ORBITAL INTEGRALS FOR $\text{GL}_2(\mathbb{R})^*$

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We report briefly on the characterization of orbital integrals of smooth ($C^\infty$) functions of compact support on $\text{GL}_2(\mathbb{R})$, following [3]. A similar argument applies to $\text{GL}_d(\mathbb{C})$ [3]. We begin by recalling some well-known properties of these integrals in a form convenient for the characterization, indicating the proof afterwards; a more elegant formulation is given in [3].

We fix an invariant 4-form $\omega_G$ on $G = \text{GL}_2(\mathbb{R})$. If $T$ is a Cartan subgroup of $G$ we take $\omega_T$ to be the form $C_T d\gamma_1 d\gamma_2 / \gamma_1 \gamma_2$ where $\gamma_1, \gamma_2$ are the eigenvalues of $\gamma$ under some order (prescribed by a diagonalization of $T$) and $C_T$ is a constant as follows:

$C_T = 1$ if $T$ is split, and $C_T = i$ otherwise. If $f \in C_c^\infty(G)$ and $\gamma \in T_{\text{reg}}$, the set of regular elements of $G$ lying in $T$, we set

$$\Phi_T^f(\gamma) = \int_{G/T} f(g \gamma g^{-1}) \frac{dg}{dt}$$

where $dg$, $dt$ are the Haar measures defined by $\omega_G$, $\omega_T$ respectively. Then $\Phi_T^f$ is a well-defined $C^\infty$ function on $T_{\text{reg}}$, invariant under the Weyl group and vanishing off some set relatively compact in $T$. Let $Z$ be the group of scalar matrices in $G$; thus $Z = T - T_{\text{reg}}$. The behavior of $\Phi_T^f$ near $z \in Z$ is described as follows: there exist a neighborhood $N_z$ of $z$ in $T$ (invariant under the Weyl group) and $C^\infty$ functions $A^T_j(z, \ )$ and $A^T_j(z, \ )$, each defined on $N_z$ and invariant under the Weyl group, such that

$$\Phi_T^f(\gamma) = A^T_j(z, \gamma) + A^T_j(z, \gamma) \ | D(\gamma)|^{-1/2}$$

for $\gamma \in N_z \cap T_{\text{reg}}$. Here $D(\gamma) = (\gamma_1 - \gamma_2)^2 / \gamma_1 \gamma_2$ where, as before, $\gamma_1$ and $\gamma_2$ are the eigenvalues of $\gamma$. The functions $A^T_j(z, \ )$ and $A^T_j(z, \ )$ depend on $T$ although we omit this in notation. Note that the equation (1) determines uniquely the restriction to $Z \cap N_z$ of $A^T_j(z, \ )$, and of all its derivatives. Thus we may set $A^T_j(z) = A^T_j(z', z)$ for any $z'$ such that $z \in N_{z'}$, with a similar definition for derivatives. Further

(a) if $T$ is split then we may take $A^T_j(\ , \ ) \equiv 0$

and if $X_T$ denotes the image under the Harish-Chandra isomorphism of the operator $X$ in the center of the universal enveloping algebra of $\text{gl}_2(\mathbb{C})$ then

(b) for each $z \in Z$, $X_T A^T_j(z)$ is independent of $T$.

To determine the restriction to $Z \cap N_z$ of the derivatives of $A^T_j(z, \ )$ it is suffi-

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cient to compute $X_T \lambda_j^f(z)$ for each $X; \ X_T \lambda_j^f(z)$, which we will not need explicitly, is the appropriately defined integral of $X^f$ over the conjugacy class of $(\lambda \lambda)$ and if $T$ is not split then $X_T \lambda_j^f(z) = c_\lambda X^f(z)$, where $c_\lambda$ is a constant depending only on our choice of Haar measure on $G$.

We recall the proof. It is sufficient to consider the Cartan subgroups $A$, the diagonal group, and

$$B = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) : a^2 + b^2 \neq 0 \right\}.$$ 

We find it more convenient to write an element of $B_{\text{reg}}$ as

$$\gamma(\lambda, \theta) = \left( \begin{array}{cc} \lambda \cos \theta & \lambda \sin \theta \\ -\lambda \sin \theta & \lambda \cos \theta \end{array} \right), \quad \lambda > 0, \theta \neq 0(\pi).$$

Proceeding formally, we may choose $\omega_0$ so that

$$|D(\gamma)|^{1/2} \Phi^\gamma_f(\gamma) = \frac{1}{2} \left| \frac{\gamma_2}{\gamma_1} \right|^{1/2} \int_N f_0(\eta \gamma) \, dn$$

where

$$N = \left\{ \left( \begin{array}{c} 1 \\ x \end{array} \right) \right\}, \quad dn = dx, \quad f_0(x) = \int_{K_0} f(k x k^{-1}) \, dk,$$

$$K_0 = \left\{ \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \right\}, \quad dk = d\theta$$

and

$$|D(\gamma)|^{1/2} \Phi^\gamma_f(\gamma) = \frac{1}{4} \int_0^\infty \left( H_3(e^{i\theta}, e^{-i\theta}) + H_3(-e^{i\theta}, -e^{-i\theta}) \right) \left( e^t - e^{-t} \right) \, dt$$

where

$$H_3(u, v) = \int_{K_0} f\left( \lambda k \exp \left( \begin{array}{cc} 0 & u \\ -v & 0 \end{array} \right) k^{-1} \right) \, dk.$$ 

We obtain (2) from the Iwasawa decomposition of $SL_2(R)$ and (3) from the Cartan decomposition. The function on the right-hand side of (3) can be analyzed as in [1]. It is easy then to see that this function can be expanded as on the right-hand side of (1). The proof is now straightforward. To compute $X_T \lambda_j^f(z)$ note that $|D|^{1/2} \Phi^\gamma_f = X_T (|D|^{1/2} \Phi^\gamma_f)$. This is essentially Harish-Chandra’s formula $F^\gamma_{X_T} = X_T F^\gamma_f$ [2]; $|D|^{1/2} \Phi^\gamma_f$ is the function $F^\gamma_f$ and $(e^{i\theta} - e^{-i\theta}) \Phi^\gamma_f (\gamma(\lambda, \theta)) = F^\gamma_f (\gamma(\lambda, \theta)).$

We come then to the characterization. Suppose that for each Cartan subgroup $T$ we are given a function $\Phi^T$, defined and $C^\infty$ on $T_{\text{reg}}$, invariant under the Weyl group and vanishing off some set relatively compact in $T$. Suppose that $\Phi^T$ and $\Phi^{T'}$ satisfy the obvious consistency requirements when $T$ and $T'$ are conjugate. Finally, suppose that for each $T$ and $z \in Z$ there exist a neighborhood $N_z$ of $z$ in $T$ invariant under the Weyl group and $C^\infty$ functions $A^0(z, \cdot)$ and $A^1(z, \cdot)$ on $N_z$, also invariant under the Weyl group, such that

$$\Phi^T(\gamma) = A^0(z, \gamma) + \frac{A^1(z, \gamma)}{|\Phi(\gamma)|^{1/2}}$$

for $\gamma \in N_z \cap T_{\text{reg}}$; the functions $A^i(z, \cdot)$ are assumed to have the following two properties:
(a) $\mathcal{A}(\ , \ ) \equiv 0$ if $T$ is split and
(b) for each $X$ in the center of the universal enveloping algebra of $\mathfrak{gl}_d(C)$ the restriction to $Z \cap N_z$ of $X_T\mathcal{A}(z, \ )$ is independent of $T$.

Then Lemma 4.1 of [3] asserts that there exists $f \in C^\infty_c(G)$ such that $\Phi_T^f = \Phi_T$ for each $T$. We sketch the argument.

Let $G_\gamma = G - Z$ and $Y = \{(x_1, x_2) \in \mathbb{R}^2; x_2 \neq 0\}$. Define $\pi: G_\gamma \to Y$ by $\pi(g) = (\text{trace } g, \det g)$; $\pi$ is submersive and each fiber is a conjugacy class in $G$. Let $S = \{(2x_1, x_2^2); x_1 \neq 0\}$. Then we define a function $\psi$ on $Y - S$ by $\psi(\pi(\gamma)) = |D(\lambda)|^{1/2} \Phi_T(\gamma)$, $\gamma \in T_{\text{reg}}$, allowing $T$ to vary among the Cartan subgroups of $G$. If $\psi$ vanishes near $S$, that is, if each $\Phi_T$ vanishes in a neighborhood of $Z$, then it is easy to find $f \in C^\infty_c(G)$ such that $\Phi_T^f = \Phi_T$ for all $T$ (via the coverings $T_{\text{reg}} \times G/T \to T_{\text{reg}}^n$). Suppose now that $\psi$ extends to a smooth function on $Y$. Since $\psi$ has compact support we may apply a partition of unity argument on $Y$ and assume that $\psi$ has support in some neighborhood (to be specified) of a point in $S$.

Fix $a \in S$ and $g \in \pi^{-1}(a)$. We can find a neighborhood $N_1$ of $g$ in $G_\gamma$ with a coordinate system $y_1, \ldots, y_4$ such that $y_1 = x_1 \circ \pi$, $y_2 = x_2 \circ \pi$; we may as well assume that $(y_2)$ maps $N_1$ to a cube in $\mathbb{R}^4$. Set $N_2 = \pi(N_1)$ and assume that $\psi \in C^\infty_c(N_2)$. We lift the form $|x_2|^{-3/2} dx_1 dx_2$ to $N_1$. Using this and the invariant form $\omega_G$ we construct a $G$-invariant measure on each fiber of $\pi$. It is easy to find $f \in C^\infty_c(N_1)$ such that $\int_{\pi^{-1}(a)} f = \psi(x)$, $x \in N_2$. On the other hand, suppose that $\gamma \in N_1 \cap T$ and that $x = \pi(\gamma)$. Then we find that $\int_{\pi^{-1}(x)} h = |D(\lambda)|^{1/2} \Phi_T(\gamma)$, $h \in C^\infty_c(N_1)$. This is a straightforward computation with coordinates. Hence $\psi(x) = |D(\lambda)|^{1/2} \Phi_T(\gamma)$ and our argument is complete in the case that $\psi$ is smooth.

We observe next that $\psi$ extends smoothly to $Y$ when the functions $\mathcal{A}(z, \ )$ attached to the Cartan subgroup $B$ satisfy
\[
\lim_{\theta \to 0; (\theta \to \infty)} \frac{d^n}{d\theta^n} \left( |D(\gamma(\lambda, \theta))|^{1/2} \mathcal{A}(\pm \lambda, \gamma(\lambda, \theta)) \right) = 0
\]
for each $n$, or, more simply, $X_B \mathcal{A}(z', z) \equiv 0$ for each $X$, $z'$ and $z$. As before we suppress $z'$ and write just $X_B \mathcal{A}(z)$. To compute $X_B \mathcal{A}(z)$ in general we resort to rapidly decreasing functions and their orbital integrals.

On $G$, or any real reductive group, we may introduce the space of rapidly decreasing (Schwartz) functions, as defined by Harish-Chandra [2]; [2], together with earlier papers listed there, contains an extensive analysis of the "$F_f$" (normalized orbital integral) transform on this space. If now $f$ is rapidly decreasing on $G$ then $\Phi_T^f$ has the properties listed earlier, except that in place of the statement about the support of $\Phi_T^f$ we have that $|D|^{1/2} \Phi_T^f$ is "rapidly decreasing on $T_{\text{reg}}" [2]. For the characterization we take $\{\Phi_T^f\}$ as before, but allow $|D|^{1/2} \Phi_T$ to be just rapidly decreasing on $T_{\text{reg}}$. The argument is straightforward (since there are many rapidly decreasing functions with computable orbital integrals [2]), but lengthy. Here is our procedure. Consider the rapidly decreasing function $|D|^{1/2} \Phi^A$ on $A$. Using wave-packets [2] we find $f_1$ such that $F_1^f = |D|^{1/2} \Phi^A$; then $\Phi_T^f = \Phi^A$. Consider $\sin \theta \Phi^A(\gamma(\lambda, \theta)) - F_1^f(\gamma(\lambda, \theta))$. From (1) we see that this function is $C^\infty$ on $B$ (and rapidly decreasing as a function of $\lambda$). Then using essentially matrix coefficients of the discrete series representations of $\{x \in G; |\det x| = 1\}$ we find $f_2$ such that $F_2^f = \sin \theta \Phi^B - F_1^f$ and $F_2^f \equiv 0$. If $f = f_1 + f_2$ then $\Phi_T^f = \Phi^T$ for all $T$ (for a
general group this argument characterizes only stable orbital integrals). We refer to [4] for details.

Returning to our original family \( \{ \Phi^T \} \), where \( \Phi^T \) vanishes off some set relatively compact in \( T \), we can find a rapidly decreasing function \( f \) such that \( \Phi^T_\gamma = \Phi^T \). Then \( X_B/\Lambda^0(z) = c_0 Xf(z), z \in Z \). Multiplying \( f \), if necessary, by a suitable function of \( \det \), we may assume that \( \{ x \in R^*; x = \det g, f(g) \neq 0 \} \) is relatively compact in \( R^* \). This allows us to find in \( C_c^\infty(G) \) a function \( f_1 \) which coincides with \( f \) on a neighborhood of \( Z \). Then \( Xf(z) = Xf_1(z) \) for all \( z \) and \( X \), and the function \( \psi \) attached to \( \{ \Phi^T - \Phi^T_\gamma \} \) is smooth on \( Y \). We now argue as earlier and the proof of the characterization is complete.

Finally, fix a quasi-character \( \chi \) on \( Z \). Suppose that \( f \in C^\infty(G) \) satisfies \( f(zg) = \chi(z)f(g) \) for \( z \in Z, g \in G \) and has support compact modulo \( Z \). Then \( \Phi^T_\gamma \) is well defined and has the properties listed earlier, with the necessary modifications concerning support and transformation under \( Z \). To characterize \( \{ \Phi^T_\gamma \} \) we can argue as follows. Let \( \{ \Phi^T \} \) have those properties. We can easily find \( \Phi^T_0, C^\infty \) on \( T_{\text{reg}} \), invariant under the Weyl group, vanishing off a set relatively compact in \( T \), satisfying (4) and such that \( \Phi^T(\gamma) = \int_Z \chi(z^{-1})\Phi^T_0(\gamma) \, dz \), \( \gamma \in T_{\text{reg}} \). For example, we may take \( \Phi^T_0 \) as \( \Phi^T \) multiplied by a suitable function of \( |\det| \). We pick \( f_0 \in C_c^\infty(G) \) such that \( \Phi^T_\gamma = \Phi^T_0 \) for each \( T \). Then \( f \) defined by \( f(g) = \int_Z \chi(z^{-1})f_0(zg) \, dz \), \( g \in G \), satisfies \( \Phi^T_\gamma = \Phi^T \) for each \( T \) and is of the desired form.

REFERENCES


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