$ , so that if we replace  $\phi(\mu, \lambda)$  by  $\phi(\mu', \lambda')$  we obtain the same pairing provided  $\omega_*$  is replaced by  $\omega_*^!$  as above.

The pairing  $\langle \cdot, \cdot \rangle$  identifies  $\mathbb{I}$  as a subset of  $\mathbf{S}^V$ . For inversion of character identities, etc., it is convenient to work with the full dual of  $\mathbf{S}$ . Thus we define  $\bar{\mathbb{I}}$  by adding to  $\mathbb{I}$  one element  $\bar{\pi}$  for each  $\chi \in \mathbf{S}^V - \mathbb{I}$ . We define  $\langle \mathbf{x}, \bar{\pi} \rangle = \chi(\mathbf{x})$ ,  $\mathbf{x} \in \mathbf{S}$ . Then  $\langle \cdot, \cdot \rangle : \bar{\mathbb{I}} \times \mathbf{S} \rightarrow \{\pm 1\}$  identifies  $\bar{\mathbb{I}}$  as the dual of  $\mathbf{S}$ . The character of  $\bar{\pi} \in \bar{\mathbb{I}} - \mathbb{I}$  is defined to be the zero distribution. Note that in the case that  $\mathbb{I}$  consists of discrete series representations  $[\bar{\mathbb{I}} - \mathbb{I}]$  is  $[\mathbb{E}(T) - \mathcal{D}(T)]$  for any  $T$  anisotropic modulo center.

Our discussion in the next section will show that, in general, if  $\Pi$  consists of the constituents of  $[\mathfrak{A}(\bar{T})]$  principal series representations then  $[\bar{\Pi} - \Pi]$  is  $[\xi(\bar{T}) - \mathfrak{A}(\bar{T})]$  times the number of constituents of each of the principal series representations, so that  $\bar{\Pi} - \Pi$  is accounted for by the failure of  $\mathfrak{A}(-)$  to equal  $\xi(-)$ , rather than the vanishing of well-defined limits of discrete series characters.

### 3.7 Simultaneous decomposition of principal series

We will start with a remark on the results in Langlands' (unpublished) notes on Knapp-Zuckerman theory ([S4, Section 5.3] reports most of the material in the notes). We then parallel arguments of [KZ] for the decomposition of unitary principal series, using roots of  $L_T^0$  in place of roots of  $S_T$ . The result is an uniform decomposition for all principal series contributing to an L-packet. Our conclusions can, of course, be obtained directly from [KZ]. We will omit direct reference to the theory of intertwining operators where  $S_\phi$  first appears.

Given  $\Pi$  with parameter  $\{\phi\}$ , we fix  $(T, \eta)$  and representatives  $\phi = \phi(\omega\mu, \lambda)$  [data defined relative to  $(T, \eta)$ ,  $\mu$   $L_B^0 \cap L_M^0$ -dominant,  $\omega \in \Omega(L_M^0, L_T^0)$ ] such that  $\phi$  is discrete relative to  $M$  [i.e.  $T$  is maximal with respect to (3.6.2) and (3.6.3), as an argument with Cayley transforms shows; see Theorem 3.5.1]. As before,  $S_\phi$  is the centralizer of  $\phi(W)$  in  $L_G^0$ , and  $S_\phi^0$  is the identity component of  $S_\phi$ . From Langlands' notes we deduce easily that  $L_T^0 \cap S_\phi^0$  is a maximal

torus in  $S_\phi^0$ ,  $L_B^0 \cap S_\phi^0$  is a Borel subgroup of  $S_\phi^0$ , and  $S_\phi^0$  is independent of  $\phi$ . i.e. of the choice of  $\omega \in \Omega(L_M^0, L_T^0)$ .

On the other hand, consider the subgroup of  $W_T = \text{Norm}(M(\mathbb{R}), G(\mathbb{R})) / M(\mathbb{R})$  consisting of the elements fixing  $\Theta_\omega(\mu, \lambda)$ . This group is independent of  $\omega$  also ... a straightforward argument for this uses [S1, Lemma 3.2]. Following the notes still, we have an isomorphism of the group with a subgroup of  $\Omega_0(G, T)$  and thence with a subgroup of  $\Omega(L_G^0, L_T^0)$ ; this latter subgroup consists of the elements of  $\Omega(L_G^0, L_T^0)$  realized in  $S_\phi$  (see [S4, Proposition 5.3.1, Lemma 5.3.14]). If  $s \in S_\phi$  realizes an element of  $\Omega(L_G^0, L_T^0)$  then  $s$  normalizes both  $S_\phi^0$  and  $L_T^0 \cap S_\phi^0$ . Langlands' version of the Knapp R-group is  $\mathbf{R}_\phi$ , the subgroup of  $\Omega(L_G^0, L_T^0)$  of elements realized by elements of  $S_\phi$  normalizing  $L_B^0 \cap S_\phi^0$  as well.

Comparing the results of the notes with Section 40 of [HCIII] we observe that the corollary to Lemma 40.3 of [HCIII] says that the dimension of the commuting algebra of  $\Theta_\omega^G(\mu, \lambda)$  is not greater than  $[\mathbf{R}_\phi]$  (see the last part of Section 5.3. in [S4]). Thus to accomplish the decomposition of  $\Theta_\omega^G(\mu, \lambda)$  into irreducible constituents it is sufficient to furnish  $[\mathbf{R}_\phi]$  constituents.

Continuing, we recall Langlands' analogue of the set " $\mathcal{X}$ " ([KZ]) of superorthogonal roots, namely the set  $R_\phi^\vee = \{\alpha^\vee \in R(L_G^0, L_T^0) : \langle \omega\mu, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle = 0 \text{ and } \sum r\alpha^\vee = 0\}$ , where the summation is over  $r \in \mathbf{R}_\phi$ . Properties of  $R_\phi^\vee$  are proved by imitation of the techniques in [KZ]. Of particular interest to us are the following (see [S4,

Lemma 5.3.13]):

(3.7.1) If  $\alpha^V \in R_\phi^V$  then  $\sigma_{(T, \eta)} \alpha^V = \alpha^V$  and  $\phi(1 \times \sigma) X_{\alpha^V} = -X_{\alpha^V}$  (recall (3.5.2)).

(3.7.2)  $R_\phi^V$  is of type  $(A_1)^n$ .

(3.7.3)  $R_\phi$  is contained in  $\Omega(R_\phi^V)$ , the group generated by the reflections with respect to the roots in  $R_\phi^V$ , and each  $\alpha^V \in R_\phi^V$  appears in the expression of some  $r \in R_\phi$  as a product of distinct reflections in  $\Omega(R_\phi^V)$ .

Recall the discussion of (3.6). Fix  $\phi = \phi(\omega\mu, \lambda)$  discrete relative to  $M$  ( $\omega \in \Omega(L_M^0, L_T^0)$ ). Our aim then is to construct  $(\bar{T}, \bar{\eta})$  such that  $\bar{M} = M_{\bar{T}}$  contains  $M = M_T$ ,  $\bar{\phi}$  with data  $(\omega\mu, \lambda)$  relative to  $(\bar{T}, \bar{\eta})$  represents  $\{\phi\}$ , and  $R_{\omega\mu}^V = \{\alpha^V \in R(L_M^0, L_T^0) : \langle \omega\mu, \alpha^V \rangle = 0\}$  coincides with  $R_\phi^V$ . The construction will not depend on  $\omega$  (recall that  $\langle \mu, \alpha^V \rangle = 0$  if and only if  $\langle \omega\mu, \alpha^V \rangle = 0$ ). Suppose that it has been done. Fix  $\bar{\omega}_*$  in  $\Omega(L_{\bar{M}}^0, L_{\bar{T}}^0)$  such that  $\bar{\omega}_*\mu$  is  $L_B^0 \cap L_{\bar{M}}^0$ -dominant, and set  $\bar{\theta}_{\bar{\omega}, \bar{\omega}}(\mu, \lambda) = \theta_\tau(\bar{\omega}_*\mu, \lambda)$ , where  $\tau = \bar{\omega}_*^{-1} \bar{\omega}$ , for  $\bar{\omega} \in \Omega(L_{\bar{M}}^0, L_{\bar{T}}^0)$ . Then application of the Schmid identities yields a decomposition of  $\theta_\omega^G(\mu, \lambda)$  as a sum of  $\bar{\theta}_{\bar{\omega}, \bar{\omega}}^G(\mu, \lambda)$ , with  $\bar{\omega}$  in  $\Omega(R_{\omega\mu}^V) = \Omega(R_\phi^V)$ . For each such  $\bar{\omega}$ ,  $\bar{\theta}_{\bar{\omega}, \bar{\omega}}^G(\mu, \lambda)$  is seen to appear in the sum. We will show also that:

(3.7.4)  $[\Omega(R_\phi^V) / \Omega(R_\phi^V) \cap \Omega(\bar{M}(\mathbb{R}), \bar{T}(\mathbb{R}))] = [R_\phi]$  and

(3.7.5) every element of  $R_\phi^V$  is noncompact and does not lie in  $(I - \sigma_{\bar{T}})X_*(\bar{T})$

(here we have transferred data to  $\bar{T}$  via  $\bar{\eta}$ ).

We may then conclude that  $\theta_{\omega}^G(\mu, \lambda) = \sum \bar{\theta}_{\omega, \bar{\omega}}^G(\mu, \lambda)$ , the summation being over  $\omega \in \Omega(R_{\phi}^V) / \Omega(R_{\phi}^V) \cap \Omega(\bar{M}(\mathbb{R}), \bar{T}(\mathbb{R}))$ .

Each term in the sum is an irreducible character and no two are equivalent.

The construction of  $\bar{T}$ ,  $\bar{\eta}$  and  $\bar{\phi}$  is an exercise with inverse Cayley transforms and their analogues in  $L_G^0$ . To obtain  $\bar{T}$  from  $T$  we perform a sequence of such transforms, one for each positive  $\alpha^V \in R_{\phi}^V$ ;  $\bar{\eta}$  is defined in the obvious way. Then  $\sigma(\bar{T}, \bar{\eta}) = \zeta \sigma(T, \eta)$ , where  $\zeta$  is the product of the reflections with respect to the positive roots in  $R_{\phi}^V$ . The homomorphism  $\bar{\phi}$  is of the form  $\text{ads} \circ \phi$ , where  $s^2$  realizes  $\zeta$  ...  $s$  is a product of transforms in  $L_G^0$ , one for each positive root in  $R_{\phi}^V$ . See [S4, Section 5.4, esp. Propositions 5.4.2, 5.4.3] for details (and precise statements).

The verification of (3.7.4), in a stronger form, brings us back to the proof of Theorem 3.6.8, at least for the parameter  $\bar{\phi}$ . From the definition of  $\mathbf{R}_{\phi} = \mathbf{R}$  we have Langlands' exact sequence:

$$1 \rightarrow S_{\phi} \cap L_T^0 / Z_G^W(S_{\phi}^0 \cap L_T^0) \rightarrow S_{\phi} / Z_G^W S_{\phi}^0 \rightarrow \mathbf{R}_{\phi} \rightarrow 1,$$

(note that  $S_{\phi} \cap L_T^0$  acts on  $S_{\phi}^0$  as  $S_{\phi}^0 \cap L_T^0$ ) or, since  $Z_G^W(S_{\phi}^0 \cap L_T^0) = Z_M^W$  (see [S4, Lemma 5.4.1]),

$$1 \rightarrow \xi(T)^V \rightarrow \mathbf{S} \rightarrow \mathbf{R} \rightarrow 1.$$

In the proof of [S4, Theorem 5.4.4] we use this sequence to find representatives for  $S_{\bar{\phi}} / (S_{\bar{\phi}} \cap L_T^0) S_{\bar{\phi}}^0$  in  $L_T^0$ . Then (i) of Theorem 3.6.8 follows. Further arguments along the same

lines yield (ii) of Theorem 3.6.8 and an exact sequence:

$$1 \rightarrow \cap \text{Ker } \alpha^{\vee} \rightarrow S_{\bar{\phi}} \cap L_{\bar{T}}^0 \rightarrow \mathbf{R}_{\bar{\phi}} \rightarrow 1,$$

where the intersection is over  $\alpha^{\vee} \in R_{\bar{\phi}}^{\vee}$  ([S4, Corollary 5.4.8]), and hence a homomorphism  $\xi(\bar{T})^{\vee} \rightarrow \mathbf{R}_{\bar{\phi}} \rightarrow 1$  with kernel equal to the annihilator of  $R_{\bar{\phi}}^{\vee}$  in  $\xi(\bar{T})^{\vee}$ . The dual of this homomorphism is an inclusion of  $\mathbf{R}_{\bar{\phi}}^{\vee}$  in  $\xi(\bar{T})$ . Then (3.7.3) shows that the image of  $\mathbf{R}_{\bar{\phi}}^{\vee}$  is identified by a Tate-Nakayama isomorphism with  $\Omega(R_{\bar{\phi}}^{\vee}) / \Omega(R_{\bar{\phi}}^{\vee}) \cap \Omega(\bar{M}(\mathbb{R}), \bar{T}(\mathbb{R}))$  [S4, Proposition 5.4.12]; this is our stronger version of (3.7.4). Also (3.7.5) now follows.

The "essential uniqueness" of the data  $\bar{T}$  and  $\bar{\phi}$  can be obtained directly from [KZ] or by parallel arguments with parameters in the L-group. The assertions of (3.6) follow without difficulty.

## 4. ENDOSCOPIC GROUPS

### 4.1 Definitions

We will follow [S4, Section 2]; since that discussion is more or less self-contained we will omit or just sketch much of the material. There are differences in the presentation of [L5], but the end results are essentially the same, as we shall indicate below.

A set of endoscopic data is a pair  $(s, L_{H_S}^0)$ ; as usual,  $L_{H_S}^0$  comes with ancillary data. To describe these, we recall that  $s$  is a coset of  $Z_G^W$  in  $L_G^0$  consisting of semisimple elements (in [L5]  $s$  is a single semisimple element in  $L_G^0$ ). Then  $L_{H_S}^0$  is the connected component of the identity in the centralizer of  $a$  in  $L_G^0$ , for any  $a \in s$ ; we write  $L_{H_S}^0 = \text{Cent}^0(s, L_G^0)$  (an abuse of notation). Next,  $L_{B_S}^0$ ,  $L_{T_S}^0$  and  $\{Y_{\alpha V}\}$  are as usual. We require that  $\sigma_{H_S} = \rho_S(1 \times \sigma)$  be an automorphism of all data so far; in particular, we require that  $\sigma_{H_S}$  fix each element of  $s$ . There is the further restriction on  $\sigma_{H_S}$  that it be "realized in" in  $L_G$ , and hence in  $\text{Cent}(s, L_G)$ ; i.e. that  $\sigma_{H_S}$  be the restriction to  $L_{H_S}^0$  of some  $\text{ad}(n \times 1 \times \sigma)$ , with  $n \in L_G^0$ . Finally,  $L_{H_S} = L_{H_S}^0 \rtimes W$  with the action of  $W$  given by  $\rho_S$ . This does not imply that that  $L_{H_S}^0$  is a subgroup of  $L_G$ .

An admissible embedding of  $L_{H_S}$  in  $L_G$  will be a homomorphism  $\xi$  of  $L_{H_S}$  into  $L_G$  which is the identity on  $L_{H_S}^0 \times 1$  and is of the form  $\xi(1 \times w) = \xi_0(w) \times w$ ; with  $\xi_0(w)$  in  $L_G^0$ , for  $w \in W$ . Clearly,  $z \rightarrow \xi_0(z \times 1)$  then defines a homomorphism of  $\mathbb{C}^\times$  into  $Z_H$ , the center of  $L_{H_S}^0$ , and  $\xi_0(1 \times \sigma) \times 1 \times \sigma$  acts on  $L_{H_S}^0$  as  $\sigma_{H_S}$ . In (4.3) we will discuss the existence and construction of admissible embeddings. For now it is sufficient to note that if  $\xi, \xi'$  are two admissible embeddings of  $L_{H_S}$  in  $L_G$  then  $\xi' = a\xi$ , where  $a$  is a 1-cocycle of  $W$  in  $Z_H$ . This implies that the image of  $L_{H_S}$  is independent of  $\xi$  and so may be written simply as  $\text{Image}(L_{H_S})$ .

We will call  $(s, L_{H_S})$  and  $(s', L_{H_{S'}})$  equivalent if the ancillary data for  $L_{H_S}$  is conjugate to that for  $L_{H_{S'}}$  by an element of  $L_G^0$ . This does not imply that  $s$  is conjugate to  $s'$  by  $g$ . We may show, however, that  $L_{H_S}$  embeds admissibly in  $L_G$  if and only if  $L_{H_{S'}}$  does, and that then  $\text{Image}(L_{H_S})$  is conjugate to  $\text{Image}(L_{H_{S'}})$  by  $g$ . Conversely, if the images are conjugate under  $L_G^0$  then  $(s, L_{H_S})$  is equivalent to  $(s', L_{H_{S'}})$ .

Our notion of equivalence is intended for groups over  $F = \mathbb{R}$  only. In [L5] Langlands gives a definition for all fields  $F$ , local or global and of characteristic zero. In the case  $F = \mathbb{R}$  there is a condition relating, in the notation above,  $s'$  and the conjugate of  $s$  by  $g$  (see

Chapter II, Section 1 of [L5]). At least in the case of "equal rank" groups (i.e.  ${}^L H_S$  contains an element acting on the roots of  ${}^L T_S^0$  as  $(-1)$ , or the associated group  $H_S$  contains maximal tori anisotropic modulo center) we may show that the condition is automatically satisfied. To verify this we need additional material (especially from [L2]); thus we will omit the proof and be content with our own definition of equivalence.

The set of equivalence classes of pairs  $(s, {}^L H_S)$  will be denoted  $\mathcal{G}(G)$ , or  $\mathcal{G}({}^L G)$  since it depends only on  ${}^L G$ , i.e. on the class of  $G$  modulo inner forms. We remark that the set  $\mathcal{G}({}^L G)$  is finite ([S4, Lemma 2.3.3]); this, like many of the properties of  $\mathcal{G}({}^L G)$ , is seen quite easily using the first construction of the next section.

A quasi-split group  $H$  over  $R$  is an endoscopic group for  $G$  (or the class of  $G$  modulo inner forms) if its  $L$ -group is among the  ${}^L H_S$  above.

#### 4.2 Two constructions

Let  $T$  be a maximal torus over  $R$  in  $G$ , and  $\kappa \in \hat{R}(T)$ . Then to  $(T, \kappa)$  there is associated an element of  $\mathcal{G}(G)$  and hence an endoscopic group  $H = H(T, \kappa)$ . This is the construction of [L1]; we will recall it below. If  $G$  is quasi-split over  $R$  then every endoscopic group is an  $H(T, \kappa)$  (see [S4, Proposition 2.3.2]). In general, an endoscopic group which is not an  $H(T, \kappa)$  "shares" no maximal torus over  $R$  with  $G$  (see (4.4)).

Continuing with  $(T, \kappa)$ , we fix a p-d.  $\eta$  of  $T$  (or any map as in the remark at the end of (3.2)). Via  $\eta$  we transfer  $\kappa$  to  $L_T^0$ ; we obtain a coset in  $L_T^0 / Z_G^W$  (recall (3.6), although we are now using  $\mathcal{K}(T)$  in place of  $\mathcal{E}(T)^\vee$ ) and, moreover, each element of this coset is fixed by  $\sigma_{(T, \eta)}$ . Let  $s$  be the coset and  $L_{H_S}^0 = \text{Cent}^0(s, L_G^0)$ ; set  $L_{B_S}^0 = L_B^0 \cap L_{H_S}^0$  and  $L_{T_S}^0 = L_T^0$ ; fix root vectors  $Y_{\alpha^\vee}$ . Then  $\sigma_{H_S}$  is the automorphism of  $L_{H_S}^0$ , etc., which acts on  $L_{T_S}^0$  as an element of  $\Omega(L_{H_S}^0, L_{T_S}^0)$   $\sigma_{(T, \eta)}$ . It is easily checked that  $\sigma_{H_S}$  is realized in  $\text{Cent}(s, L_G)$ . Note [S4, Lemma 2.1.7]. From  $\sigma_{H_S}$  we obtain  $L_{H_S}$  in the usual way. The class of  $(s, L_{H_S})$  is independent of the choice for  $\eta$ ;  $H(T, \kappa)$  is the corresponding endoscopic group.

There is another basic construction of endoscopic groups. Recall the group  $\mathbf{S}$  attached to a parameter in  $\phi_0(G)$ . If  $\phi$  represents the parameter then we may realize  $\mathbf{S}$  as  $S_\phi / Z_G^W S_\phi^0$ . Let  $\mathbf{x} \in \mathbf{S}$ . Choose a coset  $s$  of  $Z_G^W$  in  $S_\phi$  consisting of semisimple elements and mapping to  $\mathbf{x}$  under the natural projection. Let  $L_{H_S}^0 = \text{Cent}^0(s, L_G^0)$  and fix  $L_{B_S}^0$ ,  $L_{T_S}^0$  and  $\{Y_{\alpha^\vee}\}$  as usual. Note that  $\phi(1 \times \sigma)$  fixes each element of  $s$ , so that conjugation by  $\phi(1 \times \sigma)$  preserves  $L_{H_S}^0$ . Then  $\sigma_{H_S}$  is the automorphism of  $L_{H_S}^0$ , etc., which acts on  $L_{T_S}^0$  as an element of  $\Omega(L_{H_S}^0, L_{T_S}^0)$  ad  $\phi(1 \times \sigma)$ . If  $\xi$  is an admissible embedding of  $L_{H_S}$  in  $L_G$  then it follows that  $\phi(W)$  is contained in  $\xi(L_{H_S})$ . Thus  $\phi$  is of the

form  $\xi \circ \phi_s$ , where  $\{\phi_s\} \in \Phi(H_S)$ , i.e.  $\{\phi\}$  "factors through"  $\{\phi_s\}$ . In general, the class of  $(s, L_{H_S})$  is not determined by  $\mathbf{x}$ ; however, from the point of view of character liftings it is  $\mathbf{x}$  rather than  $s$  which plays the central role. We will discuss this in (5.5).

Note that the construction above can be applied to a general (unbounded) parameter. The factorings associated to a given  $\mathbf{x} \in \mathbf{S}$  may be quite different (as in the example of the Appendix where we have  $\mathbf{S} = 1$ ).

#### 4.3 Admissible embeddings

Since we are interested in  $(s, L_{H_S})$  only up to equivalence we may, and will, assume that  $L_{T_S}^0 = L_T^0$ , which implies that  $s \in L_T^0 / Z_G^W$ , and that  $L_{H_S}$  is in "standard position", i.e. that  $\sigma_{H_S}$  acts on  $L_T^0$  as  $\sigma_T$ , for some standard torus  $T$  in  $G^*$ ; this implies that  $\sigma_{H_S}$  is realized in  $L_M = L(M_T)$ . Note that the condition, from the definition of endoscopic data, that  $\sigma_{H_S}$  be realized in  $L_G$  implies, by a theorem of Steinberg, that  $\sigma_{H_S}$  acts on  $L_T^0$  as  $\text{ad } g^{-1} \circ \sigma_T \circ \text{ad } g$ , for some  $T = g^{-1}T^*g$  over  $\mathbb{R}$  in  $G^*$ . Multiplying  $g$  on the left by a suitable element of  $\text{Norm}(T^*, G^*)$ , i.e. conjugating  $L_{H_S}^0$  by a suitable element of  $\text{Norm}(L_T^0, L_G^0)$ , we may assume that  $T$  is standard and  $g \in M_T$ , i.e.  $L_{H_S}$  is in standard position (see [S3, Section 2.2]). In particular, in the case of  $G = \text{SU}(2,1)$  we will allow only one realization of the L-group of the endoscopic group  $U(1,1)$ , that of [L5] (see

[S3, Section 3.2]).

Suppose then that  $(s, {}^L H_S)$  is in standard position; to conserve notation we assume that  $\sigma_{H_S}$  acts on  ${}^L T^0$  as  $\sigma_{\bar{T}}$  and write  $\bar{M}$  for  $M_{\bar{T}}$ . Let  $\xi$  be an admissible embedding of  ${}^L H_S$  in  ${}^L G$ . We write  $\xi = \xi(\mu^*, \lambda^*)$  if  $\lambda^V(\xi_0(z \times 1)) = z^{\langle \mu^*, \lambda^V \rangle - \langle \sigma_H \mu^*, \lambda^V \rangle}$ ,  $\lambda^V \in X^*({}^L T^0)$ , and  $\lambda^V(\xi_0(1 \times \sigma)) = \exp(2\pi i \langle \lambda^*, \lambda^V \rangle)$ ,  $\lambda^V \in X^*({}^L M^0)$ . Recall that  $\xi(w) = \xi_0(w) \times w \in {}^L G^0 \times w$ ,  $w \in W$ . Then  $\mu^*$  is a well-defined element of  $X_*({}^L T^0) \otimes \mathbb{C}$  but there is ambiguity (which we will ignore) in  $\lambda^*$  as element of  $X_*({}^L T^0) \otimes \mathbb{C}$ . Since  $\xi_0(\mathbb{C}^\times \times 1)$  is contained in the center of  ${}^L H_S^0$  we have that  $\langle \mu^*, \alpha^V \rangle = 0$  for all  $\alpha^V \in R({}^L H^0, {}^L T^0)$ ; this fact will enable us to shift infinitesimal characters by  $\mu^*$  when we lift distributions from  $H_S$  to  $G$  (see (5.3)).

Suppose that  $T$  is anisotropic modulo center in  $G$  and that  $\sigma_{(T, \eta)} \in \Omega({}^L H_S^0, {}^L T^0) \sigma_{H_S}$ . Then, as will be recalled in the next section, there is  $h \in H_S$  and maximal torus  $T'$  over  $\mathbb{R}$  in  $H_S$  such that  $X_*(T) \xrightarrow{\eta} X^*({}^L T^0) \xrightarrow{\text{adh}_h} X_*(T')$  commutes with the Galois action. Then we have an isomorphism over  $\mathbb{R}$  between  $T'$  and  $T$  which we may use to identify  ${}^L T'$  with  ${}^L T$ . Since  $T'$  is anisotropic modulo center in  $H$  ( $\dots \sigma_{T'}$  acts as  $-1$  on the roots of  ${}^L T^0$ ) we may embed  ${}^L T'$  in  ${}^L H_S$  as in (3.3). The composition of this embedding with  $\xi: {}^L H_S \rightarrow {}^L G$  yields an embedding of  ${}^L T' = {}^L T$  in  ${}^L G$ . But we know all the embeddings of  ${}^L T$  in  ${}^L G$  ((3.3) and [S3, Section 1.3]). Thus if  $\lambda^* = 1/2 \sum \alpha$ , where  $\alpha = (\alpha^V)^V$  and

the summation is over roots  $\alpha^v$  of  $L_T^0$  in  $L_B^0$  but outside  $L_{H_S}^0$  then we may conclude that:

$$1/2(\mu^* - \sigma_{(T,\eta)}\mu^*) + \nu_T^* + (\lambda^* + \sigma_{(T,\eta)}\lambda^*) \in X_*(L_T^0).$$

More generally, if  $(T,\eta)$  is any pair satisfying

$$\sigma_{(T,\eta)} \in \Omega(L_{M_T}^0 \cap L_{H_S}^0, L_T^0) \sigma_{H_S}$$

and we set  $\nu_T^* = 1/2 \sum \alpha$ , where the summation is over roots  $\alpha^v$  in  $L_B^0 \cap L_{M_T}^0$  but outside  $L_{H_S}^0$ , then the same argument in  $L_{M_T}$  will give the following:

Theorem 4.3.1

$$1/2(\mu^* - \sigma_{(T,\eta)}\mu^*) + \nu_T^* + (\lambda^* + \sigma_{(T,\eta)}\lambda^*) \in X_*(L_T^0).$$

Proof: This is Theorem 3.4.1 of [S3]. For details of "reduction to  $L_{M_T}$ " see [S4, Sections (3.2) - (3.4)].

This result implies that  $(\mu^* + \nu_T^*, \lambda^*)$  is datum for a quasi-character on  $T(\mathbb{R})$ . Thus, roughly speaking, an embedding of  $L_{H_S}$  in  $L_G$  provides a way of turning "one-half the sum of the positive imaginary roots of  $T$  not from  $H$  (recall (2.10))" into a well-defined character. The precise transformation and compatibility properties (as  $T$  varies) of these quasi-characters will be used in the matching of orbital integrals.

Two different embeddings may yield the same set of quasi-characters, but then the embeddings yield the same map on parameters, i.e. are " $\phi$ -equivalent" ; conversely, any two  $\phi$ -equivalent embeddings yield the same set of quasi-characters [S3, Theorem 7.0.2]. Finally, the set of quasi-characters is unique up to "uniform multiplication by a quasi-character on

$H_S(\mathbb{R})$ ." This is discussed in detail in Sections 8, 9 of [S3].

Since we are working with bounded parameters, rather than essentially bounded ones or ones with specified central characters, we will also insist that the embedding  $\xi$  be of unitary type, that is, that  $\xi_0(W)$  be bounded; we may always replace  $\xi(\mu^*, \lambda^*)$  by  $\xi(1/2(\mu^* - \sigma_{H_S} \mu^*), \lambda^*)$  to achieve this.

Regarding the existence of admissible embeddings, we recall Langlands' result [L2] that existence is guaranteed if  $Z_G^W$  is connected. In [S3, Theorem 8.0.1] we show that, in general, existence is equivalent to the existence of a set, as above, of quasi-characters having the properties we want for transfer of orbital integrals. Using a simplified version of Theorem 4.3.1 above we may prove existence in some cases and produce a typical example where there is no embedding (see [S3, Section 9.2]).

#### 4.4 Correspondence of points

We start with notions from [S4, Section 2.4]. We write  $L_H$ ,  $\sigma_H$  and  $H$  in place of  $L_{H_S}$ , etc... We set  $\mathcal{J}_H(G) := \{(T, \eta) : \sigma_{(T, \eta)} \in \Omega(L_H^0, L_T^0) \sigma_H\}$ .

For each  $(T, \eta) \in \mathcal{J}_H(G)$  there is a maximal torus  $T'$  over  $\mathbb{R}$  in  $H$  and  $h \in H$  such that  $X_*(T') \xrightarrow{\text{adh}} X^*(L_T^0) \xrightarrow{\eta} X_*(T)$  commutes with Galois action and so lifts to  $i(h, \eta) : T' \rightarrow T$  defined over  $\mathbb{R}$ . Conversely, if such an  $i(h, \eta)$  exists then  $(T, \eta) \in \mathcal{J}_H(G)$ .

Secondly, for each  $(T, \eta) \in \mathcal{J}_H(G)$  we have the transfer of the element  $s$  of  $L_T^0 / Z_G^W$  via  $\eta$  to an element of  $\hat{K}(T)$ ,

i.e.  $\mathcal{J}_H(G)$  extracts from  $s$  a family of " $\kappa$ 's".

An element  $\gamma'$  of  $H(\mathbb{R})$  "originates from  $\gamma \in T \cap G(\mathbb{R})_{\text{reg}}$  via  $(T, \eta) \in \mathcal{J}_H(G)$ " if  $\gamma'$  is the preimage of  $\gamma$  under some map  $i(h, \eta)$  as above. Exactly one stable conjugacy class of regular semisimple elements in  $H(\mathbb{R})$  originates from given  $\gamma$  via given  $(T, \eta)$ ; if the members of this class also originate from  $\bar{\gamma}$  via  $(T, \eta)$  then  $\bar{\gamma} = \gamma^\omega$ , for  $\omega \in \Omega_0(G, T)$  "from  $H$ " (recall (2.10)). A torus  $T'$  in  $H$  originates in  $G$  if it is the preimage of some  $T$  under some  $i(h, \eta)$ ; then a Haar measure  $dt$  on  $T(\mathbb{R})$  determines a Haar measure  $dt'$  on  $T'(\mathbb{R})$ .

To formulate the matching of orbital integrals we consider first a correspondence:

$$\gamma' \rightarrow \begin{cases} \Delta_{(T, \eta)}(\gamma) \phi_T^K(\gamma, f; dt, dg) & \text{if } \gamma' \text{ originates} \\ \text{from } \gamma \in T(\mathbb{R})_{\text{reg}} \text{ via } (T, \eta) \in \mathcal{J}_H(G), & \\ 0 & \text{if } \gamma' \text{ lies in no torus originating} \\ \text{in } G. & \end{cases}$$

Here  $\kappa$  is the element of  $\mathbb{K}(T)$  attached to  $(T, \eta)$ ,  $f$  is some Schwartz function on  $G(\mathbb{R})$ , and  $dt, dg$  are given Haar measures on  $T(\mathbb{R}), G(\mathbb{R})$  respectively;  $\Delta_{(T, \eta)}$  is to be defined.

In view of (ii) at the end of (2.11) we place a condition on  $\Delta_{(T, \eta)}$  to ensure that this correspondence gives a well-defined function of  $\gamma'$  (for fixed  $(T, \eta)$ ). Our aim is then to construct  $\Delta_{(T, \eta)}$  so that this function is  $\phi_{T'}^{\text{st}}(\gamma', f_H; dt', dh)$  for some Schwartz function  $f_H$  on  $H(\mathbb{R})$  and given  $dh$ .

Before proceeding, we observe that in this formulation of

the matching our restriction on  $\eta$  is unnecessary. Recall from (3.1) that  $T^{**}$  is a standard maximal torus in  $G^*$ ,  $\psi_x = \text{ad } x^{-1} \circ \psi$  maps  $T$  to  $T^{**}$  over  $\mathbb{R}$ ,  $m^{-1}T^{**}m = T^*$ ,  $m \in M_{T^{**}}$ , and  $\eta = (\psi_x, \text{ad } m^{-1})$ , or more informally  $\eta = \text{ad } m^{-1} \circ \psi_x$ . We may as well allow  $T^{**}$  to be any torus over  $\mathbb{R}$  (for which some  $\psi_x$  exists) and replace  $m$  by any suitable element of  $G^*$ . Then all the assertions of this section remain true. Note that now  $\gamma_H \in H(\mathbb{R})$  corresponds to  $\gamma_G \in T(\mathbb{R})_{\text{reg}}$  in a diagram  $D$  from [L5] if and only if the stable conjugacy class of  $\gamma_H$  originates from  $\gamma_G$  via the  $(T, \eta)$  associated to the diagram (see [L5, III(1)] ...our  $T$  is " $T_G$ "). To conform with [L5] we will use the language of diagrams for the matching theorem (keeping in mind that attached to a diagram  $D$  is a pair  $(T, \eta)$  as above). We have first to construct the "transfer factors"  $\Delta_D$  or  $\Delta_{(T, \eta)}$  (the two are not exactly the same since in a diagram the element  $h$  in  $i(h, \eta)$  is fixed so that the  $\Delta_D$  factor can be written as a function on elements of  $H$  (as in [L5]);  $\Delta_{(T, \eta)}$ , as formulated above, is not well-defined on these elements).

#### 4.5 Transfer factors

Recall that we want  $\{\Delta_D\}$  so that  $\phi_{T'}^{\text{st}}(\gamma_H, f_H) = \Delta_D(\gamma_G) \phi_T^K(\gamma_G, f)$  if  $\gamma_H$  corresponds to  $\gamma_G$  in  $D$ , etc.. Dual to the correspondence  $(f, f_H)$  we require a dual lifting of tempered characters functorial with respect to an embedding  $\xi = \xi(\mu^*, \lambda^*)$  of  $L_H$  in  $L_G$ .

It is not difficult to describe the local form required of  $\Delta_D$ . First fix  $D$  and attached  $i(h, \eta)$  so that  $T$  is anisotropic modulo the center of  $G$ . Using  $\eta$  and  $R(L_B^0, L_T^0)$  we define a positive system for  $R(G, T)$ ; using  $h$  and  $R(L_B^0 \cap L_H^0, L_T^0)$  we get a positive system for  $R(H, T')$ , or for the roots of  $T$  which are from  $H$  in the sense of (2.10). Up to sign there is a unique local interpretation for the product, over all positive roots  $\alpha$  of  $T$  which are not from  $H$ , of  $\alpha^{1/2} - \alpha^{-1/2}$ . Suppose that  $\phi = \phi(\mu, \lambda)$  represents a discrete series parameter for  $G$ . Then  $\{\phi\}$  is the lift via  $\xi$  of the discrete series parameter  $\{\phi' = \phi(\mu - \mu^*, \lambda - \lambda^*)\}$  for  $H$ . To check the local behavior of the lifting dual to a matching of orbital integrals we use the Weyl integration formula as in [S4, 4.2, 4.4, 4.7]. Lifting the sum of the characters in the L-packet for  $\{\phi'\}$  must incorporate a shift by  $\mu^*$  in infinitesimal character. We conclude that locally  $\Delta_D$  must be of the form:

$$\text{const. } e^{\mu^*} \prod \alpha^{1/2} - \alpha^{-1/2},$$

where  $e^{\mu^*}$  is a local lifting of the linear form  $\mu^*$  to  $T$ , and the product is over positive roots not from  $H$ .

We can argue similarly within a Levi group. Thus suppose that  $D$  is such that the attached  $(T, \eta)$  is as in (3.1), i.e.  $\eta$  is a p-d. in our earlier sense. Suppose also that  $\sigma_{(T, \eta)} \in \Omega(L_H^0 \cap L_M^0, L_T^0) \sigma_H$ . Then we conclude that locally  $\Delta_D$  must be of the form:

$$(4.5.1) \quad \text{const. } e^{\mu^*} \prod_1 (\alpha^{1/2} - \alpha^{-1/2}) \prod_2 |\alpha^{1/2} - \alpha^{-1/2}|,$$

where  $\prod_1$  is over all positive imaginary roots  $\alpha$  not from  $H$  and  $\prod_2$  is over all other positive roots not from  $H$ . (See [S4, Section 3.4] for reduction to Levi groups, and [S2] for a global version of this approach).

Our first problem then is to find a global form of (4.5.1). The second product is welldefined on  $T(\mathbb{R})$  and we shall ignore it. From the first we extract  $\prod (1 - \alpha^{-1})$ , taken again over all positive imaginary  $\alpha$  not from  $H$ , which also is well-defined on  $T(\mathbb{R})$ . Thus if  $\mu^*$  denotes one-half the sum of the positive imaginary roots not from  $H$  then we have to find a global version of  $(\text{const.}) e^{\mu^* + \nu^*}$ . Here, of course, the constant is local. Recall Theorem 4.3.1. which says that  $(\mu^* + \nu^*, \lambda^*)$  is datum for a character on  $T(\mathbb{R})$ . Since  $\lambda^*$  contributes only to a signum character, we may take for the global version  $(\text{const.}) \chi(\mu^* + \nu^*, \lambda^*)$ , where now the constant is global, i.e. depends only on  $D$ . Our formulation of the matching theorem does not allow complete control over the constant. We will require it to be  $\pm(-1)^{q(G,H)}$ , with  $q(G,H)$  as in [S4, 3.7] ... the reason for the  $\pm$  will be clear when we come to compatibility problems. In conclusion, our candidate for  $\Delta_D(\gamma_G)$  is

$$(4.5.2) \quad (-1)^{q(G,H)} \varepsilon(D) \Lambda(\gamma_G, D, \xi) \Delta(\gamma_G, D),$$

where  $\varepsilon(D) = \pm 1$ ,  $\Lambda(\gamma_G, D, \xi) = \chi(\mu^* + \nu^*, \lambda^*)$  and  $\Delta(\gamma_G, D) = \prod_1 (1 - \alpha^{-1}) \prod_2 |\alpha^{1/2} - \alpha^{-1/2}|$ , where  $\prod_1$  is over all positive imaginary roots not from  $H$  and  $\prod_2$  is over all other positive roots not from  $H$ .

There is an immediate test: we must have  $\Delta_D(\gamma_G)\phi_T^K(\gamma_G, f)$  invariant under the elements of  $\Omega_0(G, T)$  which are from  $H$  (this is the condition to make our formulation of the matching with  $\Delta_{(T, \eta)}$  well-defined). Recall (ii) in (2.11) and Lemma 2.10.6. It follows that we have to verify that  $\chi(\mu^* + \nu^*, \lambda^*)$  transforms under an element  $\omega$  in  $\Omega_0(G, T)$  from  $H$  exactly as  $\nu^*$  does, i.e. that  $\omega\mu^* = \mu^*$  and  $\omega\lambda^*$  equals  $\lambda^*$  up to the allowed ambiguity. The former is immediate because  $\mu^*$  is perpendicular to the roots from  $H$ ; the proof of the latter occupies Section 5 of [S3].

Suppose that  $\gamma_H \in T'$  corresponds to  $\gamma_G \in T(\mathbb{R})_{\text{reg}}$  in  $D$  and to  $\bar{\gamma}_G \in \bar{T}(\mathbb{R})_{\text{reg}}$  in  $\bar{D}$ , where  $D$  and  $\bar{D}$  are as above (we will call such diagrams "standard"). Then the usual arguments with standard data (see [S3, 2.3] and [S4, 3.4]) show that we may as well assume that  $\bar{T} = T$  and  $\bar{\eta} = \eta \circ \omega$ , where  $\omega \in \Omega(M, T)$ . Note that  $\phi_T^K(\gamma_G, f) = \kappa(\omega)\phi_{\bar{T}}^K(\bar{\gamma}_G, f)$ . This implies that we must have  $\Delta_D(\gamma_G) = \kappa(\omega)\Delta_{\bar{D}}(\bar{\gamma}_G)$  (... and, since  $\Lambda(\bar{\gamma}_G, \bar{D}, \xi) = \Lambda(\gamma_G, D, \xi)$  and  $'\Delta(\bar{\gamma}_G, \bar{D}) = '\Delta(\gamma_G, D)$ , that  $\varepsilon(D) = \kappa(\omega)\varepsilon(\bar{D})$ ). The conclusion is then that  $\gamma_H \rightarrow \Delta_D(\gamma_G)$ , and the as yet unspecified  $\varepsilon(D)$ , will depend on the choice of (standard)  $D$ . We accommodate this by fixing a "framework of Cartan subgroups" ([S3, 2.3], [S4, 3.2], see also [S2, Section 6]) and so a certain collection of standard diagrams, say  $\mathbf{D}$ . Then  $\{\Delta_D = \varepsilon(D)\Lambda(\gamma_G, D, \xi)' \Delta(\gamma_G, D), D \in \mathbf{D}\}$  will uniquely prescribe the stable orbital integrals of the proposed

Schwartz function  $f_H$  on  $H(\mathbb{R})$ . At the heart of the matching theorem will be the fact that there are exactly two choices for  $\{\varepsilon(D): D \in \mathbf{D}\}$  for which  $f_H$  is well-defined. Write  $\Delta(\cdot, D, \xi)$  for  $\Delta_D$ ,  $D \in \mathbf{D}$ , if  $\varepsilon_D$  has been so chosen.

Returning to a general diagram  $D$ , standard or not, we observe that there is a diagram  $\bar{D} \in \mathbf{D}$  such that  $i(\bar{h}, \bar{\eta}) \circ i(h, \eta)^{-1}: T \rightarrow \bar{T}$  is defined over  $\mathbb{R}$ . This map is of the form  $\text{ad } w^{-1}|_T$ ,  $w \in \hat{G}(T)$ . To extend the matching to  $\mathbf{D}$  set:

$$(4.5.3) \quad \Delta(\gamma, D, \xi) = \kappa(w) \Delta(\gamma^w, \bar{D}, \xi), \quad \gamma \in T(\mathbb{R}).$$

Note that  $\Delta(\cdot, D, \xi)$  remains of the form  $\varepsilon(D) \Lambda(\cdot, D, \xi) \cdot \Delta(\cdot, D)$  with  $\varepsilon(D) = \pm 1$ ,  $\Lambda(\cdot, D, \xi)$  a character on  $T(\mathbb{R})$  and  $\Delta(\cdot, D)$  as defined earlier ... to obtain  $\varepsilon(D)$ ,  $\Lambda(\cdot, D, \xi)$  in terms of  $\varepsilon(\bar{D})$ ,  $\Lambda(\cdot, \bar{D}, \xi)$  when  $D$  is not standard see [S4, 3.1].

Consider again  $\{\varepsilon_D: D \in \mathbf{D}\}$ . There is a second test to apply to the candidate  $\Delta_D = \varepsilon(D) \Lambda(\cdot, D, \xi) \cdot \Delta(\cdot, D)$ . We mention it because it is central to the transition from local to global form for  $\Delta_D$ . Suppose that  $D \in \mathbf{D}$  is a diagram for  $T'$ , that  $\bar{D} \in \mathbf{D}$  is a diagram for  $\bar{T}'$  and that  $T'$  is adjacent to  $\bar{T}'$ . Then we have a jump condition to verify as  $\gamma_H$  crosses  $T'(\mathbb{R}) \cap \bar{T}'(\mathbb{R})$  (recall Lemma 2.7.1). Our (local) form for  $\Delta_D$  and  $\Delta_{\bar{D}}$  assures that this condition is satisfied up to a (possibly local) sign. The sign is in fact global in the following sense. Suppose that we adjust  $\varepsilon(D) \varepsilon(\bar{D})$  so that one jump condition for the adjacent pair  $D, \bar{D}$  is satisfied. Then all conditions for this pair are satisfied.

To prove this we argue as in Sections 9 - 11 of [S2] using  $\chi(\mu^* + \nu^*, \lambda^*)$  in place of " $\nu_m - \nu'_m$ "; we need the compatibility properties of the characters  $\chi(\mu^* + \nu^*, \lambda^*)$  proved in Section 6 of [S3]. We still have to show that the  $\varepsilon(D)$  can be chosen so that the jump conditions for all adjacent pairs hold simultaneously. If the ordering on the (stable) conjugacy classes of Cartan subgroups induced by the adjacency relation is linear then there is no problem. For the general case, see the discussion of [S2, Section 11] and the solution in [S4, Theorem 3.5.4]. It is clear that once  $\{\varepsilon(D): D \in \mathbf{D}\}$  has been chosen so that all jump conditions are satisfied the only possible replacement is  $\{-\varepsilon(D): D \in \mathbf{D}\}$ .

In summary, for the statement of the theorem in (5.2) we fix  $\mathbf{D}$  and define  $\Delta(\cdot, D, \xi)$ ,  $D \in \mathbf{D}$ , as in (4.5.2) with  $\{\varepsilon(D): D \in \mathbf{D}\}$  chosen so that all jump conditions for adjacent diagrams in  $\mathbf{D}$  are satisfied simultaneously. Then for general  $D$  we define  $\Delta(\cdot, D, \xi)$  via (4.5.3).

## 5. L-INDISTINGUISHABILITY

### 5.1 Notation

We review briefly the notation and terminology needed for the statement of the main theorems. A pair  $(s, {}^L H_S)$ , with  ${}^L H_S$  in standard position, is fixed;  $\xi = \xi(\mu^*, \lambda^*)$  is an admissible embedding, of unitary type, of  ${}^L H_S$  in  ${}^L G$  (see (4.1) - (4.3)). The corresponding endoscopic group is  $H$ . The remaining notation for Theorem 5.2.1 comes from (4.4) and (4.5) above.

Characters on the center of the universal enveloping algebra of the Lie algebra of  $G(\mathbb{C})$  will be identified by elements of  $X_*(T) \otimes \mathbb{C}$  (or their  $\Omega({}^L G^0, {}^L T^0)$ -orbits); see (3.6).

The set  $\Phi_0(G)$  consists of the parameters for the L-packets of tempered representations of  $G(\mathbb{R})$ ;  $\Pi_{\{\phi\}}$  denotes the L-packet attached to the parameter  $\{\phi\}$ . The character of  $\pi \in \Pi_{\{\phi\}}$  is written  $\chi_\pi$  and  $\chi_{\{\phi\}}$  is the sum of all  $\chi_\pi$ ,  $\pi \in \Pi_{\{\phi\}}$ . Recall from [S1] that  $\chi_{\{\phi\}}$  is a stable tempered distribution. If  $\{\phi'\} \in \Phi_0(H)$  then  $\{\xi \circ \phi'\}$  is not necessarily an element of  $\Phi_0(G)$ , i.e. is not necessarily relevant to  $G$ ; see (3.4).

## 5.2 Matching of orbital integrals

Fix Haar measures  $dg$  on  $G(\mathbb{R})$  and  $dh$  on  $H(\mathbb{R})$ .

### Theorem 5.2.1

For each Schwartz function  $f$  on  $G(\mathbb{R})$  there exists a Schwartz function  $f_H$  on  $H(\mathbb{R})$  such that:

$$(i) \quad \phi_{T'}^{st}(\gamma_H, f_H; dt', dh) = \Delta(\gamma_G, D, \xi) \phi_T^K(\gamma_G, f; dt, dg)$$

if  $\gamma_H \in T'(\mathbb{R})$  corresponds to  $\gamma_G \in T(\mathbb{R})_{reg}$  in a diagram  $D$  ( $dt$  is any Haar measure on  $T(\mathbb{R})$  and  $dt'$  is the measure on  $T'(\mathbb{R})$  associated to  $dt$  by  $D$ ) and

$$(ii) \quad \phi_{T'}^{st}(\gamma_H, f_H; dt', dh) = 0 \quad \text{if there is no diagram for } T'.$$

Proof: By the construction of  $\Delta(\gamma, D, \xi)$  we have only to find  $f_H$  so that the assertion is true for  $D \in \mathbf{D}$ , the set of diagrams attached to a framework of Cartan subgroups.

We define  $\phi_{T'}(\gamma_H, dt', dh) = \Delta(\gamma_G, D, \xi) \phi_T^K(\gamma_G, f; dt, dg)$  if  $\gamma_H$  corresponds to  $\gamma_G \in T(\mathbb{R})_{reg}$  in  $D \in \mathbf{D}$ , and  $\phi_{T'}(\gamma_H, dt', dh) = 0$  if  $\gamma_H \in T'(\mathbb{R}) \cap H_{reg}$  and no torus stably conjugate to  $T'$  has a diagram (in  $\mathbf{D}$ ).

First note that  $\Delta(\gamma, D, \xi) \phi_T^K(\gamma, f)$  extends smoothly across the walls "not from  $H$ " (recall (2.11.3)) and is invariant under the elements of  $\Omega_0(G, T)$  from  $H$  ... this was our first test for  $\Delta(\gamma, D, \xi)$  in (4.5).

Thus  $\{\phi_{T'}(\gamma, dt', dh)\}$  determines a family of functions on all stable conjugacy classes of regular semisimple elements in

$H(\mathbb{R})$ . We may now apply Theorem 2.8.1. Suppose that  $T'$  and

$\bar{T}'$  are adjacent in  $H$ , that  $D \in \mathbf{D}$  is a diagram for  $T'$  and that there is no diagram for  $\bar{T}'$ . Then [S2, Proposition 9.3]

and (2.7.2) ensure that  $\phi_{T^1}$ , suitably normalized, is smooth across the semi-regular elements in  $T^1(\mathbb{R}) \cap \bar{T}^s(\mathbb{R})$ , as needed. Note also that there are no totally compact roots in  $H$  (Theorem 2.2.4), so that (2.7.2) is vacuously satisfied by our family. The rest of the arguments have been discussed in (4.5); see Sections 9 - 11 of [S2].

### 5.3 Dual lifting of distributions

#### Theorem 5.3.1

Dual to the correspondence  $(f, f_H)$  between the Schwartz spaces of  $G(\mathbb{R})$  and  $H(\mathbb{R})$  there is a well-defined map  $\Theta_H \rightarrow \text{Lift}\Theta_H$  from the space of stable tempered distributions on  $H(\mathbb{R})$  to the space of invariant distributions on  $G(\mathbb{R})$ :

$$(\text{Lift}\Theta_H)(f) = \Theta_H(f_H).$$

If  $\Theta_H$  is an eigendistribution with infinitesimal character  $\mu_H$  then  $\text{Lift}\Theta_H$  is an eigendistribution with infinitesimal character  $\mu_H + \mu^*$ .

Proof: By definition,  $\Theta_H$  lies in the closed linear span (under simple convergence) of the tempered distributions  $f' \rightarrow \phi_{T^1}^{st}(\cdot, f')$  in the space of all tempered distributions, i.e. in the dual of the Schwartz space. It is immediate that  $\text{Lift}\Theta_H$  is well-defined, linear and invariant. Continuity of  $\text{Lift}\Theta_H$  is clear if  $\Theta_H$  is a stable orbital integral; it follows then for general stable  $\Theta_H$  by a version of the Banach-Steinhaus Theorem.

The assertion about eigendistributions follows from the differential equations for the  $F_f$  transform [HC] and

some routine arguments (see [S4, 4.2]).

If  $\theta_H$  is an eigendistribution then we may as well regard it as an analytic function on the regular semisimple elements of  $H(\mathbb{R})$  [HC]. Using the Weyl integration formula we may compute  $\text{Lift } \theta_H$ , as analytic function on the regular semisimple elements in  $G(\mathbb{R})$ , in terms of  $\theta_H$ . See [S4, Proposition 4.2.3, Lemma 4.2.4].

#### 5.4 Functoriality of the dual lifting

##### Theorem 5.4.1

Let  $\{\phi'\} \in \Phi_0(H)$ . Then there exist numbers  $\varepsilon(\pi) = \pm 1$  such that:

$$\text{Lift } \chi_{\{\phi'\}} = \begin{cases} \sum_{\pi \in \Pi_{\{\phi\}}} \varepsilon(\pi) \chi_{\pi} & \text{if } \{\phi\} = \{\xi \circ \phi'\} \text{ is} \\ & \text{relevant to } G, \\ 0 & \text{otherwise.} \end{cases}$$

The numbers  $\varepsilon(\pi)$  are computed explicitly in [S4] ... the precise form will be needed in (5.5) and (5.6).

Proof: As representative for  $\{\phi'\}$  we fix a discrete  $\phi(\mu', \lambda')$ :  $W \rightarrow L_{M'}$ , where  $L_{M'}$  is standard. Assume first that  $M' = H$ . If  $\mu'$  is sufficiently regular then  $\mu' + \mu^*$  is  $L_G^0$ -regular, i.e.  $\phi(\mu' + \mu^*, \lambda' + \lambda^*)$  is discrete for  $G$ . Thus we have to prove  $\theta = \text{Lift } \chi_{\{\phi'\}}$  is a combination of certain discrete series characters. We know that  $\theta$  is an eigendistribution of correct infinitesimal character and we can compute  $\theta$  explicitly on Cartan subgroups compact modulo the center of

$G(\mathbb{R})$ . We may assume  $G$  simply-connected and semisimple ([S4, Lemma 4.4.6]). Thus it remains to apply Harish-Chandra's characterization of (combinations of) discrete series characters. Suppose now  $M' = H$  but  $\mu = \mu' + \mu^*$  is  $G$ -singular. Then we argue by coherent continuation that  $\theta$  is a combination of the limits of discrete series characters attached to  $\{\phi(\mu, \lambda' + \lambda^*)\}$ . Then, noting that  $R_\mu^V$  is of type  $(A_1)^n$ , we apply Theorem 3.5.1 (... recall that Theorem 3.5.1 has not been proved for general  $\phi(\mu, \lambda)$ ). Thus  $\theta$  is as claimed. See [S4, 4.4].

The proof of Theorem 5.4.1 is completed by parabolic induction arguments. See [S4, 4.5].

To prove Theorem 3.5.1 for arbitrary  $\phi = \phi(\mu, \lambda)$  we write  $\{\phi\}$  as the lift of a parameter  $\{\phi'\}$ , for some suitable  $H$ . We assume the theorem for  $H$ . Then the arguments for Theorem 5.4.1 show that the lift of  $\chi_{\{\phi'\}}$  is a combination, with coefficients  $\pm 1$ , of the limit of discrete series characters associated to  $(\mu, \lambda)$ . We have then only to apply Theorem 4.5.1. See [S4, 4.6] for details.

Theorem 5.4.1 may be presented as a set of "character identities". See [S4, 4.7].

### 5.5 Factoring parameters

Let  $\phi: W \rightarrow {}^L G$  be an admissible homomorphism. Recall the "factorings" of  $\phi$  associated to a semisimple element  $x$  of  $S_\phi$  (see (4.2)):  $s = xZ_G^W$  and  $(s, {}^L H_s)$  is such that  $\phi$

$= \xi \circ \phi'$ , where  $\phi': W \rightarrow {}^L H_S$  and  $\xi: {}^L H_S \rightarrow {}^L G$  is an admissible embedding (assumed to exist). To say that  $\{\phi\}$  "factors through"  $\{\phi'\}$  we mean only that  $\phi$  is equivalent to  $\xi \circ \phi'$ . Thus we fix a complete set of representatives  $\mathcal{C} = \{(\mathbf{s}, {}^L H)\}$  for the elements of  $\mathcal{C}({}^L G)$ . We assume that each  ${}^L H$  is in standard position and embeds admissibly in  ${}^L G$ ; we fix an admissible embedding  $\xi: {}^L H \rightarrow {}^L G$ . Now each semisimple  $x \in S_\phi$  gives a factoring of  $\{\phi\}$  through some  $\{\phi''\}$ , where  $\phi'': W \rightarrow {}^L H$  and  $(\mathbf{s}, {}^L H) \in \mathcal{C}$ . We obtain  $\{\phi''\}$  from the  $\{\phi'\}$  above by  ${}^L G^0$ -conjugation of the image of  ${}^L H_S$  in  ${}^L G$  to the image of  ${}^L H$ ; the element  $\{\phi''\}$  of  $\phi(\mathbb{H})$  is not uniquely determined in general.

Note that if  $G = \mathrm{SU}(2,1)$  and  $\{\phi\}$  is discrete then there are three nontrivial elements in  $S_\phi$ ; the initial construction associates to them the three conjugate forms of  ${}^L H$ , for  $H = \mathrm{U}(1,1)$ ; we have now fixed the one form in standard position to obtain three different factorings of  $\{\phi\}$  through classes of homomorphisms into this form (... there are no ambiguities in this case).

In general, our goal is not to describe all factorings of  $\{\phi\} \in \phi_0(G)$  but only to define a "natural" family for the inversion of the liftings in Theorem 5.4.1, i.e. for the expression of an arbitrary irreducible tempered character as a combination of lifts.

Fix  $\{\phi\} \in \phi_0(G)$ . Recall that in (3.6) we gave a description of  $\mathbb{H} = \Pi_{\{\phi\}}$  in terms of certain irreducible limits of discrete

series characters. In particular, we have associated a torus  $T$  and a pairing  $\langle \cdot, \cdot \rangle: \mathbf{S} \times \overline{\mathbb{H}} \rightarrow \{\pm 1\}$  induced by one of the obvious pairings of  $\mathbb{H}$  with  $\xi(T)^V$ . While the  $G(\mathbb{R})$ - or stable conjugacy class of  $T$  is canonical the pairing is not.

Consider first the case  $\{\phi\}$  discrete. Then the attached class of tori is that of the tori anisotropic modulo the center of  $G$ . For  $\phi = \phi(\mu, \lambda)$  representing  $\{\phi\}$  we have  $S_\phi \subset L_T^0$  and  $S_\phi^0 \subset Z_G^W$ . By construction the factoring associated to  $x \in S_\phi$  is determined by the class  $\mathbf{x}$  of  $x$  in  $\mathbf{S} = S_\phi / Z_G^W$ . The attached endoscopic group  $\mathbf{H}$  contains  $T$  (more precisely, there is a torus  $T'$  in  $\mathbf{H}$  and diagram  $D$  such that  $T = i(h, \eta)(T')$ ); we are simply factoring  $\{\phi\}$  through a discrete parameter  $\{\phi_{\mathbf{x}}\}$  for  $\mathbf{H}$ . The ambiguity in  $\{\phi_{\mathbf{x}}\}$  does not matter in the following sense. Fix  $(T, \eta)$  with  $T$  anisotropic modulo the center of  $G$  and  $\eta$  any p-d. of  $T$ . We may as well take the element  $\omega_*$  of (3.6) equal to the identity in  $\Omega(G, T)$ . Then  $\langle \cdot, \cdot \rangle$  depends on  $(T, \eta)$  alone. Whatever the choice for  $\{\phi_{\mathbf{x}}\}$  we have the following:

(5.5.1) there exists  $c(\{\phi_{\mathbf{x}}\}) = \pm 1$  such that

$$c(\{\phi_{\mathbf{x}}\}) \chi_{\{\phi_{\mathbf{x}}\}}(f_{\mathbf{H}}) = \sum_{\pi \in \mathbb{H}} \langle \mathbf{x}, \pi \rangle \chi_{\pi}(f).$$

For the proof see [S4, 5.2] ... note that we have only to compare the coefficients  $\varepsilon(\pi)$  from Theorem 5.4.1 with  $\langle \mathbf{x}, \pi \rangle$ . We conclude then that the left side of the equation in (5.5.1), like the right side, is independent of the

choice for  $\{\phi_{\mathbf{x}}\}$ . We write instead  $\hat{\chi}_{(\phi, \mathbf{x})}(f)$ , keeping in mind that we have up to sign the lift of a stable discrete series character. Thus:

$$\hat{\chi}_{(\phi, \mathbf{x})}(f) = \sum_{\pi \in \mathbb{H}} \langle \mathbf{x}, \pi \rangle \chi_{\pi}(f) = \sum_{\pi \in \bar{\mathbb{H}}} \langle \mathbf{x}, \pi \rangle \chi_{\pi}(f),$$

since by definition  $\chi_{\pi}(f) = 0$  for  $\pi \in \bar{\mathbb{H}} - \mathbb{H}$  (recall (3.6)).

Then, by inversion in  $\mathbf{S}$ , we have:

$$\chi_{\pi}(f) = [\mathbf{S}]^{-1} \sum_{\mathbf{x} \in \mathbf{S}} \langle \mathbf{x}, \pi \rangle \hat{\chi}_{(\phi, \mathbf{x})}(f), \quad \pi \in \bar{\mathbb{H}}.$$

This is our final result for  $\{\phi\}$  discrete. Applied to a discrete series character  $\chi_{\pi}$  it gives a simple explicit formula for this character in terms of lifts of stable discrete series characters. Applied to  $\chi_{\pi}$ ,  $\pi \in \bar{\mathbb{H}} - \mathbb{H}$ , it gives dependence relations among lifts.

Let  $\{\phi\}$  be an arbitrary bounded parameter. We fix  $(T, \eta)$ ,  $\phi(\mu, \lambda): W \rightarrow L_M$ ,  $\omega_*$  and  $\langle, \rangle$  as at the end of (3.6). Let  $\phi = \phi(\omega_* \mu, \lambda)$ . Then we realize  $\mathbf{S}$  as  $S_{\phi} \cap L_T^0 / Z_G^W (S_{\phi}^0 \cap L_T^0)$ . Suppose that  $T$  is anisotropic modulo the center of  $G$ , i.e.  $M = G$ . We will imitate the case  $\{\phi\}$  discrete, i.e. factor  $\{\phi\}$  only through  $\mathbb{H}$  containing  $T$ . Let  $\mathbf{x} \in \mathbf{S}$ . Choose  $s \in S_{\phi} \cap L_T^0 / Z_G^W$  mapping to  $\mathbf{x}$  under the natural projection. Then the attached  $(s, L_{H_s})$  yields  $L_{\mathbb{H}}$  and a factoring of  $\{\phi\}$  through some element of  $\phi_0(\mathbb{H})$ . We call this element  $\{\phi_{\mathbf{x}}\}$  since:

(5.5.2) there exists  $c(\{\phi_{\mathbf{x}}\}) = \pm 1$  such that

$$c(\{\phi_{\mathbf{x}}\}) \chi_{\{\phi_{\mathbf{x}}\}}(f_{\mathbb{H}}) = \sum_{\pi \in \mathbb{H}} \langle \mathbf{x}, \pi \rangle \chi_{\pi}(f).$$

The proof of this rests on Lemma 3.6.9 as well as the explicit

form of the coefficients  $\varepsilon(\pi)$  in Theorem 5.4.1. See [S4, Lemma 5.4.22 and Corollary 5.4.23].

In general, we ensure that  $\{\phi_{\mathbf{x}}\}$  is well-defined for character theory (in the sense of (5.5.2)) by constructing it essentially in  $L_M$ . Thus choose  $s \in S_{\phi} \cap L_T^0 / Z_M^W$  mapping to  $\mathbf{x}$  under the natural projection. Then  $L_{H_s}$  is defined as a subgroup of  $L_M$ . After a "G(R)-conjugation" of our data for  $\{\phi\}$  we may assume that  $L_{H_s}$  is conjugate in  $L_M$  to some  $L_{M_H}$  (with  $H \in \mathcal{G}$  and  $M_H$  endoscopic for  $M$ ). See Proposition 5.4.24 of [S4] for a more precise statement. The construction of  $\{\phi_{\mathbf{x}}\}$  is now routine (see [S4, Theorem 5.4.26]). We have then a general analogue of the result for discrete parameters:

## 5.6 Inversion of character liftings

### Theorem 5.6.1

Let  $\{\phi\} \in \Phi_0(G)$ . Then:

$$\chi_{\pi} = [S]^{-1} \sum_{\mathbf{x} \in S} \langle \mathbf{x}, \pi \rangle \hat{\chi}_{\{\phi, \mathbf{x}\}}$$

for  $\pi \in \bar{\Pi}$ .

## APPENDIX

### 1. Fourier Inversion

The results of L-indistinguishability can be used to reduce the problem of Fourier inversion of ordinary orbital integrals, and more importantly for us the inversion of  $\kappa$ -orbital integrals, to the inversion of stable orbital integrals. This last problem is easier because the anti-symmetry under the imaginary Weyl group of a (normalized) stable orbital integral forces the imaginary Weyl group-singular Fourier coefficients of the integral, as function on the relevant CSG, to be zero.

We start then with the inversion for stable orbital integrals. From the parametrization of unitary principal series we obtain in the usual way a measure on tempered L-packets, i.e. a measure  $d\phi$  on  $\phi_0(G)$ . Using the normalized orbital integral  $\bar{\psi}^T$  from (2.7) we may write the inversion formula as  $\bar{\psi}^T(\gamma) = \bar{\psi}^T(\gamma, f) = \int \chi_{\{\phi\}}(f) a(\gamma, \phi) d\phi$ , where the  $a(\cdot, \cdot)$  satisfy certain growth properties which ensure absolute convergence of the integral. For this we quote Harish-Chandra's existence theorem for the inversion of ordinary orbital integrals [Supertempered distributions on real groups, preprint], noting that there are simplifications in the proof. For simply connected semisimple classical  $G$  we obtain the formula

with the  $a(,)$  in explicit form from [Herb: Fourier inversion and the Plancherel theorem for semisimple real Lie groups, preprint]. Note that the formula shows that the space of stable tempered distributions (in the sense of [S1]) is generated by  $\{ \chi_{\{\phi\}} : \{\phi\} \in \phi_0(G) \}$ .

Consider the normalized  $\kappa$ -orbital integral from (2.11). To obtain inversion for  $\bar{\Psi}^\kappa$  we apply Theorem 5.2.1 (the matching), the stable inversion for the intervening endoscopic group  $\mathbf{H}$ , Theorem 5.4.1 (dual lifting of tempered characters) and the fact that the map  $\phi_0(\mathbf{H}) \rightarrow \phi_0(G)$  induced by an embedding of  $L_{\mathbf{H}}$  in  $L_G$  has finite fibers. Then:

$$\bar{\Psi}^\kappa(\gamma, f) = \int \chi_{\{\phi'\}}(f_{\mathbf{H}}) a_*(\kappa, \gamma, \phi') d\phi' =$$

$$\int (\text{Lift } \chi_{\{\phi'\}})(f) a_*(\kappa, \gamma, \phi') d\phi' = \int \chi_\pi(f) a(\kappa, \gamma, \pi) d\pi,$$

for suitable  $d\pi$ , with the  $a(\kappa, \gamma, \pi)$  again of growth which ensures absolute convergence of the integral. Note that the  $a_*(\kappa, \gamma, \phi')$  incorporate the "shift by  $\mu^*$ " (see (5.3)) and are readily computed in terms of the stable  $a(,)$  for  $\mathbf{H}$ . So then are the  $a(\kappa, \gamma, \pi)$ .

We conclude now that the  $f \rightarrow \phi_T^\kappa(\gamma, f)$  for which  $(T, \kappa)$  is attached to  $\mathbf{H}$  (i.e. for which the attached element of  $\mathcal{C}(G)$  [see (4.2)] is the class of  $(\mathbf{s}, L_{\mathbf{H}})$ ) generate the the same space of tempered distributions as  $\{\text{Lift } \chi_{\{\phi'\}} : \{\phi'\} \in \phi_0(\mathbf{H})\}$ .

Since an ordinary orbital integral is a combination of  $\kappa$ -orbital integrals (by inversion in  $\mathfrak{E}(T)$ )

we may next recover the existence of Fourier inversion for orbital integrals from the last formula, with explicit computations following from the stable case. For classical groups Herb has since carried these out, but in more concrete terms. Note that a concrete version of Theorem 5.6.1 requires no mention of endoscopic groups. See [Herb: Discrete series characters and Fourier inversion on semisimple real Lie groups, preprint].

## 2. Base change

We use the phrase "base change" to mean the analysis associated with the  $\sigma_G$ -invariant representations of  $G(\mathbb{C})$ . For  $f \in C_c^\infty(G(\mathbb{C}))$  and  $\delta \in T(\mathbb{C})$  such that  $\delta\sigma_G(\delta)$  is regular the twisted orbital integral  $\phi_T^{\text{tw}}(\delta, f; d_{\mathbb{C}}t, d_{\mathbb{C}}g) = \int f(g^{-1}\delta\sigma_G(g)) d_{\mathbb{C}}g/d_{\mathbb{C}}t$  is well-defined ([S7]); locally it is an ordinary orbital integral (see [S7, Theorem 4.2.1]) and so we can apply the results of (2.6) and (2.7) (see [S7, 4.2, 4.3]). Using the local analysis we may then attempt to match combinations of twisted orbital integrals with stable orbital integrals on groups to be viewed as endoscopic for base change. In [S9] we have given a preliminary definition of a much more general type of endoscopic group. In the case of "endoscopic groups for base change" we obtain the endoscopic groups from L-indistinguishability. The role of these groups is now quite

different; nevertheless, some algebraic properties ([S8, Lemma 6.3], see also [S10]) reduce the problems for the matchings to precisely the problems for the L-indistinguishability matchings discussed in (4.5). This is described in detail in [S8]. Theorem 7.1 of [S8] is our matching theorem for "base change". The dual of the "stable matching" gives a lifting of stable tempered characters on  $G(\mathbb{R})$  to stable twisted tempered characters on  $G(\mathbb{C})$ ; the lifting is functorial ... this follows from, and is another way of stating, Clozel's "base-change identity" [Changement de base pour les représentations tempérées des groupes réductifs réels, preprint]. We expect that all twisted tempered characters on  $G(\mathbb{C})$  are (functorial) dual lifts from the matchings described in the theorem.

### 3. An example for $SU(2,1)$

It is more convenient to consider  $G = SU(J)$ , with  $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , and to use the notation of [S3, 4.3.1 and 3.2.1].

We are concerned with the parameters from the end of Chapter IV in [L5]; there  $G$  is the projective form, but the passage from  $SU(J)$  to  $PU(J)$  is a trivial exercise.

First we note all parameters corresponding to infinitesimal character the orbit of  $t_1 - t_2$ . Let  $\chi_0(e^{t+i\theta}) = e^{2t}$ ,  $\chi_1(e^{t+i\theta}) = e^{t+3i\theta}$  and  $\chi_2(e^{t+i\theta}) = e^{t-3i\theta}$ . Then we define  $\phi^v(z \times 1) = \text{diag}(\chi_v(z), 1, \chi_v(\bar{z})^{-1})_* \times z \times 1$ ,  $z \in \mathbb{C}^\times$ , and  $\phi^v(1 \times \sigma) = \text{diag}(1, 1, \chi_v(-1))_* \times 1 \times \sigma$ ,  $v = 0, 1, 2$ .

The equivalence class of  $\phi^v$  is uniquely determined by  $\chi_v$ , and  $\phi^1, \phi^2$  correspond to the homomorphisms similarly labelled in [L5]. Also, there is a unique discrete parameter with representative  $\phi^3$  such that  $\phi^3(z \times 1) = \text{diag}(z/\bar{z}, 1, \bar{z}/z)_* \times z \times 1$ ,  $z \in \mathbb{C}^\times$ . Then, in the notation of [BW; VI, 4.8], we may write  $\Pi_{\{\phi^0\}} = \{J_{00}\}$ ,  $\Pi_{\{\phi^1\}} = \{J_{01}\}$ ,  $\Pi_{\{\phi^2\}} = \{J_{10}\}$  and  $\Pi_{\{\phi^3\}} = \{D_0, D_1, D_2\}$ .

For any  $\{\phi\}$  we use  $\chi_{\{\phi\}}$  to denote the sum of the characters of the members of  $\Pi_{\{\phi\}}$ . While  $\chi_{\{\phi^0\}}$  and  $\chi_{\{\phi^3\}}$  are stable,  $\chi_{\{\phi^1\}}$  and  $\chi_{\{\phi^2\}}$  are not. We will "stabilize" the sum of the latter two characters using a method suggested by the tempered theory.

Note that  $S_{\phi^1} = S_{\phi^2}$  is the group of  $W$ -invariants in  $L_{T^0}$  (... so that  $S_{\phi^1} = S_{\phi^2} = 1$ ). Following the second construction of (4.2) we obtain for each  $x \in S_{\phi^v}$  a factoring of  $\{\phi^v\}$  ( $v = 1, 2$ ). The endoscopic groups for  $G$  are  $G, H$  and  $T_1$ ;  $L_G$  embeds in  $L_G$  by the identity map,  $L_{T_1}$  by the inclusion map, and  $L_H$  by the embedding  $\xi_0$  of [S3, 4.3.1]. If  $x \in S_{\phi^v}$  is different from 1 and  $\text{diag}(-1, 1, -1)_*$  then the attached factoring of  $\{\phi^v\}$  is the trivial one through  $L_{T_1}$ ; if  $x = 1$  then it is the trivial one through  $L_G$ , but if  $x = \text{diag}(-1, 1, -1)_*$  then we obtain  $\phi^v = \xi_0 \circ \phi_H^v$ , where  $\phi_H^v: W \rightarrow L_H$  is given by  $\phi_H^v(z \times 1) = \text{diag}(\chi_v(z)(\bar{z}/z)^{1/2}, \chi_v(\bar{z})^{-1}(\bar{z}/z)^{1/2})_* \times z \times 1$ ,  $z \in \mathbb{C}^\times$ , and  $\phi_H^v(1 \times \sigma) = 1 \times 1 \times \sigma$

Also,  $\Pi_{\{\phi_H^1\}}$  consists of the 1-dimensional representation of  $H(\mathbb{R})$  given by  $a(\theta, t) \rightarrow e^{2i\theta}$  and  $\Pi_{\{\phi_H^2\}}$  of the 1-dimensional representation given by  $a(\theta, t) \rightarrow e^{-4i\theta}$  (notation is from [S3, 4.3.1]). In the matching of orbital integrals between  $G$  and  $H$  we can arrange that  $f_H \in C_c^\infty(H(\mathbb{R}))$  when  $f \in C_c^\infty(G(\mathbb{R}))$  (by the characterization of stable orbital integrals for  $C_c^\infty$ -functions on  $H(\mathbb{R})$  which follows from [L3, Chapter 6], see [S6]). Thus  $\text{Lift } \chi_{\{\phi_H^v\}}$  is a welldefined invariant distribution on  $G(\mathbb{R})$ ,  $v = 1, 2$ .

Lemma A3.1

$(\chi_{\{\phi^1\}} + \chi_{\{\phi^2\}}) - 1/2(\text{Lift } \chi_{\{\phi_H^1\}} + \text{Lift } \chi_{\{\phi_H^2\}})$  is a stable combination of characters.

Proof: We compute the lifts explicitly. For the definition of  $f_H$  we fix Haar measures on  $G(\mathbb{R}), H(\mathbb{R}), T_0(\mathbb{R}) = T'_0(\mathbb{R})$  and  $T_1(\mathbb{R}) = T'_1(\mathbb{R})$ , but omit them in notation. The stable orbital integrals of  $f_H$  will be specified by:

$$\begin{aligned} \phi_{T_0}^{\text{st}}(r(\theta, \phi), f_H) = \\ e^{i\theta} (1 - e^{-i(2\theta+\phi)}) (1 - e^{i(\theta+2\phi)}) \quad \text{times} \\ \phi_{T_0}(r(\theta, \phi), f) - \phi_{T_0}(r(\theta, -(\theta+\phi)), f) + \phi_{T_0}(r(\phi, \theta), f) \end{aligned}$$

and

$$\begin{aligned} \phi_{T_1}^{\text{st}}(a(\theta, t), f_H) = \phi_{T_1}(a(\theta, t), f_H) = \\ e^{i\theta} |e^t - e^{3i\theta}|^{1/2} |e^t - e^{-3i\theta}|^{1/2} |e^{-t} - e^{3i\theta}|^{1/2} |e^{-t} - e^{-3i\theta}|^{1/2} \\ \text{times } \phi_{T_1}(a(\theta, t), f). \end{aligned}$$

Let  $I_H^1$  be the principal series representation of  $H(\mathbb{R})$  attached to the quasicharacter  $a(\theta, t) \rightarrow e^{t+2i\theta}$ ; the

1-dimensional representation given by  $a(\theta, t) \rightarrow e^{2i\theta}$  is the attached Langlands quotient. Similarly, let  $I_H^2$  be the principal series representation given by  $a(\theta, t) \rightarrow e^{t-4i\theta}$ . It is easily checked that  $\text{Lift } I_H^1 = I_{01}$  and  $\text{Lift } I_H^2 = I_{10}$  (notation from [BW]). On the level of characters we have  $I_H^1 = \chi_{\{\phi_H^1\}} + \chi_{\{\phi'\}}$  and  $I_H^2 = \chi_{\{\phi_H^2\}} + \chi_{\{\phi''\}}$ , where  $\{\phi'\}$ ,  $\{\phi''\}$  are discrete, with representatives  $\phi'$ ,  $\phi''$  such that  $\phi'(z \times 1) = \text{diag}((z/\bar{z})^{3/2}, 1, (z/\bar{z})^{1/2})_* \times z \times 1$ , and  $\phi''(z \times 1) = \text{diag}((z/\bar{z})^{-3/2}, 1, (z/\bar{z})^{-5/2})_* \times z \times 1$ ,  $z \in \mathbb{C}^\times$ . Thus we have only to calculate  $\text{Lift } \chi_{\{\phi'\}}$  and  $\text{Lift } \chi_{\{\phi''\}}$ , and examine the constituents of  $I_{01}$ ,  $I_{10}$ .

To compute the last two lifts we recall that the answer in each case will be a combination of discrete series characters; to determine the coefficients in the combinations we may argue on  $T_0(\mathbb{R})$  alone. We obtain:

$$\text{Lift } \chi_{\{\phi'\}} = D_0 + D_1 - D_2 \quad \text{and}$$

$$\text{Lift } \chi_{\{\phi''\}} = D_2 + D_1 - D_0$$

(on the level of characters) and then conclude that:

$$\begin{aligned} & (\chi_{\{\phi^1\}} + \chi_{\{\phi^2\}}) - 1/2(\text{Lift } \chi_{\{\phi_H^1\}} + \text{Lift } \chi_{\{\phi_H^2\}}) \\ &= 1/2(I_{01} + I_{10}) - (D_0 + D_1 + D_2) \end{aligned}$$

(see [BW] for the decomposition of  $I_{01}$ ,  $I_{10}$ ).

This last combination is stable (in the sense of [L5]) and so the lemma is proved.

## REFERENCES

- [A] J. Arthur, On the invariant integrals associated to weighted orbital integrals, preprint.
- [B] A. Borel, Automorphic L-functions, Proc. Sympos. Pure Math., vol. XXXIII, part 2, Amer. Math. Soc., (1979), pp. 27-61.
- [BW] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups, Annals of Math. Studies, no. 94 (1980).
- [HC] Harish-Chandra, Harmonic analysis on real reductive groups I, J. Funct. Analysis, 19 (1975), pp. 104-204.
- [HCIII] \_\_\_\_\_, Harmonic analysis on real reductive groups III, Annals of Math., 104 (1976), pp. 117-201.
- [Kn] A. Knapp, Weyl group of a cuspidal parabolic, Ann. Scient. Éc. Norm. Sup., 4<sup>e</sup> série, t.8 (1975), pp. 275-294.
- [KZ] A. Knapp and G. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, preprint.
- [Ko] R. Kottwitz, Sign changes in harmonic analysis on reductive groups, preprint.
- [L1] R. Langlands, On the classification of irreducible representations of real algebraic groups, preprint.
- [L2] \_\_\_\_\_, Stable conjugacy: definitions and lemmas, Can. J. Math., 31 (1979), pp. 700-725.

- [L3] \_\_\_\_\_, Base change for  $GL(2)$ , Annals of Math. Studies, no. 96 (1980).
- [L4] \_\_\_\_\_, On the zeta-function of some simple Shimura varieties, Canad. J. Math., 31 (1979), pp. 1121-1216.
- [L5] \_\_\_\_\_, Les débuts d'une formule des traces stable. Proceedings this conference.
- [L6] \_\_\_\_\_, lectures at the conference on Shimura varieties, Vancouver, August, 1981.
- [L7] \_\_\_\_\_, Automorphic representations, Shimura varieties, and motives, Proc. Sympos. Pure Math., vol. XXXIII, part 2, Amer. Math. Soc., (1979), pp. 205-246.
- [S1] D. Shelstad, Characters and inner forms of a quasi-split group over  $\mathbb{R}$ , Compositio Math., vol. 39 (1979) pp. 11-45.
- [S2] \_\_\_\_\_, Orbital integrals and a family of groups attached to a real reductive group, Ann. Scient. Éc. Norm. Sup.. 4<sup>e</sup> série, t. 12 (1979), pp. 1-31.
- [S3] \_\_\_\_\_, Embeddings of L-groups, Canad. J. Math., 33 (1981), pp. 513-558.
- [S4] \_\_\_\_\_, L-indistinguishability for real groups, Math. Annalen, vol. 259 (1982), pp. 385-430.
- [S5] \_\_\_\_\_, Notes on L-indistinguishability (based on a lecture of R. P. Langlands), Proc. Sympos. Pure Math., vol. XXXIII, part 2, Amer. Math. Soc., (1979) pp. 193 -203.

- [S6] \_\_\_\_\_, Orbital integrals for  $GL_2(\mathbb{R})$ , Proc. Sympos. Pure Math., vol. XXXIII, part 2, Amer. Math. Soc. (1979), pp. 107-110.
- [S7] \_\_\_\_\_, Base change and a matching theorem for real groups, in Springer Lecture Notes, vol. 880 (1981) pp. 425-482.
- [S8] \_\_\_\_\_, Endoscopic groups and base change  $\mathbb{C}/\mathbb{R}$ , to appear in Pacific J. Math.
- [S9] \_\_\_\_\_, A generalization of endoscopic groups, in preparation.
- [S10] \_\_\_\_\_, Twisted endoscopic groups in the abelian case, preprint.
- [V] D. Vogan, The algebraic structure of the representations of semisimple Lie groups I, Annals of Math., 109 (1979), pp. 1-60.
- [W2] G. Warner, Harmonic analysis on semi-simple Lie groups, vol. 2, Springer (1972).

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