ON SPECTRAL FACTORS IN REAL TWISTED ENDOSCOPY

D. SHELSTAD

1. **Introduction**

In the global theory of endoscopy, spectral transfer factors play a role in determining if certain irreducible representations $\pi = \otimes_v \pi_v$ are automorphic. Here we will consider just contributions from the infinite places, but in the general setting of twisted endoscopy [KS]. There is a very regular transfer of orbital integrals, geometric transfer, from a real reductive group to each of its twisted endoscopic groups [S1]. We may begin then to examine coefficients for the dual transfer of stable traces on an endoscopic group to twisted traces on the ambient group: these coefficients are the spectral transfer factors in this setting. First, we limit our attention to the tempered spectrum, and mainly to the discrete series, as it is even here that questions remain from current global theory. For example, in Arthur's forthcoming *Endoscopic Classification of Representations:Orthogonal and Symplectic Groups,* there are precise demands on twisted factors for real special orthogonal groups. Second, we will emphasize a quasi-split setting where Whittaker normalization of transfer factors, both spectral and geometric, is available and promises to simplify the results.

In the case of ordinary real endoscopy and tempered spectrum, our project has already been completed; see [S3] and [S4], also [S7]. It begins with definitions for the spectral transfer factors that parallel the definitions for the geometric factors, together with some algebraic properties that organize and simplify the character analysis needed for well-structured dual spectral identities. Our goal here is to start similarly in a setting where there is twisting by a character on the real points of the group as well as by an $\mathbb{R}$-automorphism of the group. Analysis of twisted characters is at present less developed, and for the steps we do here we need an assumption further to the results of [B1], [BC] (but see remark at the end of Section 8). Mezo [M] has announced a proof of a broad family of character identities using results of Duflo and Bouaziz, but with coefficients intrinsic to that method. With his result in mind, we will show that, under our assumption, if spectral identities do exist then the coefficients must coincide with those we propose. We also introduce Whittaker normalizations, and formulate a strong base point property analogous to that known for ordinary endoscopy [S4]. The results of this paper of course give only an incomplete picture. They do suggest that, as in ordinary endoscopy, it will be possible to recover geometric transfer for a pair of test functions related by tempered spectral transfer.

Twisted character identities for general tempered families in the setting of real twisted endoscopy are to be found in [C] on stable base change for a real reductive group (there is another proof in [B1]) and in [B2] on unstable base change. Here the twisting is by the Galois automorphism as algebraic automorphism of the group obtained by restriction of scalars $\mathbb{C}/\mathbb{R}$. Another approach to stable base change is given in [J1, J2]. See [H] for automorphic induction $\mathbb{C}/\mathbb{R}$. With limited exceptions (mainly abelian cases), these various examples do not arise in our setting until the later inductive step because the ambient group has no discrete series. In regard to compact groups, see [KLP] on a transfer for finite dimensional representations of a simply-connected almost simple complex group and an automorphism that preserves a splitting.

This paper is organized as follows. In Sections 2 - 5 we lay the groundwork for a setting where all sufficiently regular discrete series for a twisted endoscopic group $H_1(\mathbb{R})$ transfer to discrete series for $G(\mathbb{R})$, at least on the level of Langlands parameters. A central notion is that of fundamental splitting as it will allow us to work more easily with elliptic data when dealing with an automorphism that preserves an $\mathbb{R}$-splitting. In Section 4 we pause to describe in more detail what we mean by twist-packets, dual transfer and spectral transfer factors. As in the case of ordinary endoscopy, the geometric-spectral compatibility factor is critical. In Section 5 we describe twisted transfer for Langlands parameters and then apply this to twist-packets. Finally, we mention a type of automorphism ignored for the rest of the paper.

Sections 6 - 10 are concerned with the definitions for spectral transfer factors in the setting we have established, along with some basic properties. The term Δ_I is defined, as usual, in terms of the splitting invariant and compensates for the need to choose, via toral data, a base point in defining the positional term Δ_{III}. We use the same hypercohomology groups to define Δ_{III} as for the geometric factors in [KS]. For the spectral version of Δ_{III} in Section 7 we introduce a construction based on the fundamental splittings of Section 3. This construction often simplifies but we find it helpful to work as generally as possible to avoid later difficulties with central elements. The term Δ_{II} is the most interesting as there is no guidance from the geometric analogue. Following the case of ordinary endoscopy, we define Δ_{II} in terms of local character expansions in Section 8, and we are forced, for the present at least, to introduce an assumption on the form for twisted characters of discrete series representations. The spectral Whittaker normalization is defined at the end of Section 8 after we have introduced geometric-spectral compatibility factors. Sections 9 and 10 have the main technical results.

Section 11 concerns a problem with central elements in norm maps and transfer outside the quasi-split setting, the main point being that a simple convention for normalizing geometric factors ensures that it does not affect spectral transfer factors. There are also some comments on parabolic induction, mainly in response to a question from Mezo. We use Section 12 to review the spectral transfer statement and to check the uniqueness property of our factors. Finally, as another check on definitions, we note a property that implies a crucial sign is correct for Arthur's demands on the structure of real packets in [A].

2. **The cuspidal-elliptic setting**

We consider for our quasi-split data, a triple (G^*, θ^*, a), where G^* is connected, reductive, and quasi-split over $\mathbb{R}$, θ^* is an $\mathbb{R}$-automorphism of G^* that preserves an $\mathbb{R}$-splitting of G^*, and a is a 1-cocycle of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ in the center of the connected dual group $G^\vee$. Then ϖ will denote the character on $G^*(\mathbb{R})$, or on the real points of an inner form of G^*, attached to a. To provide an explicit transition to G^*, or to an inner form of G^*, of data attached to the Langlands dual

${}^{L}G = G^{\vee} \rtimes W_{\mathbb{R}}$, we fix an $\mathbb{R}$-splitting $spl^{*} = (B^{*}, T^{*}, \{X_{\alpha}\})$ of G^{*} preserved by θ^{*} and a Γ-splitting $spl^{\vee} = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^{\vee}}\})$ for the dual group $G^{\vee}$. Here $\Gamma = Gal(\mathbb{C}/\mathbb{R})$; the action of $W_{\mathbb{R}}$ on $G^{\vee}$ factors through $W_{\mathbb{R}} \to \Gamma$. Then $\theta^{\vee}$ is the Γ-automorphism of $G^{\vee}$ preserving $spl^{\vee}$ and dual to θ^{*} as automorphism of the dual based root data. Relative transfer factors will be independent of this choice of splittings, while absolute factors will sometimes be normalized by the choice of spl^{*}. We write ${}^{L}\theta_{a}$ for the extension

$$g \times w \to a(w).\theta^{\vee}(g) \times w$$

of $\theta^{\vee}$ to an automorphism of ${}^{L}G$.

Let $\mathfrak{e} = (H, \mathcal{H}, s)$ be a set of endoscopic data for (G^{*}, θ^{*}, a), and (H_{1}, ξ_{1}) be a z-pair for $\mathfrak{e}$. Here, as in [S1] to which we will refer frequently, we follow the definitions of [KS], although we avoid the additional choice a' from Section 2.1 of [KS] by adjusting s. For the *cuspidal-elliptic* case, we require that G^{*} is cuspidal, *i.e.* a fundamental (minimally $\mathbb{R}$-split) maximal torus defined over $\mathbb{R}$ is $\mathbb{R}$-anisotropic modulo the center $Z_{G^{*}}$ of G^{*}, and that $\mathfrak{e}$ is elliptic in the sense that the identity component of the Γ-invariants in the center of $H^{\vee}$ lies in the center of $G^{\vee}$[KS]. We then call H_{1} an elliptic endoscopic group.

Lemma 2.1. *(i) Assume G^{*} is cuspidal. Then $(G^{*})^{\theta^{*}}$ is cuspidal and there exists a θ^{*}-stable pair (B, T) for G^{*} such that T is defined over $\mathbb{R}$ and anisotropic modulo the center $Z_{G^{*}}$ of G^{*}. (ii) Assume also that $\mathfrak{e}$ is elliptic. Then H_{1} is cuspidal and a fundamental maximal torus T_{1} in H_{1} is a θ^{*}-norm group for each T as in (i).*

Proof. There is no harm, for both (i) and (ii), in assuming that G^{*} is semisimple and simply-connected, so that $I = (G^{*})^{\theta^{*}}$ is connected (as well as reductive). Consider a pair (B^{1}, T^{1}), where T^{1} is a fundamental maximal torus defined over $\mathbb{R}$ in I and B^{1} is any Borel subgroup of I containing T^{1}. Set $T = Cent(T^{1}, G^{*})$ and $B = Norm(B^{1}, G^{*})$, so that (B, T) is a θ^{*}-stable pair for G^{*}. Then T must be fundamental, for otherwise T would have a real root and then a multiple of the restriction of this root to $T^{1} = T^{\theta^{*}}$ would provide us with a real root for T^{1} in I: no such root exists since T^{1} is fundamental. Because G^{*} is cuspidal, (i) follows. For (ii), let T_{H} be a fundamental maximal torus in H. Then there is some admissible isomorphism $T_{H} \to T_{\theta^{*}}$, where T is θ^{*}-admissible, *i.e.* T is defined over $\mathbb{R}$ and is a component of some θ^{*}-stable pair. Attach to H the ordinary endoscopic group J for I as in Section 4.2 of [KS]. Then T^{1} is (isomorphic to) a fundamental maximal torus in J, and moreover J is elliptic because H is. Thus, by the familiar analogue of (ii) from ordinary endoscopy, T^{1} is anisotropic modulo Z_{I}. Since T is then anisotropic modulo $Z_{G^{*}}$ as in (i), T_{H} is anisotropic modulo Z_{H}, and (ii) follows. □

3. Inner twists in the cuspidal-elliptic setting

We continue with the setting of the last section. Consider now an inner twist (G, θ, ψ) of (G^{*}, θ^{*}). Thus G is a connected reductive group over $\mathbb{R}$, $\psi : G \to G^{*}$ is an inner twist, and $\psi \circ \theta \circ \psi^{-1}$ coincides with θ^{*} up to an inner automorphism. An isomorphism of (G, θ, ψ) with an inner twist (G', θ', ψ') is an $\mathbb{R}$-isomorphism $\alpha : G \to G'$ such that $\psi' \circ \alpha$, ψ differ by an inner automorphism of G^{*} and $\alpha^{-1} \circ \theta' \circ \alpha$, θ differ by an inner automorphism induced by an element of $G(\mathbb{R})$ ([KS], Appendix B). We refer to (G^{*}, θ^{*}, id) as the trivial twist.

The group G is cuspidal because G^{*} is. A splitting $(B, T, \{X_{\alpha}\})$ of G will be called *fundamental* if T is anisotropic modulo Z_{G} and for each B-simple root α, the root vector X_α for α is chosen as follows. First we identify the Lie algebra of T as $X_*(T) \otimes \mathbb{C}$ and an element H_α of the Lie algebra of T as the coroot of α. Then we choose both X_α and $X_{-\alpha}$ so that $B(X_\alpha, X_{-\alpha}) = 2/C(\alpha, \alpha)$, where B is the Killing form and C is the canonical form on $X^*(T) \otimes \mathbb{Q}$, to arrange that $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ is a simple triple. Here, and throughout, we will use σ to denote the action of the nontrivial element of the Galois group. Then $\sigma H_\alpha = -H_\alpha$, $\sigma X_\alpha = \lambda X_{-\alpha}$ and $\sigma X_{-\alpha} = \lambda^{-1} X_{-\alpha}$, where λ is real. So far we have placed no restriction on X_α. For a fundamental splitting $(B, T, \{X_\alpha\})$ we require that $\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$ and $\sigma X_{-\alpha} = \varepsilon_\alpha X_\alpha$, where $\varepsilon_\alpha = \pm 1$. The triple $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ determines homomorphisms over $\mathbb{R}$ from a real form of $SL(2)$ into G (examples are given in [S6]). Then we see that $\varepsilon_\alpha = 1$ if and only if α is noncompact, *i.e.* the triple determines a homomorphism over $\mathbb{R}$ from $SL(2)$ itself. Clearly a fundamental splitting exists for any pair (B, T) with T anisotropic modulo Z_G (we may adjust λ above by a square). Moreover, any two such splittings with same pair (B, T) are conjugate under $T_{sc}(\mathbb{R})$. To check this, we first notice that they are conjugate under $T_{ad}(\mathbb{R})$ since if X_α is replaced by $Int(t)X_\alpha$ for each B-simple α then our requirements on the action of σ imply that $\alpha(\sigma(t)t^{-1}) = 1$. Because T_{sc}, T_{ad} are anisotropic over $\mathbb{R}$, $T_{sc}(\mathbb{R})$ and $T_{ad}(\mathbb{R})$ are connected and thus the projection $T_{sc}(\mathbb{R}) \to T_{ad}(\mathbb{R})$ is surjective, so that the desired conjugation exists. For any fundamental splitting we also have the following.

Lemma 3.1. *Suppose that G is a cuspidal connected reductive group defined over $\mathbb{R}$ and that θ is an automorphism of G preserving a fundamental splitting. Suppose that the restriction of θ to Z_G is defined over $\mathbb{R}$. Then θ is defined over $\mathbb{R}$.*

Proof. Suppose θ preserves the fundamental splitting $(B, T, \{X_\alpha\})$. Then since σ acts by $-I$ on $X^*(T_{der})$ it is clear that the restriction of θ to T is defined over $\mathbb{R}$. Next we observe that, by our construction of X_α and $X_{-\alpha}$, we have that $\theta X_{-\alpha} = X_{-\theta\alpha}$ for all B-simple roots α as well as the assumed $\theta X_\alpha = X_{\theta\alpha}$. We conclude that $\sigma(\theta)$ agrees with θ on both X_α and $X_{-\alpha}$, for all B-simple roots α, and the lemma follows. $\square$

Lemma 3.2. *Suppose that G is a cuspidal connected reductive group defined over $\mathbb{R}$, that θ is an $\mathbb{R}$-automorphism of G, and that spl_f is a fundamental splitting of G. Then θ preserves spl_f up to an inner automorphism defined over $\mathbb{R}$, i.e.*

$$\theta = Int(g) \circ \theta_f,$$

where θ_f preserves spl_f and $g \in G$ is such that $g^{-1}\sigma(g)$ is central.

Proof. We may choose $g \in G$ so that $Int(g)(spl_f) = \theta(spl_f)$. Then $\theta_f = Int(g)^{-1} \circ \theta$ preserves spl_f and agrees with θ on Z_G. So θ_f is defined over $\mathbb{R}$. Then $Int(g)$ is also defined over $\mathbb{R}$. $\square$

We return to the automorphism θ^* of G^* provided by quasi-split data. Recall that θ^* fixes the $\mathbb{R}$-splitting spl^*. Let (B, T) be a θ^*-stable pair for G^* with T anisotropic modulo Z_{G^*}, and let $spl_f = (B, T, ...)$ be a fundamental splitting. Suppose $Int(h)$ maps spl_f to spl^*, where $h \in G^*_{sc}$. Then $t^* = \theta^*_{sc}(h^{-1})h$ lies in T_{sc} since the θ^*-stable pairs (B^*, T^*) and (B, T) are conjugate under $(G^*_{sc})^{\theta^*_{sc}}$. We write

$$\theta^* = Int(t^*) \circ \theta^*_f,$$

where

$$\theta^*_f = Int(h) \circ \theta^* \circ Int(h)^{-1}$$

preserves spl_f. Then θ^*_f is defined over $\mathbb{R}$. Thus the automorphism $Int(t^*)$ is also defined over $\mathbb{R}$. We may then adjust h and assume that t^* lies in $T_{sc}(\mathbb{R})$.

Lemma 3.3. *Suppose (B,T) is a θ^*-stable pair for G^* with T anisotropic modulo Z_{G^*}. Then the automorphism θ^* preserves a fundamental splitting $(B,T,\{X_\alpha\})$.*

Proof. We just have to show that we may choose the element t^* (from the last paragraph) in $(1-\theta^*_{sc})[T_{sc}(\mathbb{R})]$: if $t^* = u\ \theta^*_{sc}(u)^{-1}$ then θ^* preserves the fundamental splitting $Int(u)(spl_f)$. First, the natural projection $T_{sc} \to T_{ad}$ restricts to a surjective map $(T_{sc})^{\theta^*_{sc}} \to (T_{ad})^{\theta^*_{ad}}$ and then it follows from the snake lemma that $(1-\theta^*_{sc})Z_{G^*_{sc}}$ is the kernel of $(1-\theta^*_{sc})T_{sc} \to (1-\theta^*_{ad})T_{ad}$. This last map is surjective on real points since $(1-\theta^*_{sc})T_{sc}$, $(1-\theta^*_{ad})T_{ad}$ are anisotropic over $\mathbb{R}$. Now, since t^* lies in $(1-\theta^*_{sc})T_{sc}$ and its image in $(1-\theta^*_{ad})T_{ad}$ is real, it is clear that $t^*(1-\theta^*_{sc})Z_{G^*_{sc}}$ contains a real point of $(1-\theta^*_{sc})T_{sc}$, *i.e.* a point of $(1-\theta^*_{sc})[T_{sc}(\mathbb{R})]$ since T_{sc} is anisotropic, and we are done. ☐

Corollary 3.4. *We may adjust the inner twist ψ within its inner class so that the automorphism θ_f of Lemma 3.2 coincides with $\psi^{-1} \circ \theta^* \circ \psi$.*

We continue with an inner twist (G,θ,ψ) of (G^*,θ^*). Norm correspondences will be discussed in some detail in Section 11. Meanwhile we will work with data that determine such a correspondence. As in Section 3.1 of [KS], we choose elements $g_\theta, u(\sigma)$ of G^*_{sc} such that

$$\theta^* = Int(g_\theta) \circ \psi \circ \theta \circ \psi^{-1}$$

and

$$\psi\sigma(\psi)^{-1} = Int(u(\sigma)).$$

The cochain

$$z_\sigma = g_\theta.u(\sigma).\sigma(g_\theta)^{-1}.\theta^*_{sc}(u(\sigma))^{-1}$$

in $Z_{G^*_{sc}}$ attached to $g_\theta, u(\sigma)$ governs rationality questions for norm maps (see Lemma 3.1.A of [KS]).

Lemma 3.5. *Suppose (G,θ,ψ) is such that θ preserves a fundamental splitting. Then we may adjust ψ within its inner class to arrange that $g_\theta = 1$ and $u(\sigma)$ is fixed by θ^*_{sc}, so that $z_\sigma = 1$.*

Proof. Suppose θ preserves $spl_{f,G} = (B_G, T_G, \{X_\alpha\})$. Let (B,T) be a θ^*-stable pair in G^* with T anisotropic modulo Z_{G^*}. We may adjust ψ within its inner class to assume that ψ maps (B_G,T_G) to (B,T). Note that the restriction of ψ to T_G is then defined over $\mathbb{R}$: the restriction to Z_G is defined over $\mathbb{R}$ since ψ is an inner twist, and the restriction to $(T_G)_{der}$ is defined over $\mathbb{R}$ by an argument with rational characters since $(T_G)_{der}, T_{der}$ are anisotropic. Choose a fundamental splitting $spl_f = (B,T,...)$ in G^* preserved by θ^* and further adjust ψ to assume

$$spl_f = \psi(spl_{f,G}).$$

We choose $u(\sigma)$ in T_{sc} such that

$$\psi\sigma(\psi)^{-1} = Int(u(\sigma)).$$

Since both $\psi \circ \theta \circ \psi^{-1}$ and θ^* preserve spl_f and differ by an inner automorphism they must coincide. So we may take $g_\theta = 1$. On the other hand, the image of $u(\sigma)$ in G^*_{ad} is fixed by θ^*_{ad} (see Lemma 11.1). Since $(T_{sc})^{\theta^*_{sc}} \to (T_{ad})^{\theta^*_{ad}}$ is surjective we may adjust $u(\sigma)$ harmlessly by an element of $Z_{G^*_{sc}}$ to arrange also that $u(\sigma)$ is fixed by θ^*_{sc}. Then $z_\sigma = 1$, as desired. $\square$

Corollary 3.6. *With the choices of Lemma 3.3, the norm correspondence attached to*

$$m : g \to \psi(g).(g_\theta)^{-1} = \psi(g)$$

carries strongly θ-regular stable θ-conjugacy classes in $G(\mathbb{R})$ to strongly G-regular stable conjugacy classes in $H_1(\mathbb{R})$.

We will find it useful to broaden the setting of Corollary 3.5, as follows. Returning to the proof of the last lemma, we change notation to set

$$\theta_f = \psi^{-1} \circ \theta^* \circ \psi,$$

and now consider

$$\theta = Int(g) \circ \theta_f,$$

where $g^{-1}\sigma(g) \in Z_G(\mathbb{R})$. We leave ψ and $u(\sigma)$ unchanged. Choose $g_{sc} \in G_{sc}$ with $Int(g_{sc}) = Int(g)$. Denote by $\mathbf{z}_{sc}$ the element of $H^1(\Gamma, Z_{G_{sc}})$ defined by $g_{sc}^{-1}\sigma(g_{sc}) \in Z_{G_{sc}}$. It is independent of the choice for g_{sc}.

We now assume that g may be chosen in $G(\mathbb{R})$ or, equivalently, that the image of $\mathbf{z}_{sc}$ under $H^1(\Gamma, Z_{G_{sc}}) \to H^1(\Gamma, Z_G)$ is trivial. Following the definition of Appendix B of [KS], this amounts to requiring that (G, θ, ψ) be isomorphic to (G, θ_f, ψ).

We have arranged that the cochain z_σ for θ_f is 1. Rewrite this statement as $z_\sigma(\theta_f) = 1$. We calculate a cochain $z_\sigma(\theta)$ by replacing $g_{\theta_f} = 1$ with $g_\theta = \psi_{sc}(g_{sc})^{-1}$. We have chosen $u(\sigma)$ in T_{sc} and fixed by θ^*_{sc}. Thus

$$z_\sigma(\theta) = \psi_{sc}(g_{sc})^{-1}u(\sigma)\ \sigma(\psi_{sc}(g_{sc}))(u(\sigma))^{-1} = \psi_{sc}(g_{sc}^{-1}\sigma(g_{sc})).$$

Then $z_\sigma(\theta)$ is a cocycle representing $\mathbf{z}_{sc}$, or more precisely, representing the image of $\mathbf{z}_{sc}$ in $H^1(\Gamma, Z_{G^*_{sc}})$ under the isomorphism induced by ψ_{sc} as $\mathbb{R}$-isomorphism from $Z_{G_{sc}}$ to $Z_{G^*_{sc}}$. Since $\mathbf{z}_{sc}$ maps to the identity under

$$H^1(\Gamma, Z_{G^*_{sc}}) \to H^1(\Gamma, Z_G) \to H^1(\Gamma, (Z_{G^*})_{\theta^*}),$$

we may take θ_H and θ_{H_1} in Section 5.4 of [KS] to be the identity automorphisms of H and H_1 respectively, so that the assertion of Corollary 3.5 remains true. In Section 11, we will discuss norm maps in more detail. All we will need until then is that the cocycle $z_\sigma(\theta)$ appears in the computation of $\sigma(m)(x)$ for $x \in G(\mathbb{R})$:

$$\sigma(m)(x) = z_\sigma(\theta).u(\sigma)^{-1}m(x)\theta^*_{sc}(u(\sigma)) = z_\sigma(\theta).u(\sigma)^{-1}m(x)u(\sigma).$$

Notice that here we have identified $z_\sigma(\theta)$ with its image in Z_{G^*}.

4. **Geometric factors and dual transfer**

In this section we introduce notation and include some preparation for the constructions of the next several sections. We recall the main theorem of [S1] for geometric transfer just in the case that the norm correspondence involves no twisting in the endoscopic group H_1. There is also the minor assumption that θ is strongly semisimple on Z_G (see Section 6 of [S1]) and we require harmlessly that the character ϖ on $G(\mathbb{R})$ is trivial on θ-invariants in $Z_G(\mathbb{R})$. Test functions are Harish Chandra Schwartz functions: we consider functions $f \in \mathcal{C}(G(\mathbb{R}), \theta, \varpi)$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$, as in Section 1 of [S1]. We could also use $C_c^\infty(G(\mathbb{R}), \theta, \varpi)$ and $C_c^\infty(H_1(\mathbb{R}), \varpi_1)$. Measures and integrals will be defined and normalized as there, also. To be more careful, we should use measures in place of functions throughout in order to have transfer depend only on the normalization of transfer factors, but this will be ignored here (see instead the note [S7]). The geometric transfer factors $\Delta(\gamma_1, \delta)$, where γ_1 is strongly G-regular in $H_1(\mathbb{R})$ and δ is strongly θ-regular in $G(\mathbb{R})$, are from [KS]. Then Theorem 2.1 of [S1] provides us with very regular geometric transfer: for all $f \in \mathcal{C}(G(\mathbb{R}), \theta, \varpi)$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ such that

$$SO(\gamma_1, f_1) = \sum_{\delta,\ \theta\text{-}conj} \Delta(\gamma_1, \delta) O^{\theta,\varpi}(\delta, f)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. We write $f_1 \in Trans_{\theta,\varpi}(f)$.

Suppose π_1 is a tempered irreducible admissible representation of $H_1(\mathbb{R})$ and Π_1 is its packet. We will assume, usually without further mention, that the subgroup $Z_1(\mathbb{R})$ acts by the character ϖ_1, where Z_1 is the central torus $Ker(H_1 \to H)$. Let $St\text{-}Tr\ \pi_1$ be the stable tempered distribution

$$f_1 \to \sum_{\pi_1' \in \Pi_1} Trace\ \pi_1'(f_1).$$

Because $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ we have taken $\pi_1(f_1)$ as the operator

$$\int_{H_1(\mathbb{R})/Z_1(\mathbb{R})} f_1(h_1)\ \pi_1(h_1) \frac{dh_1}{dz_1}.$$

Following the case of ordinary endoscopy, we consider the linear form

$$f \to St\text{-}Tr\ \pi_1(f_1)$$

on $\mathcal{C}(G(\mathbb{R}), \theta, \varpi)$, where f_1 is attached to f by the very regular geometric transfer. It is well-defined by Lemma 5.3 of [S6] and, according to general principles for twisted endoscopy (see [KS]), it should be expressed as a linear combination of twisted traces of representations in the packet with matching Langlands parameter. We first describe how we specify twisted traces.

Let π be a tempered irreducible admissible representation of $G(\mathbb{R})$ and Π denote its packet. We use the same notation for a representation and its isomorphism class; we may also work with unitary representations on Hilbert space and unitary isomorphisms. If the Langlands parameter of Π matches that of a packet Π_1 for $H_1(\mathbb{R})$ with correct behavior on $Z_1(\mathbb{R})$ (in a sense analogous to that for ordinary endoscopy [S3], see next section) then the construction of endoscopic data ensures that the packet Π is preserved under the map $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$. This last property is a simple condition on the Langlands parameter of Π, as we will recall in the next section. Whenever it is satisfied we call the attached packet (θ, ϖ)-*stable.* Thus we may define a twisted trace on $\oplus_{\pi' \in \Pi}\ \pi'$. Only those π' fixed by the map will contribute nontrivially. We then define $\Pi^{\theta,\varpi}$ to be the (possibly empty) subset of Π consisting of such π'. We will call $\Pi^{\theta,\varpi}$ a *twist-packet* for (θ, ϖ).

Suppose π belongs to the twist-packet $\Pi^{\theta,\varpi}$ and that the unitary operator $\pi(\theta, \varpi)$ on the space of π interwines $\pi \circ \theta$ and $\varpi \otimes \pi$ or, more precisely, that

$$\pi(\theta(g)) \circ \pi(\theta, \varpi) = \varpi(g).(\pi(\theta, \varpi) \circ \pi(g)),$$

for $g \in G(\mathbb{R})$. Then by the twisted trace of π we mean the linear form

$$f \to Trace\ \pi(f) \circ \pi(\theta, \varpi)$$

on $\mathcal{C}(G(\mathbb{R}),\theta,\varpi)$, where $\pi(f)$ is the operator

$$\int_{G(\mathbb{R})/(1-\theta)(Z(\mathbb{R}))} f(g)\pi(g)\frac{dg}{dz}.$$

We will discuss analysis of this form in Section 8. Notice that if f is replaced by $g\to f(xg\theta(x)^{-1})$ then $Trace\ \pi(f)\pi(\theta,\varpi)$ is multiplied by $\varpi(x)$, $x\in G(\mathbb{R})$, as is necessary for the following transfer statement to be well-defined.

Recall that (f_1,f) is a matching pair of test functions from geometric transfer. Spectral transfer factors will be nonzero complex coefficients $\Delta(\pi_1,\pi)$ such that

$$St\text{-}Tr\ \pi_1(f_1)=\textstyle\sum_{\pi\in\Pi^{\theta,\varpi}}\Delta(\pi_1,\pi)Trace\ \pi(f)\pi(\theta,\varpi),$$

provided, of course, that they exist. Uniqueness of the coefficients will be a consequence of linear independence for twisted characters. We may change the summation to $\sum_{\pi,temp}$ over all tempered irreducible admissible representations π of $G(\mathbb{R})$ by setting $\Delta(\pi_1,\pi)=0$ if $\pi\notin\Pi^{\theta,\varpi}$. In particular, if $\Pi^{\theta,\varpi}$ is empty then the right side is zero.

The factors $\Delta(\pi_1,\pi)$ depend on how we normalize the geometric factors $\Delta(\gamma_1,\delta)$ that prescribe the correspondence (f,f_1). Following the case of ordinary endoscopy, we will introduce a compatibility factor. For ordinary endoscopy this factor was canonical. Now there is a new dependence: the choice of normalization for the operators $\pi(\theta,\varpi)$, $\pi\in\Pi^{\theta,\varpi}$. We may multiply $\pi(\theta,\varpi)$ by a nonzero complex number λ (of absolute value one since we have required unitarity). In ordinary endoscopy, the term Δ_{II} in $\Delta(\pi_1,\pi)$ comes from the explicit local representation of $f\to Trace\ \pi(f)$ around the identity. In the twisted case, we will consider a similar term from the assumed explicit local representation of $f\to Trace\ \pi(f)\pi(\theta,\varpi)$ around a certain point, in general, not the identity element. It will be immediate then that multiplying $\pi(\theta,\varpi)$ by λ has the effect of dividing Δ_{II} by λ. No other term in $\Delta(\pi_1,\pi)$ will depend on $\pi(\theta,\varpi)$, and so we conclude that

$$\Delta(\pi_1,\pi)Trace\ \pi(f)\pi(\theta,\varpi)$$

is independent of the choice for $\pi(\theta,\varpi)$. Now the compatibility factor

$$\Delta(\pi_1,\pi;\gamma_1,\delta)$$

will depend on $\pi(\theta,\varpi)$, but the quotient

$$\Delta(\pi_1,\pi)/\Delta(\pi_1,\pi;\gamma_1,\delta)$$

will not, and so we may prescribe compatibility following the ordinary case. We will return to the precise issues for Δ_{II} at the end of Section 8, and to compatibility factors in Section 10.

5. **Regular elliptic related pairs**

Regular elliptic Langlands parameters are attached to packets of discrete series representations. Matching parameters of this type for $H_1(\mathbb{R})$ and $G(\mathbb{R})$ occur if and only if G is cuspidal and H_1 is elliptic. First, in the general setting we may define *matching* tempered parameters as in Section 2 of [S3] for the ordinary case: the arguments there, and accompanying definitions, apply word for word (apart from the shift in notation to ϖ_1 for the character on the central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$). Notice that we have used *tempered* rather than *essentially tempered* throughout. This is simply to avoid another layer of notation: all our results generalize immediately to the essentially tempered setting.

We assume then, harmlessly and without further mention, that a, ξ_1 are of unitary type, so that ϖ, ϖ_1 are unitary characters. Recall that $\mathfrak{e} = (H, \mathcal{H}, s)$, the embedding ξ of [S3] being inclusion and dropped from notation, and (H_1, ξ_1) is an attached z-pair. Now following Section 2 of [S3], $\Phi_{temp}(H_1, \varpi_1)$ consists of the Langlands parameters for tempered irreducible admissible representations for which $Z_1(\mathbb{R})$ acts by ϖ_1. Each such parameter is represented by a homomorphism $\varphi_1 : W_{\mathbb{R}} \longrightarrow {}^L H_1$ with image in $\xi_1(\mathcal{H})$. Then the homomorphism $\varphi = \xi_1^{-1} \circ \varphi_1$ represents a tempered parameter for G, *i.e.* an element of $\Phi_{temp}(G)$, provided it is relevant to G. In that case we call φ_1 and φ *matching parameters.* Then if π_1 belongs to the packet Π_1 attached to φ_1 and π belongs to the subset $\Pi^{\theta,\varpi}$ of the packet Π attached to φ we call (π_1, π) a *related pair.*

We return to the cuspidal-elliptic setting of Sections 2 and 3. Thus (G^*, θ^*, a) is cuspidal quasi-split data, $\mathfrak{e}$ is a set of elliptic endoscopic data for (G^*, θ^*, a) with attached z-pair (H_1, ξ_1). A tempered packet Π_1 for $H_1(\mathbb{R})$ consists of discrete series representations if and only its parameter φ_1 is regular elliptic. If φ_1 satisfies the stronger requirement of G-regularity [S3] then there is a matching parameter φ which is also regular elliptic, and we call any (π_1, π), where $\pi_1 \in \Pi_1$ and $\pi \in \Pi^{\theta,\varpi}$, a *$G$-regular elliptic related pair.* We recall explicit data for such pairs.

A Γ-splitting $spl_{G^\vee} = (\mathcal{B}, \mathcal{T}, \{X\})$ has been fixed. To this we attach a Γ-splitting $spl_{G^\vee}^{\theta^\vee}$ for the identity component of $(G^\vee)^{\theta^\vee}$ in the standard manner (see, for example, p.61 of [KS]). We adjust $\mathfrak{e} = (H, \mathcal{H}, s)$ within its isomorphism class so that $s \in \mathcal{T}$, and then choose Γ-splitting $spl_{H^\vee} = (\mathcal{B}_{H^\vee}, \mathcal{T}_{H^\vee}, \{Y\})$, where $\mathcal{B}_{H^\vee} = \mathcal{B}_{H^\vee} \cap H^\vee$ and $\mathcal{T}_{H^\vee} = \mathcal{T} \cap H^\vee = (\mathcal{T}^{\theta^\vee})^0$. Embed $H^\vee$ in $H_1^\vee$ and extend $spl_{H^\vee}$ to $spl_{H_1^\vee} = (\mathcal{B}_1, \mathcal{T}_1, \{Y\})$ by taking $\mathcal{B}_1 = Norm(\mathcal{T}_{H^\vee}, H_1^\vee)$ and $\mathcal{T}_1 = Cent(\mathcal{T}_{H^\vee}, H_1^\vee)$. None of these choices will matter for transfer factors. Nor will the choice of χ-data (this choice does matter for geometric Δ_{II} and Δ_{III}). We will thus define all Langlands data μ_1, λ_1, *etc.* for packets in familiar terms [L]; this amounts to the choice of χ-data auch that $\chi_{(\alpha^\vee)_{res}} = (\frac{z}{\bar{z}})^{1/2}$, where $(\alpha^\vee)_{res}$ denotes the restriction to $(\mathcal{T}^{\theta^\vee})^0$ of the root $\alpha^\vee$ of $\mathcal{T}$ in $\mathcal{B}$. Then we will note at the appropriate places in Sections 7 and 8 why the choice is harmless.

We follow the approach of Section 11 of [S2] for ordinary endoscopy. To $spl_{H^\vee}$ we attach the representative $\varphi_1 = \varphi(\mu_1, \lambda_1)$ as in the ordinary case. The element $\varphi_1(1 \times \sigma)$ of ${}^L H_1$ acts on

$$\mathcal{T}_H \cap H_{der}^\vee = \mathcal{T}_1 \cap H_{der}^\vee$$

as $t \rightarrow t^{-1}$. We then set

$$\varphi_1(1 \times \sigma) = e^{2\pi i \lambda_1}.n_H(\sigma) \times (1 \times \sigma)$$

and

$$\varphi_1(z \times 1) = z^{\mu_1}.\bar{z}^{\varphi_1(1 \times \sigma)\mu_1} \times (z \times 1),$$

where $n_H(\sigma)$ is attached to the root vectors in $spl_{H^\vee}$ in the usual manner. Also

$$(\mu_1, \lambda_1) \in (X_*(\mathcal{T}_1) \otimes \mathbb{C})^2,$$

μ_1 is strictly $\mathcal{B}_1$-dominant, and

$$\frac{1}{2}(\mu_1 - \varphi_1(1 \times \sigma)\mu_1) - \iota_1 + (\lambda_1 + \varphi_1(1 \times \sigma)\lambda_1) \in X_*(\mathcal{T}_1),$$

where ι_1 is one-half the sum of the coroots of $\mathcal{T}_1$ in $\mathcal{B}_1$. Then μ_1 is uniquely determined, while λ_1 is uniquely determined modulo the sum of $X_*(\mathcal{T}_1)$ and the (-1)-eigenspace of $\varphi_1(1 \times \sigma)$.

Assume that $\varphi_1 = \varphi(\mu_1, \lambda_1)$ and φ are matching parameters. To fix an element of ${}^L G$ acting on $\mathcal{T} \cap G^\vee_{der}$ as $t \to t^{-1}$, we may replace $n_H(\sigma) \times (1 \times \sigma)$ by either $n_G(\sigma) \times (1 \times \sigma)$ defined relative to $spl_{G^\vee}$ or $n_{G,\theta}(\sigma) \times (1 \times \sigma)$ defined relative to $spl^{\theta^\vee}_{G^\vee}$ without affecting the conjugacy class of φ. It is more convenient to choose the latter. Then $\varphi = \varphi(\mu, \lambda)$ means that

$$\varphi(1 \times \sigma) = e^{2\pi i \lambda}.n_{G,\theta}(\sigma) \times (1 \times \sigma)$$

and

$$\varphi(z \times 1) = z^{\mu}.\overline{z}^{\varphi(1\times\sigma)\mu} \times (z \times 1),$$

etc.. In particular,

$$(\mu, \lambda) \in (X_*(\mathcal{T}) \otimes \mathbb{C})^2.$$

We have assumed, without harm, that μ_1 is $\mathcal{B}_1$-dominant. While G-regularity requires that μ be regular, $\mathcal{B}_1$-dominance of μ_1 does not ensure that μ is $\mathcal{B}$-dominant. That case, however, is the only one that matters to us here (in general, an extra sign is needed in transfer factors, see Sections 7, 9 of [S3]). Thus we call φ_1 *well-positioned relative to* φ if μ is $\mathcal{B}$-dominant, and make that our assumption throughout (given φ we can always find such φ_1 and it is unique). It is not difficult to check that this notion is independent of the choices made in its formulation (again see [S3]).

We note now necessary (and sufficient) conditions on (μ_1, λ_1) and (μ, λ) from (for) the matching of $\varphi_1(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$. First pick $u(w) \in \mathcal{H}$ projecting to w, as follows. For $w = z \times \sigma$, $u(w)$ is to act on $\mathcal{T}_H$ and $\mathcal{T}_1$ as $n_H(\sigma) \times (1 \times \sigma) \in {}^L H$. For $w = z \times 1$, $u(w)$ is to act trivially. Since ξ_1 embeds $\mathcal{H}$ in ${}^L H_1$ we may define

$$\xi_1(u(z \times \sigma)) = t_{\xi_1}(z \times \sigma).n_H(\sigma) \times (1 \times \sigma)$$

and

$$\xi_1(u(z \times 1)) = t_{\xi_1}(z \times 1) \times (z \times 1),$$

where each $t_{\xi_1}(w)$ lies in $\mathcal{T}_1$. On the other hand, in ${}^L G$ we have that $u(1 \times \sigma)$ acts as $n_{G,\theta}(\sigma) \times (1 \times \sigma)$. Write

$$u(w) = t(w).u'(w),$$

where

$$u'(z \times \sigma) = n_{G,\theta}(\sigma) \times (1 \times \sigma)$$

and

$$u'(z \times 1) = 1 \times (z \times 1),$$

so that $t(w) \in \mathcal{T}$. Let

$$\mathcal{T}_2 = \mathcal{T}_1 \times \mathcal{T}/\mathcal{T}_H,$$

where $\mathcal{T}_H$ is embedded by $t \to (t^{-1}, t)$. On $\mathcal{T}_2$ we use the elliptic action σ_2 of Γ inflated to $W_{\mathbb{R}}$: σ_2 acts as $n_H(\sigma) \times (1 \times \sigma)$ on the first component and as $n_{G,\theta}(\sigma) \times (1 \times \sigma)$ on the second. Let $t_2(w)$ be the image in $\mathcal{T}_2$ of $(t_{\xi_1}(w)^{-1}, t(w)) \in \mathcal{T}_1 \times \mathcal{T}$, and define

$$(\mu^*, \lambda^*) \in (X_*(\mathcal{T}_2) \otimes \mathbb{C})^2$$

by

$$t_2(z \times 1) = z^{\mu^*}.\overline{z}^{\sigma_2 \mu^*} \times (z \times 1)$$

and

$$t_2(1 \times \sigma) = e^{2\pi i \lambda^*} \times (1 \times \sigma).$$

Notice that we have constructed (μ^*, λ^*) independently of φ_1, φ. The cochain $t_2(w)$ is *not* the cocycle $a_T(w)$; $a_T(w)$ requires a ρ-shift (ι-shift in our notation) be applied to the datum μ^*. See Section 11 of [S2] for the same approach to ordinary endoscopy, the torus $\mathcal{T}_2$ collapsing there to $\mathcal{T}_1$.

Recall that $\varphi_1(W_{\mathbb{R}})$ is assumed to lie in $\xi_1(\mathcal{H})$. Identify μ_1, μ with their images in $X_*(\mathcal{T}_2) \otimes \mathbb{C}$ under the componentwise embeddings. Since we may write

$$\varphi_1(w) = t_H(w).\xi_1(u(w))$$

and

$$\varphi(w) = t_H(w).t(w).u'(w),$$

where $t_H(w) \in \mathcal{T}_H$, we conclude that $\mu_1 + \mu^* = \mu$ and that we may take $\lambda_1 + \lambda^*$ for λ.

So far we have considered parameters φ_1 for H_1 and their transport φ to G. By design, these transports provide all ${}^L\theta_a$-stable parameters for G as we vary the elliptic sets of endoscopic data. Now we write down the conditions on (μ, λ) entailed by ${}^L\theta_a$-stability of a regular elliptic parameter $\varphi = \varphi(\mu, \lambda)$.

First, the discrete series packet attached to φ is preserved by (θ, ϖ) if and only if

$$S_\varphi^{tw} = \{s \in G^\vee : \ {}^L\theta_a \circ \varphi = Int(s^{-1}) \circ \varphi\}$$

is a nonempty subset of $\mathcal{T}$. To justify the use of $\mathcal{T}$ in place of the usual $G^\vee$ [KS], we note that the definition of regular elliptic parameters requires that s act as an element of the Weyl group of $\mathcal{T}$ and then that s preserve $\mathcal{B}$, so that $s \in \mathcal{T}$. For the action of $\theta^\vee$ on (μ, λ), the equations

$$a(z \times 1)^{-1}\theta^\vee(\varphi(z \times 1)) = \varphi(z \times 1), \ z \in \mathbb{C}^\times,$$

and

$$a(1 \times \sigma)^{-1}.e^{2\pi i(\theta^\vee\lambda - \lambda)} = s^{-1}.\varphi(1 \times \sigma).s$$

imply that

$$\theta^\vee\mu = \mu + \mu_\varpi$$

and that

$$\theta^\vee\lambda - \lambda - \lambda_\varpi \in X_*(\mathcal{T}) + [1 - \varphi(1 \times \sigma)]X_*(\mathcal{T}) \otimes \mathbb{C},$$

where $(\mu_\varpi, \lambda_\varpi)$ are elliptic data (in the obvious sense) for the twisting character ϖ on $G(\mathbb{R})$.

We turn now to calculations with data for G. Suppose that π is a discrete series representation of $G(\mathbb{R})$. By the Harish Chandra characterization of these representations we may associate to π a pair (B_π, T_π), where T_π is anisotropic modulo Z_G. First we choose any T_π anisotropic modulo Z_G. Then B_π is chosen so that μ_π is B_π-dominant, where $(\mu_\pi, \lambda_\pi) \in (X^*(T_\pi) \otimes \mathbb{C})^2$ is character data for π (see Section 7b of [S3] for notation). The pair (B_π, T_π) is unique up to $G(\mathbb{R})$-conjugacy.

We will supplement (B_π, T_π) by the choice of root vectors for a fundamental splitting $spl_\pi = (B_\pi, T_\pi, \{X_\alpha\})$. This splitting is also unique up to $G(\mathbb{R})$-conjugacy. Write $x.spl_\pi$ for the (fundamental) splitting $Int(x)(spl_\pi)$, $x \in G(\mathbb{R})$.

Lemma 5.1. *Suppose that π is a discrete series representation of $G(\mathbb{R})$ such that $\pi\circ\theta\approx\varpi\otimes\pi$. Then there exists $\delta_\pi\in G(\mathbb{R})$ such that $Int(\delta_\pi)\circ\theta$ preserves spl_π. If spl_π is replaced by the fundamental splitting $x.spl_\pi$, where $x\in G(\mathbb{R})$, then δ_π is replaced by an element δ'_π of the form $zx\delta_\pi\theta(x)^{-1}$, where $z\in Z_G(\mathbb{R})$.*

Proof. Since $(\theta(B_\pi),\theta(T_\pi))$ serves as pair for $\pi\circ\theta$ and (B_π,T_π) as pair for $\varpi\otimes\pi$, the existence of δ_π is clear. Now, with spl_π fixed, δ_π may be replaced only by an element of $Z_G(\mathbb{R})\delta_\pi$. Next replace spl_π by $x.spl_\pi$, where $x\in G(\mathbb{R})$. Then

$$Int(x)\circ(Int(\delta_\pi)\circ\theta)\circ Int(x)^{-1}=Int(x\delta_\pi\theta(x)^{-1})\circ\theta$$

preserves $x.spl_\pi$, and the lemma follows. □

In Section 7 we will show that δ_π has central norm. More precisely, the following is true.

Lemma 5.2. *(i) An element $\delta_\pi\in G(\mathbb{R})$ from Lemma 5.1 has a norm γ_π in $H_1(\mathbb{R})$. (ii) This element γ_π lies in $Z_{H_1}(\mathbb{R})$ and its image in $Z_H(\mathbb{R})$ is uniquely determined by δ_π. (iii) If δ_π is replaced by $\delta'_\pi=z\delta_\pi$, where $z\in Z_G(\mathbb{R})$, then γ_π is replaced by an element $\gamma'_\pi=z_1\gamma_\pi$, where the image of z_1 under $Z_{H_1}(\mathbb{R})\to Z_H(\mathbb{R})$ coincides with the image of z under*

$$Z_G(\mathbb{R})\xrightarrow{\psi}Z_{G^*}(\mathbb{R})\longrightarrow Z_H(\mathbb{R}).$$

For central subgroups we will often drop the twist ψ, or ψ_{sc}, in notation. In particular, we will identify Z_G with Z_{G^*}. The next result will be useful in later sections.

Lemma 5.3. *Suppose that θ preserves a fundamental splitting in G and that Π is a (θ,ϖ)-stable discrete series packet. Then (i) $\Pi^{\theta,\varpi}$ is nonempty, and (ii) we may take $\delta_\pi=1$ for each $\pi\in\Pi^{\theta,\varpi}$.*

Proof. Suppose spl is a fundamental splitting in G preserved by θ. We choose ψ within its inner class so that the splitting $\psi(spl)$ in G^* is fundamental and θ^*-stable. Then $\psi\circ\theta\circ\psi^{-1}=\theta^*$. Moreover, there is $\pi\in\Pi$ such that spl serves as spl_π. We claim that $\pi\circ\theta\approx\varpi\otimes\pi$. For this, we have that the parameter $\varphi=\varphi(\mu,\lambda)$ for Π satisfies ${}^L\theta_a(\varphi(w))=s^{-1}\varphi(w)s$, $w\in W_{\mathbb{R}}$, where $s\in\mathcal{T}$. The transformation of $(\mu,\lambda)\in(X_*(\mathcal{T})\otimes\mathbb{C})^2$ under $\theta^\vee$ is written above. The equivariance of ψ for the actions of θ,θ^* ensures that when (μ,λ) is transported to $(X^*(T_\pi)\otimes\mathbb{C})^2$, where it serves as character data for π, we have the same equations with θ in place of $\theta^\vee$. These imply immediately that $\pi\circ\theta\approx\varpi\otimes\pi$. Thus it is clear that we may take $\delta_\pi=1$ for this member π of the packet. Consider another member π' such that $\pi'\circ\theta\approx\varpi\otimes\pi'$. Its character data is of the form $(\omega\mu,\lambda)$, where ω lies in the Weyl group of T_π. From regularity we conclude that ω and θ commute as automorphisms of T_π. Since θ preserves the splitting spl_π we may represent ω by an element x of $(G_{sc})^{\theta_{sc}}$ normalizing T_π (see, for example, [KS]). To conclude that we may take $\delta_{\pi'}=1$ it is sufficient to arrange that $Int(x)$ carries spl_π to a fundamental splitting which we then use for $spl_{\pi'}$. From spl_π we make a splitting spl_π^θ for $(G_{sc})^{\theta_{sc}}$ in the usual manner (see p. 61 of [KS]); it is also fundamental. We may then adjust x by an element of $((T_\pi)_{sc})^{\theta_{sc}}$ so that it carries spl_π^θ to a fundamental splitting which then lifts back to θ-stable fundamental splitting $spl_{\pi'}$ for G. □

Notice that, in general, if an automorphism $\widehat{\theta}$ has the form $Int(g)\circ\theta$, where $g\in G_{ad}(\mathbb{R})$, then a discrete series packet is $(\widehat{\theta},\varpi)$-stable if and only if it is (θ,ϖ)-stable.

Assume that (θ,ϖ)-stable discrete series packets Π do exist for if that fails then there is no elliptic endoscopy for (θ,ϖ) or $(\widehat{\theta},\varpi)$. It may happen that the twist-packet $\Pi^{\theta,\varpi}$ for (θ,ϖ) is empty but that the corresponding twist-packet $\Pi^{\widehat{\theta},\varpi}$ for $(\widehat{\theta},\varpi)$ is nonempty: for example, if θ is conjugation by $diag(1,-1)$ on $SL(2)$ and $\widehat{\theta}$ is the identity. This of course affects twisted traces. According to Lemmas 3.2 and 5.3, there is always $\widehat{\theta}=Int(g)\circ\theta$ with the property that for every (θ,ϖ)-stable discrete series packet Π, the corresponding twist-packet $\Pi^{\widehat{\theta},\varpi}$ is nonempty. For the purposes of this paper only, we will call such an automorphism $\widehat{\theta}$ *acceptable.* In particular, the automorphism θ^* of G^* that is part of our quasi-split data is acceptable (we could also argue this with generic representations; see Section 10). On the other hand, if G is $\mathbb{R}$-anisotropic modulo Z_G then any $\mathbb{R}$-automorphism of G is acceptable since packets are singletons.

For the remaining sections (6 - 10) on the cuspidal elliptic setting we restrict our attention to acceptable θ (although we have done the needed extension to the general setting of geometric transfer in Section 12 of [S1], and it is clear that if we have a nontrivial twist of the norm on $H_1(\mathbb{R})$ then spectral transfer for discrete series reduces to $0=0$). Thus, as at the end of Section 3, we assume that up to an inner automorphism induced by an element of $G(\mathbb{R})$, θ preserves a fundamental splitting. We take $u(\sigma)$ fixed by θ^*_{sc} (this places a constraint on the inner twist ψ within its inner class), and $z_\sigma(\theta)$ to be the attached cocycle in $Z_{G^*_{sc}}$ that becomes trivial, *i.e.* a coboundary, in Z_{G^*}.

6. Spectral transfer factors: definitions

Let $(\pi_1,\pi),(\pi_1',\pi')$ be G-regular elliptic related pairs. Following the ordinary case, we will define terms

$$\Delta_I(\pi_1,\pi),\ \Delta_{II}(\pi_1,\pi),\ \Delta_{III}(\pi_1,\pi;\pi_1',\pi')$$

and then set

$$\Delta(\pi_1,\pi;\pi_1',\pi')=\frac{\Delta_I(\pi_1,\pi)}{\Delta_I(\pi_1',\pi')}.\frac{\Delta_{II}(\pi_1,\pi)}{\Delta_{II}(\pi_1',\pi')}.\Delta_{III}(\pi_1,\pi;\pi_1',\pi').$$

As in the geometric case, we make three choices that play a role in the construction of these terms but will be seen not to affect the product: (elliptic) toral data, a-data and χ-data. Consider first the toral data. These fix how we view in G^* various objects from the endoscopic group and the dual groups. The data consist of a θ^*-stable pair (B,T) in G^* with T anisotropic modulo Z_{G^*}, and a pair (B_1,T_1) with T_1 anisotropic modulo Z_{H_1}. There is attached to the pairs an admissible isomorphism $T_1/Z_1\to T_{\theta^*}$. With spl^* and $spl^\vee$ fixed, the crucial datum for $T_1/Z_1\to T_{\theta^*}$ (and the transport of various objects in the dual group to G^*) is the isomorphism $Int(h)$ from (B,T) to (B^*,T^*). Here we take h in G^*_{sc}. Recall that (B,T) extends to a fundamental splitting spl_{td} and, without affecting the action of $Int(h)$ on T, we may further require that $Int(h)$ carries spl_{td} to spl^*. As a computation aid alone, we set $t_{td}=\theta^*_{sc}(h)^{-1}h$ and $\theta_{td}=Int(t_{td})\circ\theta^*$ (the subscript td indicates that these objects change with different choices of toral data). For definitions and a discussion of a-data and χ-data for our setting, see Sections 3 and 9 of [S1].

We also introduce also the operators $\pi(\theta,\varpi)$, $\pi'(\theta,\varpi)$ following the discussion in Section 4, as well as elements δ_π,γ_π and $\delta_{\pi'},\gamma_{\pi'}$ as prescribed in Lemmas 5.1 and 5.2.

Term $\Delta_I(\pi_1, \pi)$: To begin the definitions, $\Delta_I(\pi_1, \pi)$ will be the same as the corresponding geometric term for this choice of toral data and a-data. It depends on the $\mathbb{R}$-splitting spl^* for G^* preserved by θ^* but then, as shown in the geometric case [KS], the quotient $\Delta_I(\pi_1, \pi)$ / $\Delta_I(\pi_1', \pi')$ does not.

7. **Term** $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$

It is convenient to define the relative factor $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ before the factor $\Delta_{II}(\pi_1, \pi)$ since the needed proof of Lemma 5.2 comes from preparation for Δ_{III}.

Recall that $\delta_\pi \in G(\mathbb{R})$ is such that $Int(\delta_\pi) \circ \theta$ preserves a fundamental splitting $spl_\pi = (B_\pi, T_\pi, ...)$. We now make use of the chosen elliptic toral data. Suppose $g \in G^*_{sc}$ is such that $Int(g) \circ \psi$ maps (B_π, T_π) to (B, T). Since T_π, T are anisotropic modulo Z_G, the restriction of $Int(g) \circ \psi$ to T_π is defined over $\mathbb{R}$. Define

$$\delta^*_\pi = g.m(\delta_\pi).\theta^*(g)^{-1}$$

and

$$v_\pi(\sigma) = g.u(\sigma).\sigma(g)^{-1}.$$

In the first formula we have identified g with its image in G^*, but in the second we have not: $v_\pi(\sigma) \in G^*_{sc}$. For the definition of m see Corollary 3.4.

Since

$$(Int(g) \circ \psi) \circ \sigma(Int(g) \circ \psi)^{-1} = Int(v_\pi(\sigma)),$$

we have immediately that $v_\pi(\sigma) \in T_{sc}$. Also,

$$\begin{aligned} &(Int(g) \circ \psi) \circ (Int(\delta_\pi) \circ \theta) \circ (Int(g) \circ \psi)^{-1} \\ &= Int(\delta^*_\pi) \circ \theta^* \end{aligned}$$

preserves the θ^*-stable pair (B, T), so that $\delta^*_\pi \in T$. We may insist further that $g \in G^*_{sc}$ is chosen so that $Int(g) \circ \psi$ maps spl_π to spl_{td}. Then $Int(\delta^*_\pi) \circ \theta^*$ preserves spl_{td}, as does $Int(t_{td}) \circ \theta^*$. Thus δ^*_π coincides with t_{td} up to an element of Z_{G^*}. Then, for general $g \in G^*_{sc}$ such that $Int(g) \circ \psi$ maps (B_π, T_π) to (B, T), we have that δ^*_π lies in $Z_{G^*}(1 - \theta^*)T$.

Recall from Section 3 that

$$\sigma(m)(\delta_\pi) = z_\sigma(\theta).u(\sigma)^{-1}.m(\delta_\pi).\theta^*(u(\sigma)),$$

and so we compute $\sigma(\delta^*_\pi)$ as

$$\begin{aligned} & z_\sigma(\theta).\sigma(g).\sigma(m)(\delta_\pi).\theta^*(\sigma(g))^{-1} \\ = \quad & z_\sigma(\theta).\sigma(g).u(\sigma)^{-1}.m(\delta_\pi).\theta^*(u(\sigma)).\theta^*(\sigma(g))^{-1} \\ = \quad & z_\sigma(\theta).v_\pi(\sigma)^{-1}.g.m(\delta_\pi).\theta^*(g^{-1}v_\pi(\sigma)) \\ = \quad & z_\sigma(\theta).v_\pi(\sigma)^{-1}.\delta^*_\pi.\theta^*(v_\pi(\sigma)). \end{aligned}$$

Each term in this last product lies in T. We conclude then the following analogue of Lemma 4.4.A of [KS], or more precisely the slight generalization of that lemma in Section 5.4 of [KS].

Lemma 7.1. *The element* $v_\pi(\sigma)$ *lies in* T_{sc}, *the element* δ^*_π *lies in* $Z_{G^*}(1 - \theta^*)T$, *and*

$$z_\sigma(\theta).\sigma(\delta^*_\pi)^{-1}.\delta^*_\pi = (1 - \theta^*)v_\pi(\sigma).$$

Here we have again used the notation $1 - \theta^*$ also for the composition of $1 - \theta^* : T \to T$ with the natural projection $T_{sc} \to T$.

We return to Lemma 5.2.

Proof. (Lemma 5.2) As usual, we regard the coinvariants $(Z_{G^*})_{\theta^*}$of Z_{G^*} as a subgroup of the coinvariants T_{θ^*} of T. Then under the projection $N : T \to T_{\theta^*}$, δ_π^* maps into $(Z_{G^*})_{\theta^*}$. Since $N(z_{sc}(\theta).\sigma(\delta_\pi^*)^{-1}.\delta_\pi^*) = 1$, we have that $N(\delta_\pi^*) \in (Z_{G^*})_{\theta^*}(\mathbb{R})$. We identify $(Z_{G^*})_{\theta^*}(\mathbb{R})$ as a subgroup of $Z_H(\mathbb{R})$ or, using our toral data, of $T_H(\mathbb{R})$. Let γ_π be an element of $Z_{H_1}(\mathbb{R})$ whose image under $p : H_1 \to H$ coincides with the image of $N(\delta_\pi^*)$ in $Z_H(\mathbb{R})$. Then γ_π is a T_1-norm of δ_π in the sense of Section 6 of [S1]. In general, γ_π is determined up to stable conjugacy by δ_π [S1]. Since $\gamma_\pi \in Z_{H_1}(\mathbb{R})$, it is uniquely determined by δ_π. The rest is immediate. □

The term $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ is now defined in the same manner as the geometric term $\Delta_{III}(\gamma_1, \delta; \gamma_1', \delta')$ of [KS]. We use the constructions starting at the bottom of p. 41. We again write

$$\Delta_{III}(\pi_1, \pi; \pi_1', \pi') \quad = < \mathbf{V}_1, \mathbf{A}_1 > .$$

Here $\mathbf{V}_1$ is an element of the hypercohomology group $H^1(\Gamma, U \xrightarrow{1-\theta^*} S_1)$, where the tori U, S_1 are attached to the given tori T, T' as in [KS]. While we use similar notation as in the geometric case, our construction of the spectral ingredients $v_\pi(\sigma)$ and $v_\pi(\sigma)$ is of course different. We also use the pair $(\delta_\pi^*, \gamma_\pi)$ in the pullback torus $T_2 = \{(t, t_1) \in T \times T_1 : N(t) = p(t_1)\}$ (denoted T_1 in [KS]) in place of the very regular pair (δ^*, γ_1) on p. 42, and the pair $(\delta_{\pi'}^*, \gamma_{\pi'})$ in T_2' in place of $(\overline{\delta}^*, \overline{\gamma}_1)$. Then $\mathbf{V}_1$ is the class of the 1-hypercocycle

$$((v_\pi(\sigma)^{-1}, v_{\pi'}(\sigma)), ((\delta_\pi^*, \gamma_\pi), (\delta_{\pi'}^*, \gamma_{\pi'})^{-1})),$$

where our notation identifies the element $(v_\pi(\sigma)^{-1}, v_{\pi'}(\sigma))$ of $T_{sc} \times T_{sc}'$ with its image in U, and the element $((\delta_\pi^*, \gamma_\pi), (\delta_{\pi'}^*, \gamma_{\pi'})^{-1})$ of $T_2 \times T_2'$ with its image in S_1. Notice that $z_\sigma(\theta)$ no longer plays any role. We take the class $\mathbf{A}_1$ to be exactly that of the geometric case, so that $\mathbf{A}_1 \in H^1(W_{\mathbb{R}}, \widehat{S}_1 \xrightarrow{1-\widehat{\theta}} \widehat{U})$. The pairing in the definition of Δ_{III} is described explicitly in Appendix A of [KS].

It is clear that $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ depends on the choice of δ_π, γ_π and $\delta_{\pi'}, \gamma_{\pi'}$. Notice that if we replace δ_π, γ_π by $z\delta_\pi, z_1\gamma_\pi$, with z, z_1 as in Lemma 5.2, we replace $(\delta_\pi^*, \gamma_\pi) \in T_2$ by $(z, z_1).(\delta_\pi^*, \gamma_\pi)$, where (z, z_1) belongs to the pullback C defined on p. 53 of [KS]. Here we have embedded C (naturally) in T_2. Write ϖ_C for the character λ_C on $C(\mathbb{R})$ also defined there.

Lemma 7.2. *(i) Suppose $\delta_\pi, \delta_{\pi'}$ are replaced by $x\delta_\pi\theta(x)^{-1}$, $x'\delta_{\pi'}\theta(x')^{-1}$, where $x, x' \in G(\mathbb{R})$, and $\gamma_\pi, \gamma_{\pi'}$ remain fixed. Then $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ is multiplied by*

$$\varpi(x)^{-1}.\varpi(x').$$

(ii) Suppose δ_π, γ_π are replaced by $z\delta_\pi, z_1\gamma_\pi$, and $\delta_{\pi'}, \gamma_{\pi'}$ by $z'\delta_{\pi'}, z_1'\gamma_{\pi'}$, where $(z, z_1), (z', z_1') \in C(\mathbb{R})$. Then $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ is multiplied by

$$\varpi_C((z, z_1))^{-1}.\varpi_C((z', z_1')).$$

Proof. For (i) we argue the same way as for (1) in Theorem 5.1.D: since $v_\pi(\sigma), v_{\pi'}(\sigma)$ are defined a priori, there is no need for $\delta_\pi^*, \delta_{\pi'}^*$ to be strongly θ^*-regular. For (ii) we refer to the argument of the first half of p. 54 of [KS] supplementing the proof of Lemma 5.1.C. □

Recall from Lemma 5.1.C of [KS] that if z_1 lies in the subgroup $Z_1(\mathbb{R}) = Ker(H_1(\mathbb{R}) \to H(\mathbb{R}))$ of $Z_{H_1}(\mathbb{R})$, so that $(z_1, 1) \in C(\mathbb{R})$, then $\varpi_C((z_1, 1)) = \varpi_1(z_1)$, where ϖ_1 is the character on $Z_1(\mathbb{R})$ provided our choice of z-pair (H_1, ξ_1). Now we need to identify ϖ_C on the entire group $C(\mathbb{R})$. Let $\varpi_{\pi_1}, \varpi_\pi$ denote the central characters of π_1, π respectively. Then $\varpi_{\pi_1}(z_1) = \varpi_1(z_1)$ for $z_1 \in Z_1(\mathbb{R})$, by the z-pair construction, so that $\varpi_C((z_1, 1)) = \varpi_{\pi_1}(z_1)$. More generally:

Lemma 7.3.

$$\varpi_C((z_1, z)) = \varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1},$$

for all $(z_1, z) \in C(\mathbb{R})$.

Proof. In principle, this should be clear from general considerations (see Section 2 of [L]). However our proof will be by explicit calculation with the pairs (μ_1, λ_1), (μ, λ). There is no harm in arguing in G^* since ϖ_{π_1}, ϖ_π may be calculated there. Thus we embed Z_{G^*} in T, part of our toral data. Write $z \in Z_{G^*}(\mathbb{R})$ in the form $z = \exp Y. \exp i\pi\lambda^\vee$, where Y lies in the Lie algebra $\mathfrak{z}_{G^*}(\mathbb{R})$ viewed as a subspace of $X_*(T) \otimes \mathbb{C}$ and $\lambda^\vee \in X_*(T)$ is σ_T-invariant. Then it follows easily from the Langlands parametrization that

$$\varpi_\pi(z) = e^{<\mu, Y>}.e^{2i\pi<\lambda,\lambda^\vee>}.$$

Here of course we have used toral data to identify $X^*(T) \otimes \mathbb{C}$ with $X_*(\mathcal{T}) \otimes \mathbb{C}$. Similarly, for $z_1 = \exp Y_1. \exp i\pi\lambda_1^\vee$, with $Y_1 \in \mathfrak{z}_{H_1}(\mathbb{R}) \subset X_*(T_1) \otimes \mathbb{C}$ and $\lambda_1^\vee \in X_*(T)^{\sigma_{T_1}}$, we have

$$\varpi_{\pi_1}(z_1) = e^{<\mu_1, Y_1>}.e^{2i\pi<\lambda_1,\lambda_1^\vee>}.$$

Using the toral data, identify the torus $\mathcal{T}_2$ from Section 5 as the dual of the torus T_2. Then if z, z_1 have same image in $Z_H(\mathbb{R})$, *i.e.* if $(z_1, z) \in C(\mathbb{R})$, it follows from our remarks in Section 5 that

$$\varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1} = e^{-\ <\mu^*, Y_2>}.e^{-\ 2i\pi<\lambda^{*\cdot},\lambda_2^\vee>},$$

where $Y_2 = (Y_1, Y)$ and $\lambda_2^\vee = (\lambda_1^\vee, \lambda^\vee)$. It is clear from the definitions of ϖ_C and (μ^*, λ^*), and from the relation of the cochain $t_2(w)$ to the cocycle $a_T(w)$ of p. 45 of [KS], that this last expression is the same as $\varpi_C((z_1, z))$. □

Corollary 7.4. *In (ii) of Lemma 7.2 we may replace* $\varpi_C((z, z_1))^{-1}.\varpi_C((z', z_1'))$ *by*

$$\varpi_\pi(z).\varpi_{\pi_1}(z_1)^{-1}.\varpi_{\pi'}(z')^{-1}.\varpi_{\pi_1'}(z_1').$$

In general, the geometric factor $\Delta_{III}(\gamma_1, \delta; \gamma_1', \delta')$ depends on the choice of χ-data used in the construction of $\mathbf{A}_1$. The spectral side is simpler in this regard (a different spectral use for χ-data will be noted in Section 8). We will also consider the affect of replacing toral data, a-data and χ-data by (simultaneously) conjugate data as defined in (4.6.1) of [KS].

Lemma 7.5. *(i)* $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ *is unchanged if* χ*-data alone is changed. (ii) the relative term*

$$\frac{\Delta_I(\pi_1, \pi)}{\Delta_I(\pi_1', \pi')}.\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$$

is unchanged if the toral data, a-data and χ*-data are replaced by conjugate data. (iii)* $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ *is unchanged if* g_θ *or* $u(\sigma)$ *is multiplied by an element of* $Z_{G^*_{sc}}$*, or if the inner twist* ψ *is changed within its inner class.*

Proof. For (i), the effect of changing χ-data is the same as that described on pp. 48, 49 of [KS] for the geometric case. Now, however, strongly θ-regular δ is replaced by δ_π with central norm. Then, in the notation of the cited argument, $N\alpha(\delta^{(\alpha_{res})^\vee}) = 1$ for all roots α. This implies that $B_{(\alpha_{res})^\vee} = 1$ for all α, and there is no change to $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$. For (ii), the initial step of the argument for Theorem 4.6.A of [KS] applies ((i) used the final step). For (iii), the argument is the same as in the geometric case, but we have no citation. So we include an (easy) argument for the sake of completeness. A change in g_θ or $u(\sigma)$ has the same effect as the appearance of the cochain $z_\sigma(\theta)$ earlier, namely it disappears at the relativization step. Finally, if ψ is replaced by $Int(x) \circ \psi$, where $x \in G_{sc}^*$, then g_θ and $u(\sigma)$ must be changed to, say, g_θ' and $u(\sigma)'$. Acceptable choices are $g_\theta' = \theta^*(x) g_\theta x^{-1}$ and $u(\sigma)' = xu(\sigma)\sigma(x)^{-1}$. Then in the definition of $\delta_\pi^* = g.m(\delta_\pi).\theta^*(g)^{-1}$ and $v_\pi(\sigma) = g.u(\sigma).\sigma(g)^{-1}$ we replace g by gx^{-1}. A simple calculation shows that δ_π^* and $v_\pi(\sigma)$ are unchanged, and the lemma follows. $\square$

8. **Term** $\Delta_{II}(\pi_1, \pi)$

Consider now the remaining term $\Delta_{II}(\pi_1, \pi)$. It will depend on the a-data and again on the elements δ_π, γ_π which we use to center local character expansions, as well as our choice of the operator $\pi(\theta, \varpi)$. We have also fixed χ-data and toral data, but those choices will turn out not to matter. We should also point out that if the parameter for π_1 is not well-positioned relative to that for π (recall Section 5) then another sign must be included in $\Delta_{II}(\pi_1, \pi)$; for the purposes of this paper we may ignore that case.

According to Harish Chandra's regularity theorem, the stable tempered distribution $St\text{-}Tr\ \pi_1$ is represented by a locally L^1 function $St\text{-}Ch_{\pi_1}$ real analytic on the regular set of $H_1(\mathbb{R})$. Here we are interested only in the restriction of $St\text{-}Ch_{\pi_1}$ to the regular elliptic set. Our elliptic toral data provides T_1 anisotropic modulo Z_{H_1}. Here $(T_1)_{der} = (T_1)_{sc}$ since $H_1 \to H$ is a z-extension. Also $(T_1)_{der}(\mathbb{R})$ is connected and

$$T_1(\mathbb{R}) = Z_{H_1}(\mathbb{R}).(T_1)_{der}(\mathbb{R}).$$

It is enough to examine the Harish Chandra formula for $St\text{-}Ch_{\pi_1}$ on the regular set of $T_1(\mathbb{R})$. We consider (strongly) G-regular elements

$$\gamma_1 = \gamma_{\pi_1} \exp X_1,$$

with $X_1 \in \mathfrak{t}_1(\mathbb{R})$ near 0, as this is sufficient to characterize the term υ_{a_1} we need for transfer factors. At the same time, we will see immediately that while the choice of a-data matters, the choice of χ-data does not. For X_1 sufficiently close to 0, we may write $St\text{-}Ch_{\pi_1}(\gamma_1)$ as

$$\upsilon_{a_1}(\gamma_{\pi_1}).\Delta_{a_1}(X_1)^{-1}.\sum_{w_1} det(w_1).e^{w_1^{-1}\mu_1(X_1)}.$$

Here $\Delta_{a_1}(X_1)$ is, by definition,

$$\textstyle\prod_{\mathcal{O}_1}\ sign\ (\dfrac{e^{\alpha_1(X_1)/2}\ -\ e^{-\alpha_1(X_1)/2}}{a_{\alpha_1}}).\left|\det[I - e^{-ad(X_1)}]_{\mathfrak{h}_1/\mathfrak{t}_1}\right|^{1/2},$$

where the product is over representatives α_1 for the Galois orbits $\mathcal{O}_1 = \{\pm\alpha_1\}$ of roots of T_1 in H_1. Next, the summation $\sum_{w_1}$ is over the full (complex) Weyl group of T_1 in H_1 and $\mu_1 \in X^*(T_1) \otimes \mathbb{C}$ is the strictly B_1-dominant linear form on $\mathfrak{t}_1 =$

$X_*(T_1) \otimes \mathbb{C}$ attached to the Langlands parameter φ_1 for Π_1 (by transport from $X_*(\mathcal{T}_1) \otimes \mathbb{C}$ using the chosen toral data). A comparison with Harish Chandra's character formula shows that the remaining term $\upsilon_{a_1}(\gamma_{\pi_1})$ is determined as follows. First, it is clear that we must have

$$\upsilon_{a_1}(z_1) = \varpi_{\pi_1}(z_1).\upsilon_{a_1}(1),$$

for all $z_1 \in Z_{H_1}(\mathbb{R})$. Second, $\upsilon_{a_1}(1)$ is the constant written $\upsilon(\varphi_1, \{a_{\alpha_1}\})$ in the notation of Section 9 of [S3], where it is computed explicitly using Harish Chandra's character formula. It is a fourth root of unity which appears in the spectral term Δ_{II} for ordinary endoscopy.

The formula we have written for $St\text{-}Ch_{\pi_1}(\gamma_1)$ is purely local. We may use χ-data to make it global: each choice of χ-data provides us with a different way of writing the (same) extension of this local formula to Harish Chandra's global formula on the regular set in $T_1(\mathbb{R})$ (for the routine extension of the original global formula to the present setting, see [S5]). To write this extension, our first step is to observe that the local term $\Delta_{a_1}(X_1)$ coincides with

$$\Delta_{a_1,\chi_1}(\gamma_1).\textstyle\prod_{\mathcal{O}_1} \chi_{\alpha_1}(e^{\alpha_1(X_1)/2}),$$

where

$$\Delta_{a_1,\chi_1}(\gamma_1) = \Delta_{a_1,\chi_1}(\gamma_{\pi_1} \exp X_1) = \Delta_{a_1,\chi_1}(\exp X_1)$$

is the right Weyl denominator attached to the a- and χ-data a_1, χ_1, both arbitrary. Then the Langlands parameter $\varphi_1 = \varphi(\mu_1, \lambda_1)$ determines uniquely, for each Weyl group element w_1, an extension of the term

$$e^{w_1^{-1}\mu_1(X_1)}.\textstyle\prod_{\mathcal{O}_1} \chi_{\alpha_1}(e^{\alpha_1(X)/2})^{-1}$$

to a character on $T(\mathbb{R})$, and we are done; see [S3].

We turn to the twisted character of π and consider the distribution

$$f \to Tr[\pi(f) \circ \pi(\theta, \varpi)]$$

on $G(\mathbb{R})$. By Bouaziz's extension of Harish Chandra's regularity theorem [B1] (by assumption* if the twisting character ϖ is nontrivial), this distribution is represented by a locally L^1 function $Tw\text{-}Ch_\pi$ real analytic on the twisted regular set of $G(\mathbb{R})$. Recall that $\theta_\pi = Int(\delta_\pi) \circ \theta$ fixes the fundamental splitting $spl_\pi = (B_\pi, T_\pi, \{X_\alpha\})$. We consider a local expression for $Tw\text{-}Ch_\pi$ around δ_π. To transport the datum $\mu \in X_*(\mathcal{T}) \otimes \mathbb{C}$ to G we supplement the chosen toral data with the $\mathbb{R}$-isomorphism $Int(g) \circ \psi : T_\pi \to T$ mapping B_π to B. Then we regard μ as element of $X^*(T_\pi) \otimes \mathbb{C}$. Write T^{δ_π} for the identity component of the fixed points of θ_π in T_π. Then μ determines a linear form on the Lie algebra $\mathfrak{t}^{\delta_\pi}(\mathbb{R})$. **Our assumption is*** that for strongly θ_π-regular $\delta = \delta_\pi \exp X$, with $X \in \mathfrak{t}^{\delta_\pi}(\mathbb{R})$ sufficiently close to zero, we may write

$$Tw\text{-}Ch_\pi(\delta) = \upsilon_a(\delta_\pi).\Delta_a(X)^{-1}.\sum_{w,\mathbb{R}} \det w.e^{w^{-1}\mu(X)}.$$

We include some remarks as we explain notation.

(i) First, $\upsilon_a(\delta_\pi)$ is a nonzero complex number. If spl_π is replaced by $x.spl_\pi$ and δ_π by $\delta'_\pi = zx\delta_\pi\theta(x)^{-1}$, where $z \in Z_G(\mathbb{R})$ and $x \in G(\mathbb{R})$, then

$$\upsilon_a(\delta'_\pi) = \varpi_\pi(z).\varpi(x)^{-1}.\upsilon_a(\delta_\pi).$$

(ii) The summation $\sum_{w,\mathbb{R}}$ is over the quotient

$$Norm(T^{\delta_\pi}, G(\mathbb{R}))/Cent(T^{\delta_\pi}, G(\mathbb{R})) = Norm(T^{\delta_\pi}, G(\mathbb{R}))/T_\pi(\mathbb{R}).$$

Since θ_π preserves a (fundamental) splitting, this quotient embeds naturally in the complex Weyl group of T^{δ_π} in $G^\theta_{\delta_\pi} = Cent_\theta(\delta_\pi, G)^0$.

(iii) We define the factor $\Delta_a(X)$ as

$$\textstyle\prod_{\mathcal{O}_{red}} \; sign(\dfrac{e^{rN\alpha(X)/2} \; - \; e^{-rN\alpha(X)/2}}{a_{\alpha_{res}}}). \left|\det[I - (Ad(\exp X) \circ \theta_\pi)^{-1}]_{\mathfrak{g}/\mathfrak{t}_\pi}\right|^{1/2},$$

where the product is over the set of Galois orbits $\mathcal{O}_{red} = \{\pm\alpha_{res}\}$ of the reduced roots among the restrictions to T^{δ_π} of the roots α of T_π in G. Here $r = 1, 2$ according as α_{res} is of type R_1 or R_2 (defined in Section 1.3 of [KS]). Notice that $\Delta_a(X)$ may also be computed in G^* using θ^* in place of θ_π and $\delta^* = \delta^*_\pi \exp X^*$ in place of δ. Here $\delta^*_\pi = g.m(\delta_\pi).\theta^*(g)^{-1}$, as in Section 7, and X^* is the image of X under the $\mathbb{R}$-isomorphism of $\mathfrak{t}^{\delta_\pi} = (\mathfrak{t}_\pi)^{\theta_\pi}$ with $\mathfrak{t}^{\theta^*}$. Then $\Delta_a(X)$ coincides with $\Delta_a(X^*)$ given by

$$\textstyle\prod_{\mathcal{O}_{red}} \; sign(\dfrac{e^{rN\alpha(X^*)/2} \; - \; e^{-rN\alpha(X^*)/2}}{a_{\alpha_{res}}}). \left|\det[I - (Ad(\exp X^*) \circ \theta^*)^{-1}]_{\mathfrak{g}^*/\mathfrak{t}}\right|^{1/2},$$

a form familiar from geometric transfer factors. Once again, we may use χ-data to globalize this formula, now to the θ-regular θ-elliptic set in $G(\mathbb{R})$.

The constant $\upsilon_a(\delta_\pi)$ depends on our choice of the operator $\pi(\theta, \varpi)$, but clearly

$$\upsilon_a(\delta_\pi)^{-1}.Tr[\pi(f) \circ \pi(\theta, \varpi)]$$

does not. We now define

$$\Delta_{II}(\pi_1, \pi) = \upsilon_{a_1}(\gamma_{\pi_1}).\upsilon_a(\delta_\pi)^{-1}.$$

The following is an immediate consequence of our assumption on υ_a. It is all we need (at present) from the assumption.

Lemma 8.1. *(i) Suppose δ_π is replaced by $x\delta_\pi\theta(x)^{-1}$, where $x \in G(\mathbb{R})$, and γ_π remains fixed. Then $\Delta_{II}(\pi_1, \pi)$ is multiplied by $\varpi(x)$. (ii) Suppose δ_π is replaced by $z\delta_\pi$, and γ_π by $z_1\gamma_\pi$, where $(z, z_1) \in C(\mathbb{R})$. Then $\Delta_{II}(\pi_1, \pi)$ is multiplied by $\varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1} = \varpi_C((z, z_1))$. (iii) Suppose the a-data $\{a_{\alpha_{res}}\}$ are replaced by a-data $\{a_{\alpha_{res}} b_{\alpha_{res}}\}$. Then $\Delta_{II}(\pi_1, \pi)$ is multiplied by $\prod_{\mathcal{O}_{red}} \; sign(b_{\alpha_{res}})$. (iv) $\Delta_{II}(\pi_1, \pi)$ is unaltered by a change in toral data.*

This completes our definition of the terms in $\Delta(\pi_1, \pi; \pi'_1, \pi')$. For ordinary endoscopy this factor is a canonical sign: see Theorem 11.1 of [S3]. In the present setting, it depends on the choice of operators $\pi(\theta, \varpi), \pi'(\theta, \varpi)$. Thus we may ask instead which choices of operators make it a sign. In any case, it is clear immediately that, with our assumption above, $\Delta(\pi_1, \pi; \pi'_1, \pi')$ is a nonzero complex number.

It will be useful to replace the choice of a single operator $\pi(\theta, \varpi)$ with the choice of a map $\mathcal{P}_\Pi : \pi \to \pi(\theta, \varpi)$, defined for all π in the subset $\Pi^{\theta,\varpi}$ of a (θ, ϖ)-stable tempered packet Π. We discuss this further in the next section.

(*) **Remark:** It appears routine to remove these assumptions by a reduction and then a systematic look at Harish Chandra's results on the discrete series. We will include details in an appendix to the final version of this paper.

9. Spectral transfer factors, conclusions

We continue with the cuspidal elliptic setting, as at the end of Section 5.

Theorem 9.1. *Fix G-regular elliptic related pairs (π_1, π), (π_1', π'). Then the relative factor $\Delta(\pi_1, \pi; \pi_1', \pi')$ depends only on the choice of $\mathcal{P}_\Pi$ and $\mathcal{P}_{\Pi'}$.*

Proof. Here is a summary of the data used in our constructions. Choices (made in defining the listed term) which we have shown *not* to matter are as follows.

(i) $\frac{\Delta_I(\pi_1,\pi)}{\Delta_I(\pi_1',\pi')}$ *: spl**.

(ii) $\Delta_{II}(\pi_1, \pi)$ *: toral data,* ψ *within inner class.*

(iii) $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ *:* g_θ, $u(\sigma)$, ψ *within inner class,* χ*-data.*

Remaining dependencies are as follows.

(i) $\frac{\Delta_I(\pi_1,\pi)}{\Delta_I(\pi_1',\pi')}$ *: toral data, a-data.*

(ii) $\frac{\Delta_{II}(\pi_1,\pi)}{\Delta_{II}(\pi_1',\pi')}$ *: a-data, elements* (δ_π, γ_π) *and* $(\delta_{\pi'}, \gamma_{\pi'})$, $\mathcal{P}_\Pi$ and $\mathcal{P}_{\Pi'}$.

(iii) $\Delta_{III}(\pi_1, \pi; \pi_1', \pi')$ *: toral data, elements* (δ_π, γ_π) *and* $(\delta_{\pi'}, \gamma_{\pi'})$.

Now consider

$$\Delta(\pi_1, \pi; \pi_1', \pi') = \frac{\Delta_I(\pi_1, \pi)}{\Delta_I(\pi_1', \pi')}.\frac{\Delta_{II}(\pi_1, \pi)}{\Delta_{II}(\pi_1', \pi')}.\Delta_{III}(\pi_1, \pi; \pi_1', \pi').$$

Lemma 7.5 eliminates dependence of this product on toral data. Lemma 8.1, along with Lemma 3.2.C of [LS], eliminates dependence on a-data. Finally Lemma 7.2, Corollary 7.4 and Lemma 8.1 eliminate dependence on (δ_π, γ_π) and $(\delta_{\pi'}, \gamma_{\pi'})$, and so the theorem is proved. □

In Section 11 we will use parabolic induction to extend this result to most tempered *G-regular* related pairs (π_1, π), (π_1', π'). The cuspidal-elliptic assumption on the groups G and H_1 is dropped there also. We assume for now that step has been done (and limit the definition of *G-regular related pair* to the pairs of Section 11).

We next record the dependence of $\Delta(\pi_1, \pi; \pi_1', \pi')$ on $\mathcal{P}_\Pi$, $\mathcal{P}_{\Pi'}$, writing it for now as $\Delta_{\mathcal{P},\mathcal{P}'}(\pi_1, \pi; \pi_1', \pi')$.

Lemma 9.2. *Let (π_1, π), (π_1', π'), (π_1'', π'') be tempered G-regular related pairs. Then: (i)*

$$\Delta_{\mathcal{P},\mathcal{P}}(\pi_1, \pi; \pi_1, \pi) = 1.$$

Also: (ii)

$$\Delta_{\mathcal{P},\mathcal{P}''}(\pi_1, \pi; \pi_1'', \pi'') = \Delta_{\mathcal{P},\mathcal{P}'}(\pi_1, \pi; \pi_1', \pi').\Delta_{\mathcal{P}',\mathcal{P}''}(\pi_1', \pi'; \pi_1'', \pi'').$$

Proof. This requires an argument only for the Δ_{III} term. For that we use a variant of the argument for Lemma 4.1.A of [LS]. □

Finally, we define a nonvanishing complex-valued function $\Delta_{\mathcal{P}}(\pi_1, \pi)$ on the set of tempered G-regular related pairs, also depending appropriately on $\mathcal{P}_\Pi$, to be a spectral transfer factor (more precisely, a proposed spectral transfer factor) if

$$\Delta_{\mathcal{P}}(\pi_1, \pi)/\Delta_{\mathcal{P}'}(\pi_1', \pi') = \Delta_{\mathcal{P},\mathcal{P}'}(\pi_1, \pi; \pi_1', \pi')$$

for all tempered G-regular related pairs (π_1, π), (π_1', π').

We return to the cuspidal-elliptic case and consider the case that θ preserves a fundamental splitting. Let Π be a (θ, ϖ)-stable discrete series packet. For each

$\pi \in \Pi^{\theta,\varpi}$ we may take δ_π, γ_π to be the identity elements of G, H_1 respectively. Then it is a simple matter to choose $\mathcal{P} = \mathcal{P}_\Pi$ so that

$$\Delta_{II}(\pi_1, \pi) = \Delta_{II}(\pi_1, \pi')$$

for all $\pi, \pi' \in \Pi^{\theta,\varpi}$. In that case we will call $\mathcal{P}$ a *balanced normalization* of the operators $\pi(\theta, \varpi)$. Notice that in the case θ is the identity and ϖ is trivial, choosing the identity operator for each $\pi(\theta, \varpi)$ yields a balanced normalization, and that this rests on the precise form of Harish Chandra's discrete series character formulas.

From now on, we drop $\mathcal{P}$ from notation.

10. **Compatibility factors**

Here we exploit the parallel nature of the geometric and spectral constructions. Geometric transfer factors are defined for strongly G-regular related pairs of elements (γ_1, δ) in $H_1(\mathbb{R}) \times G(\mathbb{R})$. We continue with a tempered G-regular related pair (π_1, π) of representations of $H_1(\mathbb{R}) \times G(\mathbb{R})$. Then, following the ordinary case [S3], we define a compatibility factor:

$$\Delta(\pi_1, \pi; \gamma_1, \delta) = \tfrac{\Delta_I(\pi_1,\pi)}{\Delta_I(\gamma_1,\delta)} \cdot \tfrac{\Delta_{II}(\pi_1,\pi)}{\Delta_{II}(\gamma_1,\delta)} . \Delta_{III}(\pi_1, \pi; \gamma_1, \delta).$$

Here $\Delta_{III}(\pi_1, \pi; \gamma_1, \delta)$ is defined in the now obvious way using the cochains $v_\pi(\sigma), v(\sigma)$ etc., the same norm correspondence being used for both the geometric and spectral pieces.

Theorem 10.1. *Fix related pairs (γ_1, δ), (π_1, π). Then $\Delta(\pi_1, \pi; \gamma_1, \delta)$ depends only on $\mathcal{P}_\Pi$.*

Proof. This is immediate from the arguments for Theorem 4.6.A of [KS] and Theorem 9.1. □

We note separately the following.

Corollary 10.2. *(i) If $\Delta(\pi_1, \pi)$ is a spectral transfer factor in the sense of Section 9 then*

$$\Delta(\pi_1, \pi)/\Delta(\pi_1, \pi; \gamma_1, \delta)$$

does not depend on $\mathcal{P}_\Pi$. (ii) $\Delta(\pi_1, \pi; \gamma_1, \delta)$ is independent of the norm correspondence chosen.

Proof. (i) is immediate from definitions. (ii) will be clear following our discussion of norm correspondences at the start of Section 11. □

We next label the geometric, spectral and compatibility factors with the appropriate subscripts, and use $'$ to indicate the choice of a second related pair. We also choose absolute factors $\Delta_{spec}(\pi_1, \pi)$ and $\Delta_{geom}(\gamma_1, \delta)$, using the definition of Section 9 and its geometric analogue. The following transitivity properties are immediate from previous arguments extending Lemma 4.1.A of [LS].

Lemma 10.3.

$$\Delta_{spec}(\pi_1, \pi; \pi_1', \pi').\Delta_{comp}(\pi_1', \pi'; \gamma_1, \delta) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$$

and

$$\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta).\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta') = \Delta_{comp}(\pi_1, \pi; \gamma_1', \delta').$$

Corollary 10.4.

$$\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)/\Delta_{comp}(\pi_1', \pi'; \gamma_1', \delta')$$
$$= \Delta_{spec}(\pi_1, \pi; \pi_1', \pi')/\Delta_{geom}(\gamma_1, \delta; \gamma_1', \delta').$$

Corollary 10.5.

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta).\Delta_{geom}(\gamma_1, \delta)$$

if and only if

$$\Delta_{spec}(\pi_1', \pi') = \Delta_{comp}(\pi_1', \pi'; \gamma_1', \delta').\Delta_{geom}(\gamma_1', \delta').$$

We say that the absolute factors Δ_{geom} and Δ_{spec}, defined using the same norm correspondence, have *compatible normalization* if for some, and hence any, choice of (π_1, π) and (γ_1, δ), we have

$$\Delta_{spec}(\pi_1, \pi) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta).\Delta_{geom}(\gamma_1, \delta).$$

We finish this section with a basic example. Namely, we introduce a Whittaker normalization for Δ_{spec} and show it is compatible with the Whittaker normalization of [KS] for Δ_{geom}. Let (B_{Wh}, λ_{Wh}) be θ^*-stable Whittaker data for G^*. This means B_{Wh} is a θ^*-stable Borel subgroup of G^* defined over $\mathbb{R}$ and λ_{Wh} is a generic θ^*-stable character on $N_{Wh}(\mathbb{R})$, where N_{Wh} is the unipotent radical of B_{Wh}. We have fixed the θ^*-stable $\mathbb{R}$-splitting $spl^* = (B^*, T^*, etc.)$; there is no harm in assuming that $B_{Wh} = B^*$, so that character λ_{Wh} is then determined by the choice of additive character $\psi_{\mathbb{R}}$ on $\mathbb{R}$.

Assume that G^* is cuspidal, $G = G^*$, $\theta = \theta^*$, and that the inner twist ψ is the identity. We consider an elliptic endoscopic group H_1, and note that there is a canonical choice for norm map in this setting (we take m to be the identity on $G(\mathbb{R})$). An *absolute* geometric factor $\Delta_{III}(\gamma_1, \delta)$, where (γ_1, δ) is a strongly G-regular related pair, is defined on p. 63 of [KS]. Then

$$\Delta_0(\gamma_1, \delta) = \Delta_I(\gamma_1, \delta)\ \Delta_{II}(\gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta)$$

is a geometric transfer factor that is independent of the choice of toral data, a-data and χ-data. It does depend (through Δ_I) on the choice of spl^*. An epsilon factor $\varepsilon_L(V, \psi_{\mathbb{R}})$ is defined on p. 65 of [KS]. Replace Δ_I by $\varepsilon_L(V, \psi_{\mathbb{R}}).\Delta_I$ to define the factor $\Delta_{_{Wh}}(\gamma_1, \delta)$ *with Whittaker normalization*. Then dependence on spl^* is replaced by dependence on the Whittaker data (B_{Wh}, λ_{Wh}) [KS].

Suppose now (π_1, π) is a G-regular elliptic related pair. The definition of $\Delta_{_{Wh}}(\pi_1, \pi)$ will follow the same pattern. First, $\Delta_I(\pi_1, \pi)$ is attached to our choice of elliptic toral data, a-data and $spl^*$from Section 6; the construction is that of the geometric case. We consider next the term $\Delta_{III}(\pi_1, \pi)$ which, as always, will depend on the choice of toral data, and as always in a Whittaker setting, there will be a particular choice for these data that simplifies the results.

Suppose Π is the packet of π. By Vogan's classification theorem there is a unique $\pi_{Wh} \in \Pi$ generic relative to (B_{Wh}, λ_{Wh}). Since (B_{Wh}, λ_{Wh}) is θ^*-stable, $\varpi^{-1} \otimes (\pi_{Wh} \circ \theta) \in \Pi$ is generic relative to (B_{Wh}, λ_{Wh}) also, and then $\pi_{Wh} \circ \theta \approx \varpi \otimes \pi_{Wh}$, *i.e.* $\pi_{Wh} \in \Pi^{\theta, \varpi}$. We attach to π_{Wh} a θ^*-stable fundamental splitting $spl_{_{Wh}} = (B_{_{Wh}}, T_{_{Wh}}, \{X_\alpha\})$ as in Lemma 3.3; $spl_{_{Wh}}$ is an $\mathbb{R}$-opp splitting, *i.e.* each simple root α of $T_{_{Wh}}$ in $B_{_{Wh}}$ is noncompact. Assume that $Int(h)$ maps $spl_{_{Wh}}$ to spl^*, where $h \in G^*_{sc}$. We will use $Int(h)$ as part of of our toral data. For the elements δ_π, γ_π we may use the identities in G, H_1. Then the constructions of Section 7 simplify and yield an absolute factor $\Delta_{III}(\pi_1, \pi)$, as follows. Using our usual notation [S3], we write Π as $\{\pi(w)\}$, where $w \in G$ represents a coset of the real Weyl group in the full Weyl group of $T_{\pi_{Wh}}$ in G. We may assume that w is the image of an element w_{sc} in G_{sc} and take $w = 1$ for the trivial coset. We have arranged that $\pi_{Wh} = \pi(1)$ and the proof of Lemma 5.3 shows that $\Pi^{\theta,\varpi}$ consists of those $\pi(w)$ for which we can find w_{sc} in the fixed points of θ_{sc}. Write T^x for the fixed points of θ_{sc} in the preimage of T_{Wh} in G_{sc}, and let $inv_\theta(\pi_{Wh}, \pi)$ denote the element of $H^1(\Gamma, T^x)$ defined by the cocycle $\sigma(w_{sc})w_{sc}^{-1}$, where $\pi = \pi(w)$. Since the dual of T^x consists of the $\theta^\vee_{ad}$-coinvariants in $(T^\vee)_{ad}$, we can use our chosen toral data and the property (4.4.1) of [KS] to attach to the endoscopic datum s a well-defined element s_{Wh} of $((T^x)^\vee)^\Gamma$. This group is finite because T^x is anisotropic over $\mathbb{R}$. Then, for π in the twist-packet $\Pi^{\theta,\varpi}$,we define

$$\Delta_{III}(\pi_1, \pi) = \langle inv_\theta(\pi_{Wh}, \pi),\ s_{Wh}\rangle^{-1},$$

using Kottwitz's version of the Tate-Nakayama pairing (the inverse sign is a formality since the term is simply a sign). In particular,

$$\Delta_{III}(\pi_1, \pi_{Wh}) = 1.$$

From the definition of the pairings in [KS] used in Section 7 it is clear that for any two discrete series packets Π, Π' we have

$$\Delta_{III}(\pi_1, \pi)\ /\ \Delta_{III}(\pi_1', \pi') = \Delta_{III}(\pi_1, \pi; \pi_1', \pi')$$

for related pairs $(\pi_1, \pi), (\pi_1', \pi')$ with π, π' in $\Pi^{\theta,\varpi}, (\Pi')^{\theta,\varpi}$ respectively.

We come then to $\Delta_{II}(\pi_1, \pi)$. Recall that we have assumed since Section 5 that the parameter for π_1 is well-positioned; this is critical for our formulas for Δ_{II} since otherwise we have to carry another sign. The term Δ_{II} also depends on the normalization $\mathcal{P}_\Pi$ for the operators $\pi(\theta, \varpi)$, $\pi \in \Pi^{\theta,\varpi}$. We will normalize $\pi_{Wh}(\theta, \varpi)$ to fix one and hence each Whittaker vector for π (as in [A1]), and then extend this to a balanced normalization $\mathcal{P}_\Pi$ (see the last paragraph of Section 8). Thus, in effect, we prescribe $\Delta_{II}(\pi_1, \pi_{Wh})$ as in Section 8, and then set

$$\Delta_{II}(\pi_1, \pi) = \Delta_{II}(\pi_1, \pi_{Wh})$$

for all $\pi \in \Pi^{\theta,\varpi}$.

It remains to assemble the spectral transfer factor

$$\Delta_0(\pi_1, \pi) = \Delta_I(\pi_1, \pi)\Delta_{II}(\pi_1, \pi)\Delta_{III}(\pi_1, \pi).$$

Like $\Delta_0(\gamma_1, \delta)$, it is independent of the choice of toral data, a-data and χ-data, but depends on spl^*. It is clear that

$$\Delta_0(\pi_1, \pi)/\Delta_0(\gamma_1, \delta) = \Delta_{comp}(\pi_1, \pi; \gamma_1, \delta),$$

so that the factors $\Delta_0(\pi_1, \pi)$ and $\Delta_0(\gamma_1, \delta)$ are compatible. Now, for Whittaker normalization $\Delta_{_{Wh}}$, we again replace Δ_I by $\varepsilon_L(V, \psi_\mathbb{R}).\Delta_I$ to shift the dependence from spl^* to the $G(\mathbb{R})$-conjugacy class of the Whittaker data (B_{Wh}, λ_{Wh}). From the compatibility for Δ_0 it is clear that geometric and spectral Whittaker normalizations are compatible also.

Since by definition, $\Delta_I(\pi_1, \pi) = \Delta_I(\pi_1, \pi_{wh})$, and by arrangement, $\Delta_{II}(\pi_1, \pi) = \Delta_{II}(\pi_1, \pi_{wh})$,we have immediately the formula

$$\begin{aligned}\Delta_{Wh}(\pi_1,\pi) &= \Delta(\pi_1,\pi;\pi_1,\pi_{wh}).\Delta_{_{Wh}}(\pi_1,\pi_{Wh})\\ &= \Delta_{III}(\pi_1,\pi;\pi_1,\pi_{wh}).\Delta_{_{Wh}}(\pi_1,\pi_{Wh})\\ &= \Delta_{III}(\pi_1,\pi).\Delta_{_{Wh}}(\pi_1,\pi_{Wh})\\ &= \langle inv_\theta(\pi_{Wh},\pi),s_{Wh}\rangle^{-1}.\Delta_{_{Wh}}(\pi_1,\pi_{Wh})\end{aligned}$$

for all discrete series π and well-positioned π_1.

If we limit our attention to ordinary endoscopy ($\theta = id$, $\varpi = triv$) then by Theorem 11.1 of [S4] we have

$$\Delta_{_{Wh}}(\pi_1,\pi_{Wh}) = 1.$$

We will refer to this as the *strong base point property.* It seems reasonable to expect this in the twisted case also, as much of a descent argument is already available (see Appendix).

11. **Norms and parabolic descent**

We start with some remarks on norm correspondences. Consider an inner twist (G,θ,ψ), where the quasi-split data (G^*,θ^*) are arbitrary. In particular, we drop the assumption G^* is cuspidal. Write

$$\psi\sigma(\psi)^{-1} = Int(u(\sigma)),$$

where $u(\sigma)\in G^*_{sc}$.

Lemma 11.1. *The following are equivalent: (i) the automorphism $\psi^{-1}\circ\theta^*\circ\psi$ of G is defined over $\mathbb{R}$, (ii) the automorphism $\psi\sigma(\psi)^{-1}$ of G^*commutes with θ^*, (iii) we may choose $u(\sigma)$ fixed by θ^*_{sc}.*

Proof. All implications are immediate from definitions except $(ii)\Rightarrow(iii)$ for which we use the result that the natural projection $(G^*_{sc})^{\theta^*_{sc}}\to(G^*_{ad})^{\theta^*_{ad}}$ is surjective. □

Lemma 11.2. *Any inner class of inner twists $\psi: G\to G^*$ has a representative ψ such that $\psi\sigma(\psi)^{-1} = Int(u(\sigma))$, where $u(\sigma)\in(G^*_{sc})^{\theta^*_{sc}}$.*

Proof. By a familiar argument we adjust ψ within its inner class to assume that $\psi: T_G\to T$ is defined over $\mathbb{R}$, where T_G, T are fundamental. Then $u(\sigma)\in T_{sc}$. Let $M_G = Cent(S_{T_G},G)$ and $M = Cent(S_T,G^*)$, where S_{T_G},S_T are the maximal $\mathbb{R}$-split tori in T_G,T respectively; we may assume that $S_T\subset S_{T^*}$, where T^* is part of our $\mathbb{R}$-splitting for G^*. We restrict ψ to M_G, obtaining an inner twist to M. Notice that we have $u(\sigma)$ in the inverse image $M_{(sc)}$ of M in G_{sc} instead of in M_{sc}. We may further adjust ψ so that it carries a fundamental splitting *spl* in M_G to a θ^*-stable fundamental splitting in M, with $u(\sigma)$ remaining in T_{sc}. Then it is clear that $\psi^{-1}\circ\theta^*\circ\psi$, as automorphism of M_G, preserves *spl* and so is defined over $\mathbb{R}$. Then we may choose $u(\sigma)\in(M_{(sc)})^{\theta^*_{sc}}\subset(G^*_{sc})^{\theta^*_{sc}}$, and the lemma is proved. □

We will assume now that $\psi\sigma(\psi)^{-1} = Int(u(\sigma))$, where $u(\sigma)\in G^*_{sc}$ is fixed by θ^*_{sc}. Returning to Section 3.1 of [KS], we change notation slightly: z_σ becomes $z_\sigma(\theta)$, and we write $\psi^{-1}\circ\theta^*\circ\psi = Int(h_\theta)\circ\theta$, where $h_\theta\in G_{sc}$. Then our assumption on $u(\sigma)$ implies that $h_\theta^{-1}\sigma(h_\theta)$ is central and $z_\sigma(\theta) = \psi_{sc}(h_\theta^{-1}\sigma(h_\theta))$, as earlier (Section 3). Again we ignore ψ_{sc} and regard $z_\sigma(\theta)$ as a cocycle in $Z_{G_{sc}}$. Our only interest in this paper is the case that the image under $Z_{G_{sc}}\to Z_G$ of $z_\sigma(\theta)$, is a coboundary in Z_G (and we continue our usual practice of not distinguishing in notation between $z_\sigma(\theta)$ and its image). Then writing $h_\theta^{-1}.\sigma(h_\theta) = z.\sigma(z)^{-1}$, we obtain an element $h_\mathbb{R} = z.h_\theta$ in $G(\mathbb{R})$ such that $Int(h_\mathbb{R}) = Int(h_\theta)$. The element z is determined uniquely up to multiplication by an element $z_\mathbb{R}$ of $Z_G(\mathbb{R})$.

It is useful now to recall precisely what we mean by norm correspondence in this setting. We appeal, but in a trivial way, to the construction of Section 5.4 of [KS] for the case $z_\sigma(\theta) \neq 1$. Thus we say strongly G-regular γ_1 in an endoscopic group $H_1(\mathbb{R})$ is a *z-norm* of strongly θ-regular $\delta \in G(\mathbb{R})$ if there are toral data including tori T, θ^*-admissible in G^*, and T_1 in H_1, along with an element $x \in G^*_{sc}$ with the following property:

$$\delta^* = x.m(\delta).\theta^*(x)^{-1} = x.\psi(\delta.h_\theta^{-1}).\theta^*(x)^{-1}$$

lies in T and $N(z^{-1}\delta^*)$ coincides with the image of γ_1 under $p : T_1 \to T_H \to T_{\theta^*}$ provided by the choice of toral data. Notice that $N(z^{-1}\delta^*)$ lies in $T_{\theta^*}(\mathbb{R})$ since

$$z_\sigma(\theta).\sigma(\delta^*)^{-1}\delta^* = \sigma(z^{-1}\delta^*)^{-1}.z^{-1}\delta^* \in (1-\theta^*)T.$$

Clearly, γ_1 is an z-norm of δ if and only if γ_1 is a $z_\mathbb{R}z$-norm of $\delta z_\mathbb{R}$.

Denote by Δ_z the geometric transfer factors defined using z-norms. The relative factors

$$\Delta_z(\gamma_1, \delta; \gamma_1', \delta')$$

and

$$\Delta_{z_\mathbb{R}z}(\gamma_1, \delta z_\mathbb{R}; \gamma_1', \delta' z_\mathbb{R})$$

are equal because the element $z_\mathbb{R}$ is inserted only into the element D constructed for Δ_{III} (see p. 33 of [KS]) where it has no affect. Thus we may (and will) normalize the absolute factors so that

$$\Delta_z(\gamma_1, \delta) = \Delta_{z_\mathbb{R}z}(\gamma_1, \delta z_\mathbb{R})$$

for all such pairs. The choice of z affects the correspondence on test functions. If $f_1 \in Trans(f)$ for z-norms (notation of Section 4) then clearly $f_1 \in Trans(f_{z_\mathbb{R}})$ for $z_\mathbb{R}z$-norms, where $f_{z_\mathbb{R}}$ denotes the translate of f by $(z_\mathbb{R})^{-1}$. A calculation using the extended version of Lemma 5.1.A of [KS] then shows that if $z_1 \in Z_{H_1}(\mathbb{R})$ has image in $Z_H(\mathbb{R})$ equal to the image of $z_\mathbb{R}$ under N, *i.e.* $(z_1, z_\mathbb{R}) \in C(\mathbb{R})$ as in Section 7, then

$$\varpi_C(z_1, z_\mathbb{R}).(f_1)_{z_1} \in Trans(f)$$

for $z_\mathbb{R}z$-norms. In view of Lemma 7.3 this will show that if the spectral factors $\Delta_z(\pi_1, \pi)$ and $\Delta_{z_\mathbb{R}z}(\pi_1, \pi)$ are compatible with geometric Δ_z and $\Delta_{z_\mathbb{R}z}$ respectively, then

$$\Delta_z(\pi_1, \pi) = \Delta_{z_\mathbb{R}z}(\pi_1, \pi)$$

for all G-regular related pairs (π_1, π), where if $\pi(\theta, \varpi)$ is used in the definition on the left then $\varpi_\pi(z_\mathbb{R}).\pi(\theta, \varpi)$ must be used on the right. Similar comments apply also to Chapter 3 of [KS] for the case $z_\sigma(\theta) = 1$, where the map m is unique only up to translation by an element $z_\mathbb{R}$ in the image of $Z_{G_{sc}}(\mathbb{R})$. The canonicity statement in Theorem 3.3.A in [KS] should be modified to take account of this (there is of course no problem for the trivial inner form, *i.e.* the quasi-split setting, where we take m to be the identity map).

We introduce another variant of the norm definition that will be helpful for parabolic descent. We will assume $z_\sigma(\theta) = 1$ for convenience. We recall once again that strongly G-regular $\gamma_1 \in H_1(\mathbb{R})$ is a norm, in the ordinary sense, of strongly

θ-regular $\delta \in G(\mathbb{R})$ if there are toral data including tori T, θ^*-admissible in G^*, and T_1 in H_1, along with an element $x \in G^*_{sc}$ with the property that

$$\delta^* = x.m(\delta).\theta^*(x)^{-1} = x.\psi(\delta.h_\theta^{-1}).\theta^*(x)^{-1}$$

lies in T and $N(\delta^*) = p(\gamma_1)$. Recall also that $N(\delta^*) \in T_{\theta^*}(\mathbb{R})$ because

$$\sigma(\delta^*)^{-1}.\delta^* = (\theta^* - 1)v(\sigma).$$

For our second variant, assume that $z^\dagger \in Z_{G^*}$ is such that that $N(z^\dagger) \in (Z_{G^*})_{\theta^*}(\mathbb{R})$. Then we will call γ_1 a $z^\dagger$-norm of δ if we have instead $N(\delta^\dagger) = p(\gamma_1)$, where $\delta^\dagger = (z^\dagger)^{-1}\delta^*$. Of course, if $z^\dagger \in Z_{G^*}(\mathbb{R})$ then γ_1 is a $z^\dagger$-norm of δ if and only if γ_1 is an ordinary norm of $(z^\dagger)^{-1}\delta$. In general, γ_1 is a $z^\dagger$-norm of δ if and only if $N(z^\dagger)^{-1}\gamma_1$ is an ordinary norm of δ, for all $z_1^\dagger \in Z_{H_1}(\mathbb{R})$ such that $p(z_1^\dagger) = N(z^\dagger)$. We attach relative geometric transfer factors to the $z^\dagger$-norm correspondence using $\delta^\dagger$ in place of δ^*. This change has no effect on Δ_{III} since $\delta^\dagger$ and δ^* determine the same element D as before, and hence no effect on the canonical factor.

We consider parabolic descent associated with θ^*-stable triples (P, M, T), where P, T are defined over $\mathbb{R}$ and $P \supseteq T$ is a parabolic subgroup of G^* that has $M = Cent(S_T, G)$ as Levi subgroup. Such a triple is prescribed by the choice of θ^*-stable pair (B, T), where B also has the property that if α is a root of T in B then so is $\sigma\alpha$ unless $\sigma\alpha = -\alpha$. There are some cases where this choice is not possible, *i.e.* we cannot attach θ^*-stable P to θ^*-stable (M, T), but we will ignore them here. We assume harmlessly that $S_T \subseteq S_{T^*}$, where T^* is part of the fixed $\mathbb{R}$-splitting of G^*.

Suppose (G, θ, ψ) is an inner twist of (G^*, θ^*) and that T_G is a maximal torus over $\mathbb{R}$ in G. Assume that ψ_M in the inner class of ψ is such that $\psi_M : T_G \to T$ is defined over $\mathbb{R}$, where (P, M, T) is a θ^*-stable triple in G^*. Let $M_G = Cent(S_{T_G}, G) = \psi_M^{-1}(M)$. Then M_G is defined over $\mathbb{R}$ and $\psi_M : M_G \to M$ is an inner twist. We may further assume that ψ_M carries a fundamental splitting spl_{fm} over T_G to a θ^*-stable fundamental splitting of G^* over T, so that $\theta_{fm} = (\psi_M)^{-1} \circ \theta^* \circ \psi_M$ preserves spl_{fm} and thus is defined over $\mathbb{R}$. Set $P_G = \psi_M^{-1}(P)$. Then P_G is defined over $\mathbb{R}$. Notice that (P_G, M_G, T_G) is not necessarily θ-stable. Its $G(\mathbb{R})$-conjugacy class is, however, θ-stable: $\theta(P_G)$ and P_G, parabolic subgroups defined over $\mathbb{R}$, are conjugate under $G(\mathbb{C})$ by construction, and hence by a theorem of Borel and Tits they are conjugate under $G(\mathbb{R})$; we then modify this conjugation by an element of $P_G(\mathbb{R})$ to obtain $Int(g) \circ \theta$, where $g \in G(\mathbb{R})$, preserving the triple. Write θ_{M_G} for its restriction to M_G. The element g is uniquely determined up to multiplication by an element of $Norm(T_G, M_G(\mathbb{R}))$. To guarantee that θ_{M_G} is acceptable we assume that when we write it in the form $Int(l) \circ \theta_{fm}$ we may take $l \in M_G(\mathbb{R})$. Then we may adjust our choice of g to assume $\theta_{M_G} = \theta_{fm}$.

By the tempered G-regular series for (P_G, M_G, T_G) we will mean the (unitarily) induced representations

$$\pi = Ind(\pi_{M_G(\mathbb{R})} \otimes I_{N_G(\mathbb{R})} : P_G(\mathbb{R}), G(\mathbb{R})) = I(\pi_{M_G(\mathbb{R})}),$$

where $\pi_{M_G(\mathbb{R})}$ belongs to the discrete series for $M_G(\mathbb{R})$, N_G is the unipotent radical of P_G, and the parameter for π is G-regular. This series is then (θ, ϖ)-stable in the sense that it is invariant under the operation $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$. In the setting of the next paragraph there will be packets Π of these representations that are themselves (θ, ϖ)-stable, because the endoscopic group provides Langlands parameters with the appropriate property.

We turn to G-regular related pairs (π_1, π) : π_1 is a tempered irreducible admissible representation of $H_1(\mathbb{R})$ under which the subgroup $Z_1(\mathbb{R})$ acts by the character ϖ_1, where Z_1 is the central torus $Ker(H_1 \to H)$, and π is a tempered irreducible admissible representation of $G(\mathbb{R})$ such that $\pi \circ \theta \approx \varpi \otimes \pi$. The representations have matching Langlands parameters φ_1 and φ, and G-regularity means that $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ is a torus. We follow closely the case of ordinary endoscopy to define descent. Suppose that φ_1 is regular elliptic in ${}^L M_1$, where (P_1, M_1, T_1) is a triple in H_1. We may define a θ^*-stable triple (P, M, T) and admissible isomorphism $T_1/Z_1 \to T_{\theta^*}$. Because the parameter φ is relevant to G we may attach (P_G, M_G, T_G) as above, after adjusting the inner twist to ψ_M. As we have noted, there is no harm in assuming this new twist maps a given fundamental splitting of M_G to a fundamental splitting of M. We regard M_1 as an endoscopic group for M_G. The endoscopic data are $\mathfrak{e}_M = (M_H, {}^L M \cap \mathcal{H}, s)$, with $\mathfrak{e} = (H, \mathcal{H}, s)$ as earlier, and the embedding in the z-pair is defined by restriction. We construct representatives $\varphi_1(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$, now regular elliptic for M_1 and M_G. These parameters have attached discrete series packets $\Pi_{M_1(\mathbb{R})}$ and $\Pi_{M_G(\mathbb{R})}$. Then π_1 and π are parabolically induced from $\pi_{M_1(\mathbb{R})} \in \Pi_{M_1(\mathbb{R})}$ and $\pi_{M_G(\mathbb{R})} \in \Pi_{M_G(\mathbb{R})}$.

We continue with the same setting. Suppose $\pi \circ \theta \approx \varpi \otimes \pi$, where π belongs to this tempered G-regular series. Then there exists $h \in G(\mathbb{R})$ such that that θ maps (P_G, M_G, T_G) to its image under $Int(h^{-1})$ and $\pi_{M_G(\mathbb{R})} \circ Int(h) \circ \theta \approx \varpi \otimes \pi_{M_G(\mathbb{R})}$. Here we do not change notation when restricting ϖ to $M_G(\mathbb{R})$. Because of G-regularity, the restriction of $Int(h) \circ \theta$ to M_G coincides with θ_{M_G} up to a conjugation in $M_G(\mathbb{R})$. Thus we may take this restriction to be θ_{M_G} itself. Then $(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})})$ is an M_G-regular elliptic related pair for the endoscopic group M_1 and twist $(M_G, \theta_{M_G}, \psi_M)$ of (M, θ^*_M).

Now we describe a parabolic descent for geometric transfer. Consider the triples (G, θ, ψ) and $(M_G, \theta_{M_G}, \psi_M)$. We will assume for convenience that $h_\theta = 1$, so that $z_\sigma(\theta) = 1$ and we have an ordinary norm correspondence for (G, θ, ψ); otherwise we would simply use a modified norm correspondence in that case also. Norms for ${}_g\theta = Int(g) \circ \theta$ as automorphism of G will be used also. Recall that ${}_g\theta$ preserves M_G and its restriction to M_G is θ_{M_G} which we write from now on as θ_{des}. Since $z_\sigma({}_g\theta) \neq 1$, we use the z_g-norm correspondence defined above. Here z_g is specified by the choice of g as $h_{\mathbb{R}}$: set $h_{{}_g\theta} = g_{sc}^{-1}$, where $g_{sc} \in G_{sc}$ is chosen with same image in G_{ad} as g, and then there exists a unique element z_g in Z_G such that $g = z_g.g_{sc}$ (in this equation g_{sc} has been identified with its image in G).

An element δ of $G(\mathbb{R})$ is strongly ${}_g\theta$-regular if and only if δg is strongly θ-regular. The norm correspondences for ${}_g\theta$ and θ have been constructed so that $\gamma_1 \in H_1(\mathbb{R})$ is a z_g-norm of δ if and only if γ_1 is an ordinary norm of δg. Also a now straightforward argument with relative factors shows that we may normalize geometric transfer factors so that

$$\Delta_{z_g}(\gamma_1, \delta) = \Delta(\gamma_1, \delta g)$$

whenever γ_1 is a z_g-norm of strongly ${}_g\theta$-regular δ. Then, with consistent normalization of Haar measures, we have

$$Trans_\theta(f) = Trans_{{}_g\theta}(f_g),$$

where f_g is the right g-translate of f. The implications for spectral factors are clear:

$$\Delta_{z_g}(\pi_1, \pi) = \Delta(\pi_1, \pi),$$

where we use operator $\pi(g) \circ \pi(\theta, \varpi)$ in the definition on the left if we use $\pi(\theta, \varpi)$ on the right.

If $\delta \in M_G(\mathbb{R})$ is strongly ${}_g\theta$-regular as element of $G(\mathbb{R})$ then δ is strongly θ_{des}-regular in $M_G(\mathbb{R})$. Suppose that δ has z_g-norm γ_1 in $M_1(\mathbb{R})$. We now define a norm correspondence for the automorphism θ_{des} of M_G under which γ_1 is again a norm of δ, at least up to $G(\mathbb{R})$-conjugacy.

Write ψ_M as the restriction to M_G of $Int(x) \circ \psi$, where $x \in G_{sc}$. From the assumption $h_\theta = 1$ we have $\theta^* = \psi \circ \theta \circ \psi^{-1}$.

Lemma 11.3. *(i) $z_M = \theta^*(x).g_{sc}^{-1}.x^{-1}$ lies in the center of $M_{(sc)}$. (ii) Let $z^\dagger = z_M.(z_g)^{-1}$. Then $N(z^\dagger) \in (Z_M)_{\theta^*}(\mathbb{R})$.*

Proof. We obtain (i) from definitions: ${}_g\theta$ acts on M_G as $(\psi_M)^{-1} \circ \theta^* \circ \psi_M$. For (ii), we start with strongly ${}_g\theta$-regular $\delta \in M_G(\mathbb{R})$. Because we have arranged that $z_\sigma(\theta_{des}) = 1$, there are toral data for M_1 and the quasi-split form M, including $p : T_1' \to T_1'/Z_1 \to (T')_{\theta^*}$, and $l \in M_{(sc)}$ such that

$$l.\psi_M(\delta).\theta_M^*(l)^{-1} = \delta_M^*$$

lies in T' and $N(\delta_M^*) \in (T')_{\theta^*}(\mathbb{R})$. We rewrite δ_M^* as

$$\begin{aligned} & lx.\psi(\delta).x^{-1}.(\theta^*(x)g_{sc}^{-1}.x^{-1})^{-1}.\theta^*(l)^{-1}.(z^M)^{-1} \\ = \quad & lx.\psi(\delta).g_{sc}.\theta^*(lx)^{-1}.(z^M)^{-1}. \end{aligned}$$

Then

$$lx.\psi(\delta).g_{sc}.\theta^*(lx)^{-1} = z^M.\delta_M^*,$$

or

$$lx.\psi(\delta).(g_{{}_g\theta})^{-1}.\theta^*(lx)^{-1} = z^M.\delta_M^*.$$

We may take the left side of this equation as our definition of δ^* in the construction for the z_g-norm of δ. Since $N((z_g)^{-1}\delta^*) \in (T')_{\theta^*}(\mathbb{R})$, we conclude that

$$N(z_M.(z_g)^{-1}) \in (Z_M)_{\theta^*}(\mathbb{R}),$$

and the lemma is proved. □

We regard $z^\dagger$ as an element of Z_{M_G}, with $N(z^\dagger)$ a real coinvariant for θ_{des}. Write Δ_{M_G} for the (relative and absolute) geometric transfer factors for the $z^\dagger$-norm correspondence. For spectral and compatibility factors our earlier constructions apply: statements are the same if we replace the coboundary $z_\sigma(\theta_{des})$ of Sections 3 and 7 by $z_\sigma = z^\dagger.(\sigma(z^\dagger))^{-1}$.

Lemma 11.4. *Suppose strongly ${}_g\theta$-regular $\delta \in M_G(\mathbb{R})$ has $z^\dagger$-norm γ_1 in $M_1(\mathbb{R})$. Then: (i) γ_1 is a z_g-norm of δ and (ii) if (γ_1', δ') is another such pair, then*

$$\Delta_{M_G}(\gamma_1, \delta; \gamma_1', \delta') = \Delta_{z_g}(\gamma_1, \delta; \gamma_1', \delta'),$$

provided the discriminant term Δ_{IV} is removed from both Δ_{M_G} and Δ_{z_g}.

Proof. The assertion (i) follows immediately from the last proof. For (ii) we may use any toral, a- and χ-data to calculate Δ_g. So we choose toral data from the definition of $z^\dagger$-norms and χ-data trivial for roots outside M. Then only Δ_{III} requires further argument. The cochain $v(\sigma)$ is

$$lx.u(\sigma).\sigma(lx)^{-1} = l.(xu(\sigma)\sigma(x)^{-1}).\sigma(l)^{-1}$$

which may be used as the corresponding cochain for M. The rest now follows. □

There is one more step in this descent for norm correspondences: to show that each ${}_g\theta$-conjugacy class in the stable ${}_g\theta$-conjugacy of δ in $G(\mathbb{R})$ contains an element in $M_G(\mathbb{R})$. In the notation of Section 7 of [S1] this says

$$\mathcal{D}_{{}_g\theta}(T^\delta) = \mathcal{D}_{\theta_{des}}(T^\delta),$$

where T^δ is a maximal torus over $\mathbb{R}$ in M_G that is $(Int(\delta)\circ\, {}_g\theta)$-admissible. By a two-step argument it is enough to do the case that δ is regular elliptic in M_G. That case is proved by an easy generalization of the argument for ordinary stable conjugacy in [S6] (Theorem 2.1 there). Further details are omitted.

Now, by (ii) of Lemma 11.4, we may normalize absolute transfer factors Δ_{z_g} for the z_g-norm correspondence and absolute transfer factors Δ_{M_G} for the $z^\dagger$-norm correspondence so that

$$\Delta_{M_G}(\gamma_1, \delta) = \Delta_{z_g}(\gamma_1, \delta),$$

provided the discriminant term Δ_{IV} is removed from both Δ_{M_G} and Δ_{z_g}, for all pairs (γ_1, δ) as in the lemma.

Recall that we have also arranged $\Delta_{z_g}(\gamma_1, \delta) = \Delta(\gamma_1, \delta g)$, where Δ is our original transfer factor for (G, θ, ψ). In Section 14 of [S2] we recalled descent for ordinary stable orbital integrals which we now denote $f_1 \to Des(f_1)$. Then

$$Des(Trans_\theta(f)) = Des(Trans_{{}_g\theta}(f_g))$$

consists of test functions on $M_1(\mathbb{R})$. Now we observe that we may define a ${}_g\theta$-twisted descent from test functions on $G(\mathbb{R})$ to test functions on $M_G(\mathbb{R})$, denoted $Des_{{}_g\theta}$, with the following property: if $Des(f) = Des_{{}_g\theta}(f_g)$ then

$$Des(Trans_\theta(f)) \subseteq Trans_{\theta_{des}}(Des(f))$$

for all test functions f on $G(\mathbb{R})$. This points us to the definition of G-regular spectral factors.

Consider a G-regular related pair (π_1, π), where $\pi_1 = I(\pi_{M_1(\mathbb{R})}), \pi = I(\pi_{M_G(\mathbb{R})})$, with $\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})}$ in the discrete series for $M_1(\mathbb{R}), M_G(\mathbb{R})$ respectively. Let $f_1 \in Trans_\theta(f)$. Then

$$St\text{-}Tr\ \pi_1(f_1) = St\text{-}Tr\ \pi_{M_1(\mathbb{R})}(Des(f_1)).$$

This standard calculation is written in the present setting in [S5]. Normalize the absolute factor $\Delta_{M_G}(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})})$ according to the definitions of Section 10: it is to be compatible with geometric Δ_{M_G} and we use the $z^\dagger$-norm correspondence. Assume, for the purpose of motivating the definitions of the next paragraph, that we have dual transfer for $(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})})$. Since

$$Des(f_1) \in Trans_{\theta_{des}}(Des(f)),$$

this means that

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi_M} \Delta_{M_G}(\pi_{M_1(\mathbb{R})}, \pi_M)\ Trace[\pi_M(Des(f)) \circ \pi_M(\theta_M, \varpi)],$$

where the summation is over all π_M in the packet of $\pi_{M_G(\mathbb{R})}$ such that $\pi_M \circ \theta_M \approx \varpi \otimes \pi_M$. A calculation shows that for suitably related normalizations of $\pi(\theta, \varpi)$, $\pi_M(\theta_M, \varpi)$ we have

$$\begin{aligned} & Trace[\pi_{M_G(\mathbb{R})}(Des(f)) \circ \pi_M(\theta_M, \varpi)] \\ = \ & Trace[\pi(f_g) \circ \pi({}_g\theta, \varpi)] \\ = \ & Trace[\pi(f) \circ \pi(\theta, \varpi)]. \end{aligned}$$

Then to obtain a transfer identity for (π_1, π) we must arrange that

$$\Delta(\pi_1, \pi) = \Delta_{M_G}(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})}).$$

First we define $\Delta_{spec}(\pi_1, \pi; \pi_1', \pi')$ using $v_{\pi_{M_G(\mathbb{R})}}(\sigma)$, $v_{\pi_{M_G'(\mathbb{R})}}(\sigma)$ *etc.,* for any two G-regular related pairs (π_1, π), (π_1', π'). Some checking shows that it depends only on the normalizations for $\pi(\theta, \varpi), \pi'(\theta, \varpi)$. Similarly, we define $\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$. Now we recheck the assertions in Sections 9 and 10 in this general setting, and are then done. As a final check on definitions, we see that in the descent setting of the last paragraph we do have

$$\Delta(\pi_1, \pi) = \Delta_{M_G}(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})})$$

since $\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta)$ also has an appropriate descent property:

$$\Delta_{comp}(\pi_1, \pi; \gamma_1, \delta g) = \Delta_{comp}(\pi_{M_1(\mathbb{R})}, \pi_{M_G(\mathbb{R})}; \gamma_1, \delta),$$

with (γ_1, δ) as in Lemma 11.3. On the left we use the automorphism θ of G, and on the right we use the automorphism θ_{des}, the restriction of $Int(g) \circ \theta$ to M_G.

12. **Dual transfer and structure on packets**

Let Δ_{geom} and Δ_{spec} be transfer factors with compatible normalization: Δ_{geom} comes from [KS] while Δ_{spec} and compatibility are as defined in the present paper. For each $f \in \mathcal{C}(G(\mathbb{R}), \varpi)$, define $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ by the very regular geometric transfer of [S1], or use C_c^∞ in place of $\mathcal{C}$. Then we write the dual transfer of stable tempered traces as a (proposed) identity of the form

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi)\ Trace\ \pi(f) \circ \pi(\theta, \varpi)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ under which the subgroup $Z_1(\mathbb{R})$ acts by the character ϖ_1, where Z_1 is the central torus $Ker(H_1 \to H)$.

The factors $\Delta_{spec}(\pi_1, \pi)$ have not been defined for tempered related pairs (π_1, π) that are G-singular. In the untwisted case that was done using coherent continuation principles [S3]. Also, the G-regular case itself is not complete: we have ignored some contributions from induced representations.

Notice that dual spectral transfer, so formulated, implies that we can recover geometric transfer from tempered spectral transfer: if a pair

$$(f, f_1) \in \mathcal{C}(G(\mathbb{R}), \varpi) \times \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$$

satisfies the spectral identity for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ under which the central subgroup $Z_1(\mathbb{R})$ acts by the character ϖ_1, then (f, f_1) also satisfies the very regular geometric identity

$$SO(\gamma_1, f_1) = \sum_{\delta, \theta\text{-}conj} \Delta_{geom}(\gamma_1, \delta) O(\delta, f)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. We may argue this as we did in Section 16 of [S2] for ordinary endoscopy, the only new ingredients being the (as yet incomplete) definitions of the spectral factors involved and the existence of very regular twisted geometric transfer [S1].

Mezo [M] has proved character identities that provide spectral factors defined differently from our $\Delta_{spec}(\pi_1, \pi)$ for the G-regular discrete series transfer. We observe next that, under the assumption of Section 8, his factors coincide with ours, up to normalization. The argument is based on the calculation of Section 13 of [S3]. We may assume that θ preserves a fundamental splitting, take δ_π and γ_π to be identity elements, and use a balanced normalization for the operators $\pi(\theta, \varpi)$ as at the end of Section 9.

Lemma 12.1. *Under these assumptions, suppose that* Π_1, Π *are matching packets of discrete series representations. Suppose that the complex numbers* $c(\pi_1, \pi)$ *satisfy*

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi \in \Pi} c(\pi_1, \pi)\ Trace\ \pi(f) \circ \pi(\theta, \varpi)$$

for $\pi_1 \in \Pi_1$ *and all pairs* (f, f_1) *of test functions satisfying very regular geometric transfer by means of transfer factors* Δ_{geom} *compatible with* Δ_{spec} *in the sense of Section 10. Then*

$$c(\pi_1, \pi) = \Delta_{spec}(\pi_1, \pi),$$

for $\pi_1 \in \Pi_1$, $\pi \in \Pi$.

Proof. To prove the lemma it is enough to show that $f \to St\text{-}Tr\ \pi_1(f_1)$ is represented around the identity by

$$\sum_{\pi \in \Pi} \Delta_{spec}(\pi_1, \pi)\ Tw\text{-}Ch_\pi,$$

with $Tw\text{-}Ch_\pi$ as in Section 8, because then we can appeal directly to the explicit form for $Tw\text{-}Ch_\pi$ to see that $c(\pi_1, \pi) = \Delta_{spec}(\pi_1, \pi)$, for $\pi_1 \in \Pi_1$, $\pi \in \Pi$.

Assume that the twisting character ϖ is trivial. To establish the setting, we fix elliptic toral data as in Section 6. Then we return to the setting of Lemma 6.2 of [S2], with now (G, θ) in place of (G^*, θ^*) since there is no harm in assuming $\theta = \psi^{-1} \circ \theta^* \circ \psi$. Now we may follow word-for-word the calculation of Section 13 of [S3]; we have matched notation in the present paper for this purpose. Starting on the endoscopic side, the Weyl integration formula brings us to the elliptic torus. We follow the toral calculations and transfer them to our elliptic torus in G. The crucial Remark 7.1 is replaced by the first equation from Lemma 9.6 of [S2] (it is enough to consider reflections). The transfer factor constants are simplified with the same steps, but now using the twisted analogues from the present paper. Finally the Weyl group indices match up when we use a twisted version of the Weyl integration formula, with our conventions for normalizing measures as in [S2], to lift back to the full twisted elliptic set.

For the case that ϖ is nontrivial we proceed in the same manner but with Δ_{III} handled more carefully, as described in Section 5 of [S2] for orbital integrals; see Appendix. ☐

We finish by describing how our spectral factors fit with expected structure on discrete series packets in the quasi-split setting.

Suppose that G^* is cuspidal and start with $G = G^*$ and $\theta = \theta^*$ (later we add some inner forms to complete packets). Fix a θ^*-stable fundamental splitting $spl_f = (B, T, ...)$ of G^*. We may assume that the simple roots of T in B are all noncompact (*i.e.* spl_f is an $\mathbb{R}$-opp splitting of G^*) because G^* is both cuspidal and quasi-split over $\mathbb{R}$. Assume $Int(h)$ maps spl_f to spl^*, where $h \in G^*_{sc}$.

Let Π be an L-packet of discrete series representations of $G(\mathbb{R})$. We use the (nearly) canonical representative $\varphi = \varphi(\mu, \lambda)$ for its Langlands parameter as in Section 5. Assume that the packet Π is preserved by (θ, ϖ) so that

$$S^{tw}_\varphi = S^{tw} = \{s \in G^\vee :\ {}^L\theta_a \circ \varphi = Int(s^{-1}) \circ \varphi\}$$

is a nonempty subset of $\mathcal{T}$. Let $s \in S^{tw}$. Attach endoscopic data $\mathfrak{e}(s) = (H, \mathcal{H}, s)$ for (G^*, θ^*) as on p. 24 of [KS], and let (H_1, ξ_1) be a z-pair for $\mathfrak{e}(s)$. Then $\mathfrak{e}(s)$ is elliptic and we define the well-positioned matching parameter $\varphi^s = \varphi(\mu_1, \lambda_1)$ for H_1 by $\mu_1 = \mu - \mu^*$ and $\lambda_1 = \lambda - \lambda^*$, the pair (μ^*, λ^*) being attached to the endoscopic data and z-pair as in Section 5. Let π^s denote an element of the packet attached to φ^s.

Fix θ-stable Whittaker data Wh, and let π_{Wh} denote the unique element of the twist-packet $\Pi^{\theta,\varpi}$ in Π that is generic for Wh. Then we define

$$\langle \pi, s \rangle_{tw} = \Delta(\pi^s, \pi; \pi^s, \pi_{Wh})$$

for all $\pi \in \Pi^{\theta,\varpi}$. If we assume the expected strong base-point property (see Section 8) then

$$\Delta_{Wh}(\pi^s, \pi_{Wh}) = 1,$$

so that

$$\langle \pi, s_{tw} \rangle = \Delta_{Wh}(\pi^s, \pi),$$

and the spectral transfer statement becomes simply

$$\textit{St-Trace}\ \pi^s(f^s) = \sum_{\pi \in \Pi^{\theta,\varpi}} \langle \pi, s \rangle_{tw}\ \textit{Trace}\ \pi(f) \circ \pi(\theta, \varpi),$$

where test function f^s on $H_1(\mathbb{R})$ is attached to given test function f on $G(\mathbb{R})$ by geometric transfer. Otherwise $\Delta_{Wh}(\pi^s, \pi_{Wh})$ appears outside the sum on the right.

We introduce also ordinary (*i.e.* untwisted) transfer for Π. Then S^{tw} is replaced by $S = Cent(\varphi(W_{\mathbb{R}}), G^\vee)$, again contained in $\mathcal{T}$, now as a subgroup. To $s_0 \in S$ attach ordinary endoscopic data $\mathfrak{e}_0(s_0) = (H_0, \mathcal{H}_0, s_0)$ for G^*, and let $(H_{0,1}, \xi_{0,1})$ be a z-pair for $\mathfrak{e}_0(s_0)$. Then we attach well-positioned parameter φ^{s_0} for $H_{0,1}$ matching the parameter φ for Π; π^{s_0} denotes an element of the packet attached to φ^{s_0}. We will also enlarge $G = G^*$ to a K-group without change in notation except that we will write Π_{full} for the extended packet containing Π (see [S4] for the precise setting).We use the Whittaker data already specified to define Whittaker normalizations of the ordinary geometric and spectral transfer factors. Then we attach test function f^{s_0} on $H_{0,1}(\mathbb{R})$ to f on $G(\mathbb{R})$ already given. The ordinary spectral transfer result [S4] is

$$\textit{St-Trace}\ \pi^{s_0}(f^{s_0}) = \sum_{\pi \in \Pi} \langle \pi, s_0 \rangle\ \textit{Trace}\ \pi(f).$$

Here

$$\langle \pi, s_0 \rangle = \Delta_{Wh}(\pi^{s_0}, \pi)$$

defines a perfect pairing (in an obvious sense) between Π_{full} and $S \ / \ (Z_{G^\vee})^\Gamma$ [S4].

This perfect pairing is calculated using Tate-Nakayama duality for T_{Wh} (which we will now denote simply T) or for its preimage T_{sc} in G_{sc}. Notice that it is possible to choose $T_{Wh} = T$ to be any maximal torus over $\mathbb{R}$ in G that is anisotropic mod Z_G and θ^*-admissible. It will be useful to recall some details of the pairing. First, we attach to π the element $inv(\pi_{Wh}, \pi)$ of

$$\mathcal{E}(T) = \operatorname{Im}(H^1(\Gamma, T_{sc}) \longrightarrow H^1(\Gamma, T))$$

as usual from the Harish Chandra picture (this is worked in detail in [S4] for extended packets). Second, we project s_0 as element of $(T^\vee)^\Gamma$ to

$$s_0(T) \in \pi_0(T^\vee)^\Gamma.$$

Then, by definition,

$$\langle \pi, s_0 \rangle = \langle inv(\pi_{Wh}, \pi), s_0(T) \rangle .$$

Because

$$inv(\pi_{Wh}, \pi) \in \mathcal{E}(T),$$

$\langle \pi, s_0 \rangle$ depends only in the image of $s_0(T)$ in the finite group $(T_{ad}^\vee)^\Gamma$, where by $T_{ad}^\vee$ we mean $(T_{sc})^\vee$. The image of

$$(T^\vee)^\Gamma \longrightarrow \pi_0((T^\vee)^\Gamma) \longrightarrow \pi_0((T_{ad}^\vee)^\Gamma) = (T_{ad}^\vee)^\Gamma$$

is naturally identified with $S \ / \ (Z_{G^\vee})^\Gamma$, so it is clear we may calculate the pairing with either T or T_{sc}. Recall that it is necessary to proceed more cautiously outside the quasi-split setting (in [S4] we work directly with the canonical relative spectral factor $\Delta(\pi^s, \pi; \pi^s, \pi'))$; see [A2] for what is needed in general.

Suppose π lies in the twist-packet $\Pi^{\theta,\varpi} \subseteq \Pi$ (we will ignore the rest of Π_{full} here). Then for each $s \in S^{tw}$ we have defined $\langle \pi, s \rangle_{tw}$. In Section 9, $\langle \pi, s \rangle_{tw}$ was calculated using the Tate-Nakayama pairing for the torus $(T_{sc})^{\theta_{sc}}$. Notice that if $s_1, s_2 \in S^{tw}$ then $s = s_1(s_2)^{-1}$ lies in S, so that $\langle \pi, s \rangle$ is well-defined.

Lemma 12.2. *For* $\pi \in \Pi^{\theta,\varpi}$, *we have*

$$\langle \pi, s_1 \rangle_{tw} \ = \langle \pi, s \rangle \ . \ \langle \pi, s_2 \rangle_{tw} .$$

Proof. We calculate

$$\begin{aligned} \langle \pi, s_1 \rangle_{tw} &= \langle \pi, s s_2 \rangle_{tw} = \langle inv_\theta(\pi_{Wh}, \pi), \ (s s_2)_{Wh} \rangle \\ &= \langle inv_\theta(\pi_{Wh}, \pi), \ s_{Wh} \rangle . \langle inv_\theta(\pi_{Wh}, \pi), \ (s_2)_{Wh} \rangle , \end{aligned}$$

where s_{Wh} denotes the image of s as Γ-invariant in $T^\vee$ under

$$(T^\vee)^\Gamma \longrightarrow (T_{ad}^\vee)^\Gamma \longrightarrow ((T_{ad}^\vee)_{\theta_{ad}^\vee})^\Gamma,$$

$(T_{ad}^\vee)_{\theta_{ad}^\vee}$ being the dual of $(T_{sc})^{\theta_{sc}}$. We can just as well project only as far as $(T_{ad}^\vee)^\Gamma$ and project $inv_\theta(\pi_{Wh}, \pi)$ to untwisted $inv(\pi_{Wh}, \pi)$ in $H^1(\Gamma, T_{sc})$, to compute $\langle inv_\theta(\pi_{Wh}, \pi), s_{Wh} \rangle$ as untwisted $\langle \pi, s \rangle$. This proves the lemma. □

We can interpret the last lemma very simply as follows. We will do just the case ϖ is trivial. Replace the dual group $G^\vee$ by $G^\vee \rtimes \langle \theta^\vee \rangle$ and consider instead the group

$$\overline{S} = Cent(\varphi(W_\mathbb{R}), G^\vee \rtimes \langle \theta^\vee \rangle),$$

where $\theta^\vee$ acts on $\varphi(W_\mathbb{R})$ by its action on the first component, *i.e.* by ${}^L\theta$. Then both S and S^{tw} embed in $\overline{S}$. First, $s \mapsto s \times 1$ embeds S as

$$Cent(\varphi(W_\mathbb{R}), G^\vee \times 1),$$

and then $t \mapsto t \times \theta^\vee$ embeds S^{tw} as

$$Cent(\varphi(W_\mathbb{R}), G^\vee \times \theta^\vee).$$

For each $\pi \in \Pi^{\theta,\varpi}$, Lemma 12.2 provides a necessary and sufficient condition that the function

$$t.(Z_{G^\vee})^\Gamma \mapsto \langle \pi, t \rangle_{tw}$$

on the quotient set $S^{tw} / \ (Z_{G^\vee})^\Gamma$ determines (uniquely) a sign character on the group $\overline{S} \ / \ (Z_{G^\vee})^\Gamma$ extending both itself and the character

$$s.(Z_{G^\vee})^\Gamma \mapsto \langle \pi, s \rangle$$

on the group $S\ /\ (Z_{G^\vee})^\Gamma$. In particular, in the case of quasi-split special orthogonal groups, we see that we have the correct pairing for discrete series packets in the statement of Theorem 2.2.4 in [A].

Appendix:

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