Tempered endoscopy for real groups I: geometric transfer with canonical factors

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Real groups offer many opportunities to explore Langlands' principle of functoriality in the L-group. The example we consider here begins in the paper of Labesse and Langlands [LL] on automorphic representations of $SL(2)$: certain representations with same local L-factors may be one automorphic and the other not. A little more precisely, Labesse and Langlands determined a multiplicity formula for these representations $\pi = \otimes_v \pi_v$ in terms of the *position* of the local representations π_v among representations with same L-factors, *i.e.* in the same local L-packet. See [S1] for a brief report on Langlands' lecture at Corvallis. The *tempered endoscopy for real groups* of the title refers to Langlands' proposed generalization, for real reductive groups, of the analysis of orbital integrals and tempered representations used in the $SL(2)$ proof. The Arthur conjectures [A2] pursue this beyond the tempered spectrum.

Following a recent suggestion of Arthur [A1], we present a proof of the geometric transfer in tempered ordinary endoscopy for real groups based directly on the canonical transfer factors defined by Langlands and myself in [LS1] for any local field of characteristic zero. These factors are not only independent of the way we view the Cartan subgroups of the endoscopic group as Cartan subgroups of the given group but are also given by an explicit formula on each such subgroup that appears significant for a number of problems in invariant harmonic analysis. A previous proof of the transfer of orbital integrals involved rigidly defined factors with an implicitly defined sign [S2, S3, S4, S5]. Then a limit formula for regular unipotent orbital integrals ([LS1], Theorem 5.5.A) confirmed that the canonical factors are correct and, up to a global constant, the same as the implicitly defined factors ([LS2], Theorem 2.6.A).

Once we have completed our discussion of the canonical factors and a direct, but equally long, argument for the existence of geometric transfer, we will also recall briefly the dual transfer of tempered characters from [S5]. We relabel certain welldefined coefficients from [S5] as tempered spectral transfer factors following Arthur [A1]. The implicit sign persists, however, along with questions about normalization and compatibility. In an accompanying paper [S7] we will begin again and define canonical spectral transfer factors in a simple manner that directly parallels the approach for the geometric transfer factors we discuss in the present paper, and we will show, again directly, that they are correct for transfer. Section 16 summarizes the final transfer theorem we will prove in [S7]. We should mention here that it is the *relative* transfer factors that are canonical. We will conclude with a lemma from [S7] which shows that when we normalize the spectral factors to be signs we also obtain a simple local form for the geometric factors.

We have chosen to limit our discussion to ordinary endoscopy, not only to give a more direct presentation of that topic with the canonical factors, but also to prepare a template for our approach when twisting is present. There various technical matters complicate arguments in a general setting. One feature from twisted endoscopy that we will use here is passage to a z-extension of the endoscopic group, and we prefer to label the z-extension rather the base group as the endoscopic group. This replaces passage to a z-extension of the group we start with, a device which resolves a technical problem in L-group embeddings only for the ordinary case. A minor needed modification of the transfer factor is available from [KS]. On the other hand, the norm map is simpler in the ordinary case, and we retain the older terminology of *image* from [LS1] rather than *norm (*[KS], Chapter 2).

We start then with the geometric side: conjugacy classes and orbital integrals. Our approach involves most naturally Harish Chandra's space of (complex-valued) Schwartz functions. We match the orbital integrals of an arbitrary Schwartz function f on a group $G(\mathbb{R})$ with the stable orbital integrals of a Schwartz function f_1 on an endoscopic group $H_1(\mathbb{R})$, using the canonical transfer factors. This yields a correspondence (f, f_1) of Schwartz spaces. We then obtain a welldefined dual map from tempered stable eigendistributions Θ_1 on $H_1(\mathbb{R})$ to tempered invariant eigendistributions Θ on $G(\mathbb{R}) : \Theta(f) = \Theta_1(f_1)$. The image of the stable tempered characters under the dual map has been calculated in [S5]. The starting point is of course

Harish Chandra's characterization of discrete series characters among tempered invariant eigendistributions. We take this up in [S7]; for now, we will simply rewrite results from [S5] in the language of spectral transfer factors.

The assumption of temperedness in the map on eigendistributions can be dropped if we use a theorem of Bouaziz [B] characterizing the stable orbital integrals of smooth functions of compact support. But that takes us beyond the scope of this discussion and into the realm of the Arthur conjectures [A2] and results of [ABV].

Much of the paper consists of collecting and applying results from several quite long papers, and we include introductory remarks at various points along the way. In particular, there are some informal comments on terms in the geometric transfer factors in Section 8. To begin, we will review in some detail a characterization of stable orbital integrals by their *jump conditions,* in order to make more transparent the significance of canonically defined transfer factors for our proof of geometric transfer.

1. Stable conjugacy in real groups

Throughout, G will denote a connected reductive algebraic group defined over $\mathbb{R}$, and σ (or σ_G) will denote the Galois action on $G(\mathbb{C})$, so that $G(\mathbb{R}) = \{g \in G(\mathbb{C}) : \sigma(g) = g\}$. It is sufficient for now to limit our discussion to regular semisimple elements. Thus suppose γ is regular semisimple in $G(\mathbb{R})$. Typically, the centralizer $Cent(\gamma, G)$ of γ in G is connected, and we then call γ *strongly regular*. In that case the stable conjugacy class of γ in $G(\mathbb{R})$ is simply the intersection of its conjugacy class in $G(\mathbb{C})$ with $G(\mathbb{R})$. In general, however, Langlands prescribes in [L1] that we take a smaller set of $G(\mathbb{C})$-conjugates. Let $g \in G(\mathbb{C})$. If $g^{-1}\gamma g$ lies in $G(\mathbb{R})$ then $g\sigma(g)^{-1}$ belongs to $Cent(\gamma, G)$. Then $g^{-1}\gamma g$ is called a *stable conjugate* of γ if $g\sigma(g)^{-1}$ lies in the identity component of $Cent(\gamma, G)$, a maximal torus in G which we will denote T_γ, or T if there is no confusion. Equivalently, we could require that the map $t \rightarrow g^{-1}tg$ from T_γ to $T_{\gamma'}$ be defined over $\mathbb{R}$. As a simple example, the images of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ under the natural homomorphism $GL_2(\mathbb{R}) \rightarrow PGL_2(\mathbb{R})$ are $PGL_2(\mathbb{C})$-conjugate but not stably conjugate.

Let S_T be the maximal $\mathbb{R}$-split torus in T, and M_T, or just M, be the centralizer $Cent(S_T, G)$ of S_T in G. Then T is a fundamental maximal torus in M, and an argument involving parabolic subgroups defined over $\mathbb{R}$ shows that $g^{-1}\gamma g$ is a stable conjugate of γ if and only if $gG(\mathbb{R})$ contains an element of $M(\mathbb{C})$ normalizing T ([S2], Theorem 2.1).Then to get a complete set of representatives for the conjugacy classes in the stable conjugacy class of γ, we may take the elements $w^{-1}\gamma w$,where w belongs to a complete set of representatives for the quotient of the normalizer of $T(\mathbb{R})$ in $M(\mathbb{C})$ by the normalizer in $M(\mathbb{R})$. Equally as well, we could regard the elements w as representatives for the quotient of the *imaginary* Weyl group $\Omega(T(\mathbb{C}), M(\mathbb{C}))$ of T, which we will abbreviate by $\Omega_{im}(T)$, by the subgroup $\Omega_{im,\mathbb{R}}(T)$ of those elements realized in $M(\mathbb{R})$. Notice that there may be redundancy unless γ is strongly regular but, neverless, for continuity reasons this is the set we use for indexing combinations of orbital integrals for all regular semisimple elements.

An invariant function on the regular semisimple set is thus *stably invariant* exactly when its restriction to *each* Cartan subgroup is invariant under the full imaginary Weyl group. In particular, the function Θ^* appearing in Harish Chandra's construction of discrete series characters is stably invariant: Θ^* is better behaved than the individual terms Θ_w, of which it is the sum. We will see also that stable orbital integrals are better behaved than ordinary orbital integrals.

2. Stable orbital integrals

In view of the passage to a z-extension that we will be making shortly, we work modulo a central subgroup. Thus suppose that Z_0 is a torus lying in the center of G, and that λ_0 is a character on $Z_0(\mathbb{R})$ (...there will be no harm in assuming λ_0 unitary). We denote by $\mathcal{C}(G(\mathbb{R}), \lambda_0)$ the set of all complex-valued functions f on $G(\mathbb{R})$ that, first, are Schwartz modulo $Z_0(\mathbb{R})$, *i.e.* if we factor off the split component of $Z_0(\mathbb{R})$ from $G(\mathbb{R})$ then f is Schwartz on the complementary subgroup and that, second, transform under $Z_0(\mathbb{R})$ according to λ_0^{-1}, i.e. $f(zg) = \lambda_0^{-1}(z)f(g)$ for all $z \in Z_0(\mathbb{R})$, $g \in G(\mathbb{R})$. In our application, we will take Z_0 trivial for the given group G, recovering the ordinary Schwartz space $\mathcal{C}(G(\mathbb{R}))$, while for an attached endoscopic group H_1 we take Z_0 to be the torus Z_1 specified in a z-extension.

In regard to normalization of Haar measures, some formulas will require consistency of choices. Thus we use invariant differential forms of highest degree to specify measures dg on $G(\mathbb{R})$ and dt on a Cartan subgroup $T(\mathbb{R})$ in a canonical manner (see Section 1.4 of [LS1]). Then if T and $T' = g^{-1}Tg$ are defined over $\mathbb{R}$ we may attach to dt a measure dt^g on $T'(\mathbb{R})$.

For γ regular semisimple in $G(\mathbb{R})$ and f in $\mathcal{C}(G(\mathbb{R}), \lambda_0)$ the orbital integral

$$O_\gamma(f, dt, dg) = \int\limits_{T_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}\gamma g)\tfrac{dg}{dt}$$

is well-defined. If γ is strongly regular then the stable orbital integral of f at γ,which we will write as $SO_\gamma(f, dt, dg)$, or $SO_\gamma(f)$ when the measures are understood, is then simply the sum of the integrals $O_{\gamma^w}(f, dt^w, dg)$ over w belonging to a complete set of representatives for the conjugacy classes in the stable conjugacy class of γ. There is no harm in assuming w normalizes T_γ. Then $dt^w = dt$, and the summation is over a complete set of representatives for the quotient $\Omega_{im}(T)/\Omega_{im,\mathbb{R}}(T)$. This is the definition, *i.e.* the summation, we will use also if regular semisimple γ fails to be strongly regular.

We recall first Harish Chandra's $'F_f^T$ transform (adjusted for conjugation as a right action on $G(\mathbb{R})$). Our main source for this topic is [HC2] ; it contains references to earlier papers where many of the proofs begin. For γ regular semisimple in $T(\mathbb{R})$,

$$'F_f^T(\gamma) = \Delta'(\gamma)O_\gamma(f),$$

where the normalizing factor Δ', a modified Weyl denominator, requires the choice of a positive system for the imaginary roots of T. Then

$$\Delta'(\gamma) = \left|\det{}_{\mathfrak{g}/\mathfrak{m}}(Ad(\gamma) - I)\right|^{1/2} \prod_{\alpha>0, imaginary} (\alpha(\gamma) - 1),$$

where $\mathfrak{g}, \mathfrak{m}$ denote the Lie algebras of G, M respectively. If we choose instead a positive system for all roots of T and use the notation $\left|z^{1/2} - z^{-1/2}\right|$ for $\left|(1-z)(1-z^{-1})\right|^{1/2}$ then we may rewrite this as

$$\Delta'(\gamma) = \prod_{\alpha>0,\text{real}} \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right| \prod_{\alpha>0,\text{complex}} \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right|$$
$$\times \prod_{\alpha>0,imaginary} (\alpha(\gamma) - 1).$$

Recall that a root α is called real if $\sigma\alpha = \alpha$, complex if $\sigma\alpha \neq \pm\alpha$, or imaginary if $\sigma\alpha = -\alpha$. Thus α is imaginary if and only if its Galois orbit is *symmetric* in the sense of [LS1].

If G is simply-connected and semisimple, and $T(\mathbb{R})$ is compact, then we may replace Δ' by the more convenient skew symmetric Weyl denominator Δ :

$$\Delta(\gamma) = \iota(\gamma)^{-1}\Delta'(\gamma),$$

where ι is one-half the sum of the positive roots, welldefined as a character on $T(\mathbb{R})$ under the given assumption. Then locally we have

$$\Delta(\gamma) = \prod_{\alpha>0} (\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}).$$

In general, while a group and one of its endoscopic groups may each fail to have welldefined symmetric denominators, there is always a welldefined term that behaves much like a quotient of symmetric denominators: the *transfer factor* of [LS1] which we will discuss in Section 8.

A theorem of Harish Chandra asserts that ${}'F_f^T$ extends to a Schwartz function on the set $T(\mathbb{R})_{I-reg}$ of all elements of $T(\mathbb{R})$ regular as elements of M. It remains then to describe its behavior of near those elements γ of $T(\mathbb{R})$ such that $\alpha(\gamma) = 1$ for at least one positive imaginary root α of T. For our purposes it will be sufficient, again by a principle of Harish Chandra, to consider only those elements γ_0 annihilated by exactly one positive imaginary root α, i.e. elements of $T(\mathbb{R})$ lying on exactly one imaginary wall $\{\gamma : \alpha(\gamma) = 1\}$. Then $Cent(\gamma_0, G)$ is of type A_1. It is either split modulo center, and $\pm\ \alpha$ are *noncompact* roots, or anisotropic modulo center, and $\pm\ \alpha$ are *compact* roots. Noncompact roots are sometimes called *nonsingular*.

Consider now the wall defined by a positive imaginary root α. For $\nu \in \mathbb{R}^\times$ and $|\nu|$ sufficiently small, the element $\gamma_\nu = \gamma_0 \exp i\nu\alpha^\vee$ of $T(\mathbb{R})$ is strongly regular. Also, let $\mathcal{S}$ be the algebra of all invariant differential operators on $T(\mathbb{R})$ and $D \to \widehat{D}$ denote the automorphism of $\mathcal{I}$ given on the Lie algebra by $H \to H - \iota(H)I$. Then both $\lim_{\nu\to 0+} \widehat{D}\ 'F_f^T(\gamma_\nu)$ and $\lim_{\nu\to 0-} \widehat{D}\ 'F_f^T(\gamma_\nu)$ are well-defined (because $'F_f^T$ is Schwartz), and if f is such that they are always equal then $'F_f^T$ extends to a Schwartz function on $T(\mathbb{R})$. For general f, Harish Chandra's descent to the identity component of $Cent(\gamma_0, G)$ shows the limits are equal if α is compact, but if α is noncompact their difference, i.e. the *jump* of $\widehat{D}\ 'F_f^T$ across the wall defined by α, is, up to a constant, the value at γ_0 of an appropriate derivative of $'F_f$ calculated on an adjacent Cartan subgroup also containing γ_0. Notice that because f lies in $\mathcal{C}(G(\mathbb{R}), \lambda_0)$ it is sufficient to consider operators D in the subalgebra $\mathcal{S}_0$ of $\mathcal{I}$ obtained by embedding the symmetric algebra on the Lie algebra of T/Z_0 in that for T, and we will often do so without comment.

We will need to apply the precise form of this jump not just for the function $\gamma \to {'F_f^T}(\gamma)$ but also for its stable conjugates, i.e. for all functions

$$\gamma \to {'F_f^{T^w}}(w^{-1}\gamma w),$$

where $w^{-1}\gamma w$ is a stable conjugate of γ. It is more useful *not* to modify the normalizing factor, i.e. to work instead with the function

$$'F_f^w(\gamma) = \Delta'(\gamma) O_{\gamma^w}(f).$$

Now we are concerned not just with an imaginary root α but with its orbit under the full imaginary Weyl group (modulo the subgroup $\Omega_{im,\mathbb{R}}(T)$). We call α totally compact if every root in this orbit is compact. Note that there are no totally compact roots if G is quasisplit over $\mathbb{R}$ (see [S3], Lemma 9.2). If α is totally compact then all the functions $\widehat{D}\ 'F_f^w$ have zero jump across the wall defined by α, by Harish Chandra descent to the groups $Cent(\gamma_0^w, G)^0(\mathbb{R})$, all of which are compact modulo center.

For the remaining orbits it will be sufficient for our purposes to consider the case that α itself is noncompact. Then $\pm\alpha$ are the only noncompact roots in the orbit (see [S2], Lemma 4.2), up to the action of $\Omega_{im,\mathbb{R}}(T)$, and so if $w\alpha \neq \pm\alpha$ modulo $\Omega_{im,\mathbb{R}}(T)$ then $\widehat{D}\ 'F_f^w$ has zero jump. We may then assume $w\alpha = \pm\alpha$, so that either w or ww_α fixes α, where w_α denotes the Weyl reflection for α. If this reflection is realized in $M(\mathbb{R})$, *i.e.* belongs to $\Omega_{im,\mathbb{R}}(T)$, and both w and ww_α index the same conjugacy class in the stable conjugacy class of a strongly regular element of $T(\mathbb{R})$, then we set $d(\alpha) = 2$; otherwise, set $d(\alpha) = 1$.

We now assume $w\alpha = \alpha$. If $d(\alpha) = 2$ then there is no harm in this, but if $d(\alpha) = 1$ we will have to consider the contribution from ww_α as well when we come to stable orbital integrals. By a Cayley transform with respect to α we will mean any map $\gamma \to s^{-1}\gamma s$ of T to T^s for which $s\sigma(s)^{-1}$ acts on T as the Weyl reflection w_α (see [S2]). Then T^s is defined over $\mathbb{R}$, and $s\alpha$ is a real root of T^s. Thus for nonzero real ν sufficiently small, we may define the strongly regular element $\gamma_\nu^s = \gamma_0^s \exp \nu(s\alpha^\vee)$ of $T^s(\mathbb{R})$. Note that γ_0^s lies in $T^s(\mathbb{R})_{I-reg}$.

For the jump formula, the positive system used in defining Δ' is required to be *adapted to* α. This ensures that if β is a positive imaginary root not perpendicular to nor equal to α then $\beta' = -w_\alpha(\beta)$ is also positive, and so both these roots appear in Δ'. Since their transport to T^s via s is a pair of complex conjugate roots, we may rewrite $(\beta(\gamma_0) - 1)(\beta'(\gamma_0) - 1)$ as $|s\beta(\gamma_0^s) - 1|\,|s\beta'(\gamma_0^s) - 1|$, simplifying the comparison of Δ' for T and T^s (see Lemma 13.2).

In the jump formula for ${}'F_f^w$ we take s to be a standard Cayley transform, i.e. given by the usual choice of root vectors, and so we have $\gamma_0^s = \gamma_0$. Then Harish Chandra's formula may be written as

$$\begin{aligned} &\lim_{\nu\to 0+} \widehat{D}\ {}'F_f^w(\gamma_\nu, dt, dg) - \lim_{\nu\to 0-} \widehat{D}\ {}'F_f^w(\gamma_\nu, dt, dg) \\ &\quad = id(\alpha) \lim_{\nu\to 0} \widehat{D^s}\ {}'F_f^{s^{-1}ws}(\gamma_\nu^s, dt^s, dg) \end{aligned}$$

(see [S2]] for a more complete discussion and a proof).

We shall normalize the stable combination $SO_\gamma(f)$ with the same factor Δ', setting

$$\Psi(\gamma) = \Delta'(\gamma) SO_\gamma(f).$$

Write Ψ^T for the restriction of Ψ to the regular elements in the Cartan subgroup $T(\mathbb{R})$. Then Ψ^T is the sum of the functions ${}'F_f^w$ over a complete set of representatives w for the conjugacy classes in the stable conjugacy class of a regular element in $T(\mathbb{R})$. Thus Ψ^T extends to a Schwartz function on the set of all points of $T(\mathbb{R})$ not annihilated by a root in the orbit of a noncompact imaginary root. To calculate the jumps across the walls attached to the orbit of a noncompact root α, it is enough to consider only the wall defined by α and then use stable invariance of $SO_\gamma(f)$ and the simple transformation rule for Δ'under the imaginary Weyl group. Examining Ψ^T near semiregular γ_0 with $\alpha(\gamma_0) = 1$, we see that only those w such that $w\alpha = \pm\alpha$ contribute to the jump. These are exactly the elements we need to construct representatives for the conjugacy classes in a stable conjugacy class of strongly regular elements in $T^s(\mathbb{R})$, and thus to form Ψ^{T^s}. If $d(\alpha) = 2$ then we can assume that $w\alpha = \alpha$, and so obtain

$$\lim_{\nu\to 0+} \widehat{D}\ \Psi^T(\gamma_\nu, dt, dg) - \lim_{\nu\to 0-} \widehat{D}\ \Psi^T(\gamma_\nu, dt, dg) \\ = 2i \lim_{\nu\to 0} \widehat{D^s}\ \Psi^{T^s}(\gamma_\nu^s, dt^s, dg).$$

If $d(\alpha) = 1$ and $w\alpha = \alpha$ then w and ww_α each contribute the same jump ($\lim_{\nu\to 0+}$ for one equals $-\lim_{\nu\to 0-}$ for the other). Thus we get the *same* final formula regardless of the value of $d(\alpha)$. We may now also allow s to be any Cayley transform for α. If D is skew with respect to w_α, then both sides of the final formula are zero, whereas if D is symmetric with respect to w_α, then $\lim_{\nu\to 0-}$ equals $-\lim_{\nu\to 0+}$ on the left, giving the simpler formula

$$\lim_{\nu\to 0+} \widehat{D}\ \Psi^T(\gamma_\nu, dt, dg) = i \lim_{\nu\to 0} \widehat{D^s}\ \Psi^{T^s}(\gamma_\nu^s, dt^s, dg)$$

for D symmetric with respect to w_α.

3. Characterization of stable orbital integrals

We consider complex-valued functions $\gamma \to \Phi(\gamma, dt, dg)$ on the regular semisimple set of $G(\mathbb{R})$ with the following properties (for all γ, dt, dg) :

(i) $\Phi(\gamma^w, dt^w, dg) = \Phi(\gamma, dt, dg)$ *for all* γ^w *stably conjugate to* γ,
(ii) $\Phi(\gamma, \alpha dt, \beta dg) = (\beta/\alpha)\ \Phi(\gamma, dt, dg)$ *for all* α, β *in* $\mathbb{C}^\times$,
(iii) $\Phi(z\gamma, dt, dg) = \lambda_0^{-1}(z)\ \Phi(\gamma, dt, dg)$ *for all* z *in* $Z_0(\mathbb{R})$.

Now suppose Φ_T denotes the restriction of Φ to the regular elements of the Cartan subgroup $T(\mathbb{R})$. We will use various objects introduced in the last section. Set $\Psi_T = \Delta'\Phi_T$. Here the choice of dt, dg and of the positive system for the imaginary roots of T used to define Δ' may be fixed arbitrarily and ignored in notation. Then we add decay and smoothness properties:

(iv) Ψ_T *extends to a Schwartz function on* $T(\mathbb{R})_{I-reg}$*,*
(v) $\lim_{\nu \to 0+} \widehat{D}\ \Psi_T(\gamma_\nu) = \lim_{\nu \to 0-} \widehat{D}\ \Psi_T(\gamma_\nu)$ *if* γ_0 *is on a single totally compact wall of* $T(\mathbb{R})$*.*

We could have combined these into a single Schwartz condition, but the given form is more useful.

Next, suppose that γ_0 lies on a single noncompact imaginary wall. Let s be a Cayley transform with respect to either of the noncompact roots annihilating γ_0. Choose a positive system for the imaginary roots of T adapted to that root when defining $\widehat{D}$ and Ψ_T, and use transport by s for $\widehat{D^s}$ and Ψ_{T^s}. Then our final condition is that if D is symmetric with respect to w_α then

$$\text{(vi)}\quad \lim_{\nu \to 0+} \widehat{D}\ \Psi_T(\gamma_\nu, dt, dg) = i \lim_{\nu \to 0} \widehat{D^s}\ \Psi_{T^s}(\gamma_\nu^s, dt^s, dg).$$

Note that γ_0^s lies in $T^s(\mathbb{R})_{I-reg}$, and so the limit on the right could be replaced by the value at γ_0^s. The number i appears on the left side in the definition of $\gamma_\nu : \gamma_\nu = \gamma_0 \exp i\nu\alpha^\vee$. There is then no harm in replacing i by $-i$ on both sides. Finally, we recall again Harish Chandra's principle that if the left side of (vi) is zero for all noncompact imaginary walls, and hence all jumps, across all walls and for all D, are zero by (i) and (iv), then Ψ_T extends to a Schwartz function on $T(\mathbb{R})$.

Theorem 3.1 ([S2],Theorem 4.7)

If $\gamma \to \Phi(\gamma, dt, dg)$ *has the properties (i) - (vi) then there exists* $f \in \mathcal{C}(G(\mathbb{R}), \lambda_0)$ *such that*

$$\Phi(\gamma, dt, dg) = SO_\gamma(f, dt, dg)$$

for all γ *regular semisimple in* $G(\mathbb{R})$*, and all* dt, dg*.*

Define a partial ordering on the set of maximal tori over $\mathbb{R}$ in G by $T \preceq T'$ if and only if S_T is, up to $G(\mathbb{R})$-conjugacy, a subtorus of $S_{T'}$.Then adjacent tori are exactly the pairs T, T^s we have described. An inductive argument shows that it is enough to prove the following theorem (assuming the theorem, start by matching Φ_T to $SO_\gamma(f_1)$ on maximally split T and then replace Φ by $\Phi - SO_\gamma(f_1)$ to apply the theorem again...).

Theorem 3.2 ([S2], Lemma 4.8)

Suppose $\Phi_{T'}$ is defined for all Cartan subgroups $T'(\mathbb{R})$ conjugate in $G(\mathbb{R})$ to a given $T(\mathbb{R})$, satisfies (i) to (iii), and $\Psi_{T'}$ extends to a Schwartz function on $T'(\mathbb{R})$. Then there exists $f \in \mathcal{C}(G(\mathbb{R}), \lambda_0)$ such that

$$\Phi_{T'}(\gamma', dt', dg) = SO_{\gamma'}(f, dt', dg)$$

for γ' regular in $T'(\mathbb{R})$ and

$$SO_{\gamma''}(f, dt'', dg) = 0$$

for all regular γ'' in $T''(\mathbb{R})$ unless $T'' \preceq T$.

Proof: Consider first the example that G is simplyconnected, semisimple and $T(\mathbb{R})$ is compact. Here, keeping in mind the paradigm of characters as orbital integrals of matrix coefficients, we look to the results of Harish Chandra on matrix coefficients of the discrete series representations. We also have the skew symmetric normalizing factor Δ, so we now set $\Psi = \Delta\Phi_T$. Then Ψ extends to a smooth function on $T(\mathbb{R})$ by the hypothesis of the theorem. The invariance of Φ_T under stable conjugacy implies that Ψ is skew symmetric relative to the full Weyl group of T. Thus if we use Fourier inversion on $T(\mathbb{R})$ to write Ψ as a Fourier series $\sum_\Lambda \Psi^\vee(\Lambda)\Lambda$ then the Fourier coefficient $\Psi^\vee(\Lambda)$ vanishes unless the (rational) character Λ is regular, and we may therefore rewrite the expansion of Ψ as a sum over regular characters Λ dominant relative to the positive system defining Δ :

$$\Psi = \sum_\Lambda \Psi^\vee(\Lambda) \sum_w (\det w) w\Lambda,$$

and so

$$\Phi_T = \sum_\Lambda \Psi^\vee(\Lambda)\ \Delta^{-1} \sum_w (\det w) w\Lambda.$$

Here the sums are over the full Weyl group of T. But $\Delta^{-1} \sum_w (\det w) w\Lambda$ is the local formula on $T(\mathbb{R})_{reg}$for the Harish Chandra's tempered distribution Θ^*_Λ and, up to a constant, Θ^*_Λ is a sum of discrete series characters ([HC, HC1]). Let be K be a maximal compact subgroup of $G(\mathbb{R})$. Then, according to theorems of Harish Chandra [HC2], for each regular dominant Λ we can find K-finite discrete series matrix coefficients f_Λ, which all lie in $\mathcal{C}(G(\mathbb{R}))$, with $SO_\gamma(f_\Lambda) = \Delta(\gamma)^{-1} \sum_w (\det w) w\Lambda(\gamma)$ for regular γ in $T(\mathbb{R})$, and also $SO_{\gamma'}(f_\Lambda) = 0$ for regular nonelliptic γ'. Moreover if the K-types of the functions f_Λ are dominated by a polynomial in the length of Λ then the series $\sum_\Lambda \Psi^\vee(\Lambda) f_\Lambda$ converges absolutely in $\mathcal{C}(G(\mathbb{R}))$ and the stable orbital integrals of the sum f satisfy

$$SO_\gamma(f) = \textstyle\sum_\Lambda \Psi^\vee(\Lambda)\; SO_\gamma(f_\Lambda) = \Phi_T(\gamma)$$

for regular γ in $T(\mathbb{R})$, with $SO_{\gamma'}(f) = 0$ for regular nonelliptic γ'. To finish this argument we may use a result of Vogan on minimal K-types. See the discussion of [S2]; we will return to K-types later.

To consider now the general case, we note first that the above argument is easily modified to apply to a Cartan subgroup compact modulo the center in a general reductive algebraic group $G(\mathbb{R})$. So it applies to any $T(\mathbb{R})$ if we replace $G(\mathbb{R})$ by $M(\mathbb{R})$, where $M = Cent(S_T, G)$. Suppose K_M is a maximal compact subgroup in $M(\mathbb{R})$. Then to adapt the above argument to general $T(\mathbb{R})$ we need to know how to pass from the the K_M-finite discrete series matrix coefficients f_Λ, now in $\mathcal{C}(M(\mathbb{R}), \lambda_0)$, to functions in $\mathcal{C}(G(\mathbb{R}), \lambda_0)$ with appropriate orbital integrals. Again we find the answer in Harish Chandra's Plancherel theory [HC3].

We recall briefly some results about tempered characters before describing the rest of our argument in the next section.

4. Stable tempered characters

We are concerned with tempered irreducible admissible representations π of $G(\mathbb{R})$ such that $\pi(zg) = \lambda_0(z)\pi(g)$, for all $z \in Z_0(\mathbb{R})$ and $g \in G(\mathbb{R})$. If $f \in \mathcal{C}(G(\mathbb{R}), \lambda_0)$ then $\pi(f)$ is the operator $\int_{G(\mathbb{R})/Z_0(\mathbb{R})} f(g)\pi(g)dg$, and $Tr\ \pi$

denotes the character of π as tempered distribution, i.e. as the continuous linear form $Tr\ \pi : f \to Trace\ \pi(f)$ on $\mathcal{C}(G(\mathbb{R}), \lambda_0)$. We write χ_π for the analytic function on the regular semisimple set of $G(\mathbb{R})$ which represents Tr π. Recall that by a theorem of Harish Chandra,

$$Tr\ \pi(f) = \int_{G(\mathbb{R})/Z_0(\mathbb{R})} f(g)\chi_\pi(g)dg$$

for any $f \in \mathcal{C}(G(\mathbb{R}), \lambda_0)$. The distribution St-$Tr\ \pi$, the stable trace of π, may be defined as the (finite) sum over representations π' in the L-packet of π of the distributions $Tr\ \pi'$. It is represented by the function

$$\chi_\pi^{st} = \sum_{\pi'} \chi_{\pi'}$$

which is invariant under stable conjugacy. When we come to the spectral side of endoscopy we see that all tempered irreducible characters on $G(\mathbb{R})$ are recovered by the transfer maps from the stable tempered characters on the endoscopic groups for G ([S5, S8]).

The definition of St-$Tr\ \pi$ is ad hoc in the sense that it depends explicitly on the classification of tempered irreducible representations of $G(\mathbb{R})$, and most particularly on Harish Chandra's construction of the discrete series characters. Thus assume that G is cuspidal, *i.e.* that G has a maximal torus T over $\mathbb{R}$ such that $T(\mathbb{R})$ is compact modulo the center of $G(\mathbb{R})$. We describe the stable discrete series characters by characters Λ on $T(\mathbb{R})$. We write Λ as $\Lambda(\mu - \iota, \lambda)$, where $(\mu - \iota, \lambda)$ are its Langlands parameters (see Section 8). Here ι is one half the sum of the roots of a positive system for which μ is, by assumption, dominant regular. For each w in the Weyl group, the character $\Lambda(w^{-1}\mu - \iota, \lambda)$ is also welldefined. Harish Chandra's distribution Θ^* is given on the regular elements γ of $T(\mathbb{R})$ by

$$\Theta^*(\gamma) = \frac{\sum_w (\det w)\Lambda(w^{-1}\mu - \iota, \lambda)(\gamma)}{\prod_{\alpha>0}(1-\alpha(\gamma)^{-1})}.$$

It does not depend on the choice of positive system, and is invariant under stable conjugacy on each Cartan subgroup ([HC], Section 24). Finally, $(-1)^{q_G}\Theta^*$ is the sum of the characters of irreducible representations attached to the real Weyl group orbits in the full orbit ([HC1], Theorem 16). Here of course we have to pass from the cited results to a general reductive algebraic group, but that is routine. These representations π form an L-packet [L3], and $St\text{-}Tr\ \pi = (-1)^{q_G}\Theta^*$ for each such π. Here $2q_G$ is the dimension of the quotient of $G_{sc}(\mathbb{R})$ by a maximal compact subgroup.

The remaining stable tempered characters are obtained by parabolic induction from cuspidal Levi groups. Thus we start with a general Cartan subgroup $T(\mathbb{R})$ and consider the packet of representations π^M contributing to the discrete series character $(-1)^{q_M}\Theta_M^*$ on $M(\mathbb{R})$ given by the same formula except that now the sum is over the full imaginary Weyl group and ι is one half the sum of the roots in a positive system of imaginary roots with respect to which μ is assumed dominant. Let P be a parabolic subgroup of G defined over $\mathbb{R}$ and N be its unipotent radical. Then the character of $\Pi = Ind(\oplus\pi^M \otimes I_{N(\mathbb{R})}; P(\mathbb{R}), G(\mathbb{R}))$ is stably invariant on $G(\mathbb{R})$. Its irreducible summands π form an L-packet and each occurs with multiplicity one in Π. Thus again $St\text{-}Tr\ \pi$ is defined appropriately as the sum of the characters in the L-packet of π. Otherwise we would count the summands with multiplicity, as it is the induced character that is stable; stability within M is due to Harish Chandra's theorem and the rest, invariance under conjugacy in $G(\mathbb{R})$, comes from the inducing process.

Returning now to the proof of Theorem 3.1, we start now with $\Psi = \Delta'\Phi_T$ which extends to a Schwartz function on $T(\mathbb{R})$. When we apply Fourier inversion on $T(\mathbb{R})$ we obtain a series indexed by stable discrete series characters on $M(\mathbb{R})$, but now each term in the series may be rewritten as an integral over the dual of the Lie algebra of the split component of $T(\mathbb{R})$ of normalized stable tempered principal series characters. We then find how to construct a suitable function from Harish Chandra wave packets of Eisenstein integrals from [HC3]. This is described in detail in [S2].

We will use the following in Section 16 to see that spectral matching of functions implies geometric matching.

Theorem 4.1 ([[S2], Lemma 5.3])

Let $f \in \mathcal{C}(G(\mathbb{R}), \lambda_0)$. *Then* $St\text{-}Tr(\pi)(f) = 0$ *for all tempered irreducible representations* π *such that* $\pi(zg) = \lambda_0(z)\pi(g)$, for all $z \in Z_0(\mathbb{R})$ and $g \in G(\mathbb{R})$, *if and only if* $SO_\gamma(f) = 0$ *for all strongly regular* γ *in* $G(\mathbb{R})$.

Proof: If the given stable orbital integrals of f are zero then the Weyl integration formula for $G(\mathbb{R})/Z_0(\mathbb{R})$ shows that for each given π the value of

St-$Tr(\pi)(f)$ is zero. For the converse, we argue as for Theorem 3.1. If the stable tempered traces of f are zero then we can conclude from Fourier inversion that the smooth function Ψ_T, made from the stable orbital integrals of f for the (strongly) regular classes meeting a maximally split Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$, vanishes. Then for T' adjacent to T, the function $\Psi_{T'}$ also extends smoothly to the whole Cartan subgroup and so again vanishes. We continue the argument by induction. See [S2] for details.

5. Endoscopy

An endoscopic group H_1 is prescribed to meet two sets of demands, one geometric and one spectral. It comes with various additional data which we will describe following [LS1] and [KS]. A homomorphism of L-groups ${}^LH_1 \to {}^LG$ *almost* exists. To deal with this minor complication for the functoriality principle we follow the approach of [KS]. There will be a group $\mathcal{H}$ and embeddings of $\mathcal{H}$ in both ${}^LH_1$ and ${}^LG$. This provides us with a map from certain Langlands parameters for H_1 to those for G that is appropriate for the transfer to $G(\mathbb{R})$ of all stable tempered characters on $H_1(\mathbb{R})$ transforming according to a fixed character λ_1 on a central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$. We discuss this further when we define spectral transfer factors [S7].

We denote by G^* a quasisplit inner form of G, with $\mathbb{R}$-splitting spl_{G^*} (a choice of Borel subgroup $\mathbf{B}^*$ defined over $\mathbb{R}$, maximal torus $\mathbf{T}^*$ over $\mathbb{R}$ in $\mathbf{B}^*$, and a root vector X_α for each simple root α of $\mathbf{T}^*$ in $\mathbf{B}^*$) and choose an inner twist $\psi : G \to G^*$. We denote by $G^\vee$ the complex dual of G, with splitting $spl_{G^\vee}$ preserved by the algebraic dual $\sigma_{G^\vee}$ of the Galois action, and by ${}^LG$ the L-group $G^\vee \rtimes W_{\mathbb{R}}$, where the Weil group $W_{\mathbb{R}}$ of $\mathbb{C}/\mathbb{R}$ acts through $W_{\mathbb{R}} \to \{1, \sigma\}$. The transfer factors will be independent of the choice of splittings and, roughly speaking, of twisting ψ within its inner class (we take this up later).

A set of endoscopic data for G is a tuple $(H, \mathcal{H}, s, \xi)$, where:

(i) H is connected, reductive and quasi-split over $\mathbb{R}$, and so has dual $H^\vee$ with splitting $spl_{H^\vee}$ preserved by dual Galois

automorphism $\sigma_{H^\vee}$,

(ii) $\mathcal{H}$ *is a split extension of* $W_\mathbb{R}$ *by* $H^\vee$, *where again* $W_\mathbb{R}$ *acts through* $W_\mathbb{R} \to \{1, \sigma\}$, *and now* σ *acts as* $\sigma_{H^\vee}$ *only up to an inner automorphism of* $H^\vee$,

(iii) $\mathfrak{s}$ *is a semisimple element of* $G^\vee$, *and*

(iv) $\xi : \mathcal{H} \to {}^L G$ *is an embedding of extensions under which the image of* $H^\vee$ *is the identity component of* $Cent(s, G^\vee)$, *and the full image lies in* $Cent(s', {}^L G)$, *for some* s' *congruent to* s *modulo the center of* $G^\vee$.

Standard constructions of endoscopic data start with conjugacy classes [L1] or with representations [LL] (see [S7] for a discussion). For simple concrete examples and counterexamples, we recall that the L-group of a maximal torus T over $\mathbb{R}$ embeds in ${}^L G$ (this is central to the construction of transfer factors). If we take $\mathcal{H}$ to be ${}^L T$, or its embedded image in ${}^L G$, we can ask if it is possible to extend $\mathcal{H}$ to a set of endoscopic data. If $G = GL(2)$, no s exists unless T splits over $\mathbb{R}$, but that is not a problem since stable conjugacy coincides with ordinary conjugacy (but the nonsplit tori do appear in twisted endoscopy). If $G = SL(2)$ and $T(\mathbb{R})$ is compact, then we can take s conjugate to the image of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ under the projection $GL(2, \mathbb{C}) \to PGL(2, \mathbb{C})$. For $G = SU(2, 1)$, the compact Cartan subgroup does not work; instead, the relevant group is larger (it is $U(1, 1)$). See also [S4] and Section 2.1 of [S5].

We will regard H as an endoscopic group only if $\mathcal{H}$ is isomorphic, as split exension of $W_\mathbb{R}$, to ${}^L H$. While the examples where there is no such isomorphism are somewhat complicated in ordinary endoscopy (see [L1], [S4]), there are ample elementary examples when twisting is present. In general, we choose a z-pair (H_1, ξ_1) following Section 2.2 of [KS]. Thus H_1 is a z-extension of H. This means that the derived group of H_1 is simplyconnected, and we have an extension $1 \to Z_1 \to H_1 \to H \to 1$, where Z_1 is a central torus defined over $\mathbb{R}$. Moreover, the torus Z_1 is induced, and then $H_1(\mathbb{R}) \to H(\mathbb{R})$ is surjective. For example, $GL(2)$ is a z-extension of $PGL(2)$, but $SL(2)$ is not. The second datum ξ_1 is an embedding of extensions $\mathcal{H} \to {}^L H_1$ that extends the embedding $H^\vee \to H_1^\vee$ (which we will write as inclusion) dual to $H_1 \to H$. If we take a section $W \to \mathcal{H}$, follow it by ξ_1 and then by ${}^L H_1 \to {}^L Z_1$ (dual to $Z_1 \to H_1$), we obtain the Langlands parameter for a welldefined quasicharacter λ_1 on $Z_1(\mathbb{R})$.As is easily seen (note Remarks 11.3, 11.4), there is no harm in assuming λ_1 is unitary. Each Langlands parameter for H_1 also determines a quasicharacter on $Z_1(\mathbb{R})$. We will be interested in those parameters for which that quasicharacter is λ_1. These are the parameters that, up to $H_1^\vee$-conjugacy, have image in $\xi_1(\mathcal{H})$. We will take this up later ([S7]). We describe parameters for ξ_1 and ξ in Section 11, and again make a harmless unitarity assumption (see Remark 11.4).

Notice that because the derived group of H_1 is simply connected, the embedding ξ_1 always exists [L1]. In descent (Section 7) we will use an embedding $\xi_{1,desc}$ provided directly by the descent. After descent we will have an extension $1 \to Z_1 \to H_{desc,1} \to H_{desc} \to 1$ with same Z_1 again, but we will not assume that the derived group of $H_{desc,1}$ is simply connected. The character λ_1 will not change in parabolic descent, but may do so in the second case, semisimple descent, that we need.

6. Endoscopy and maximal tori

Continuing with endoscopic data $(H, \mathcal{H}, s, \xi)$, we have a canonical map from certain stable conjugacy classes in $H(\mathbb{R})$, the strongly G-regular classes, to strongly regular stable conjugacy classes in the quasisplit form $G^*(\mathbb{R})$ (see [LS1], Section 1.3). At the same time, the inner twist ψ identifies the set of strongly regular stable conjugacy classes in $G(\mathbb{R})$ as a subset of those in $G^*(\mathbb{R})$. If the class of γ in $H(\mathbb{R})$ maps to that of δ in $G(\mathbb{R})$, then we call γ an image of δ (in analogy with the notion of norm [KS]). If γ_1 maps to γ under the surjective map $H_1(\mathbb{R}) \to H(\mathbb{R})$ we also call γ_1 an image of δ.

While the map on classes is canonical, for local analysis we need to be able to switch freely between Cartan subgroups in $G(\mathbb{R})$, $G^*(\mathbb{R})$ and $H_1(\mathbb{R})$ and to transport roots and other objects back and forth, though in a non-canonical way. For this we recall the isomorphisms of Cartan subgroups associated with the choice of Borel subgroups. Later we will formalize the following choices as *toral data.* They are essentially the same as the *fixed framework* of Cartan subgroups in [S3], but now transfer factors will be both explicitly defined and independent of these choices (see Section 12).

Thus suppose that $\gamma_1 \in T_1(\mathbb{R})$ is an image of strongly regular $\delta \in T(\mathbb{R})$. There are Borel subgroups B_1, B of H_1, G respectively such that the attached homomorphism $\eta = \eta_{B_1,B} : T_1 \to T$ is defined over $\mathbb{R}$, and by adjusting δ within its stable conjugacy class we may assume that η maps γ_1 to δ. The homomorphism η is the composition of the inverse of an isomorphism $\psi_T = Int(x) \circ \psi$ with a homomorphism $\eta^* = \eta_{B_1,B^*} : T_1 \to T^*$ defined over $\mathbb{R}$, where the restriction of ψ_T to T is defined over $\mathbb{R}$.

The isomorphism η embeds the coroots of T_1 in H_1 as a *subsystem* of the coroots of T in G, and any root α of T in G is transported to a rational character α_1 on T_1 (the roots of T_1 form a subset but not a subsystem in general). The G-regular elements of T_1 are those γ_1 for which $\alpha_1(\gamma_1) \neq 1$ for all roots α of T in G. The G-walls in $T_1(\mathbb{R})$ will be those $\{\gamma_1 : \alpha_1(\gamma_1) = 1\}$ for which α_1 is not a root in H_1 but α is a root in G (and we then also say that α is a root outside H_1). To detect if α_1 is a root in H_1, we return to the endoscopic datum $\mathfrak{s}$.

Since we will not use it explicitly until [S8], we forgo a detailed discussion of equivalence for endoscopic data. We do need to observe that the datum $\mathfrak{s}$ may be taken in the maximal torus $\mathcal{T}$ of $G^\vee$ provided by $spl_{G^\vee}$. The splitting also provides a Borel subgroup $\mathcal{B}$ containing $\mathcal{T}$, and then attached to $\mathcal{B}$, B we have $\eta^\vee : T^\vee \to \mathcal{T}$ to transport $\mathfrak{s}$ back and forth between $\mathcal{T}$ and $T^\vee$ as needed. If we regard the coroot $\alpha^\vee$ as a character on $T^\vee$ then α_1 is a root in H_1 exactly when $\alpha^\vee(\mathfrak{s}) = 1$. Note also if α is imaginary, so that $\sigma_T\alpha^\vee = -\alpha^\vee$, then $\alpha^\vee(\mathfrak{s}) = \pm 1$.

We recall the weights $\kappa(w)$ for κ-orbital integrals. The map $\mathcal{T} \to T^\vee$ also embeds the center $Z(G^\vee)$ of $G^\vee$ in $T^\vee$ (independently of the choice of $\mathcal{B}$, B). Set $T^\vee_{ad} = T^\vee/Z(G^\vee)$ and $\Gamma = \{1, \sigma\}$. Then using the property (iv) of endoscopic data, we may find $\mathfrak{s}' = z\mathfrak{s}$, where z is in $Z(G^\vee)$, fixed by σ_T, giving then an element of the component group $\pi_0((T^\vee)^\Gamma)$. Set $T^\vee_{ad} = T^\vee/Z(G^\vee)$. The image $\boldsymbol{s}_T$ of this element of $\pi_0((T^\vee)^\Gamma)$ in $\pi_0 = \pi_0((T^\vee_{ad})^\Gamma)$ is independent of the choice of z. By Tate-Nakayama duality, we may pair $\boldsymbol{s}_T$ with an element of $H^1(\Gamma, T_{sc})$, where T_{sc} denotes the preimage of T under $G_{sc} \to G$. If w represents an element of the imaginary Weyl group of T we may assume w is the image of w_{sc} in G_{sc} and set $\boldsymbol{w}$ equal to the element of $H^1(\Gamma, T_{sc})$ determined by the cocycle $w_\sigma = \sigma(w_{sc})w_{sc}^{-1}$. Then $< \boldsymbol{w}, \boldsymbol{s}_T >$ is what we will use as the weight for κ-orbital integrals in the setting of Lemma 12.1, and for δ and δ^w there we write $inv(\delta, \delta^w)$ in place of $\boldsymbol{w}$. This matches the definition of κ-orbital integrals in Section 4 of [S3] where we, in effect, used $\mathfrak{s}'$ to define the pairing. Notice we now replace w by w^{-1} in the summation there. The pairing depends on how we transport $\mathfrak{s}$ to $T^\vee$, *i.e.* on the choice of toral data, and has to be used with some care. Various transformation rules are given in [S3].

7. Two examples of descent in endoscopy

The simplest descent is to a cuspidal Levi group $M = M_T = Cent(S_T, G)$, when T originates in H_1, i.e when there are strongly regular elements in $T(\mathbb{R})$ with images in $H_1(\mathbb{R})$ or, equivalently there is a maximal torus T_1 over $\mathbb{R}$ in H_1 with an isomorphism $\eta_{B_1,B} : T_1 \to T$ over $\mathbb{R}$. Recall that an $\mathbb{R}$-splitting $spl_{G^*} = (\mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\})$ has been fixed, and $\eta_{B_1,B}$ is a composition $T_1 \to T_H \to T^* \to T$. We may assume that S_{T^*} is contained in $S_{\mathbf{T}^*}$, and choose g in G_{sc} such that $\psi_M = Int\ g \circ \psi$ acts on T as the inverse of $T^* \to T$. Then ψ_M carries S_T to S_{T^*} and M to $M^* = Cent(S_{T^*}, G^*)$ which will serve as quasisplit inner form for M, with inner twist ψ_M. For splitting spl_{M^*} we may use $(M^* \cap \mathbf{B}^*, \mathbf{T}^*, \{X_\alpha\})$, and the root vectors X_α for simple roots in M^*. Then we realize $M^\vee$ as the $\sigma_{G^\vee}$-invariant Levi group in $G^\vee$ with dual splitting $spl_{M^\vee} = (M^\vee \cap \mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$. For ${}^LM$ we may take $M^\vee \rtimes W_{\mathbb{R}}$, with the action of $W_{\mathbb{R}}$ on $M^\vee$ inherited from ${}^LG$. Now given endoscopic data $(H, \mathcal{H}, \mathfrak{s}, \xi)$ and z-pair (H_1, ξ_1), we define data $(M_H\ , \mathcal{H}_M, \mathfrak{s}_M, \xi_M)$ and pair $(M_{H_1}, \xi_{1,M})$ for M as follows. We may assume $\mathfrak{s} \in \mathcal{T}$, and then set $\mathfrak{s}_M = \mathfrak{s}$. We may also assume ξ is inclusion, so that $\mathcal{H}$ is a subgroup of ${}^LG$.Then we set $\mathcal{H}_M = \mathcal{H} \cap {}^LM$ and take ξ_M to be inclusion; $\mathcal{H}_M$ is a split extension of $W_{\mathbb{R}}$ by $M^\vee \cap H^\vee$. For M_H we may take a dual Levi group in H and we choose specifically $M_H = Cent(S_{T_H}, H)$, where T_H is the image of T_1 under $H_1 \to H$. Let M_{H_1} be the inverse image of M_H under $H_1 \to H$, so that $M_{H_1} = Cent(S_{T_1}, G)$ and $1 \to Z_1 \to M_{H_1} \to M_H \to 1$ is exact. For embedding $\xi_{1,M}$ of $\mathcal{H}_M$ in ${}^LM_{H_1}$ we take the restriction of ξ_1 to $\mathcal{H}_M$. The attached character on $Z_1(\mathbb{R})$ is again λ_1. If $T \preceq T'$ then, replacing T' by a $G(\mathbb{R})$-conjugate if necessary, we assume $S_{T'}$ contains S_T and descend to $M_{T'}$ through M_T. We will complete our discussion of parabolic descent at the beginning of Section 13. Meanwhile, we will *work in* M, *i.e.* with the pair (M, M_{H_1}), at various points, in preparation for the proof of the transfer theorem.

The second example we consider is *local* descent to the identity component G^{δ_0} of the centralizer of a semisimple element δ_0 of $G(\mathbb{R})$. To establish the geometric transfer theorem, we know by the characterization theorem for orbital integrals that it will be enough to consider the case that δ_0 is semiregular, i.e. that G^{δ_0} is of type A_1. The general setting of the descent in [LS2] for real groups may be used to extend the transfer to other conjugacy classes after we have established its existence on the very regular set. See, for an example, the matching of equisingular semisimple conjugacy classes in Section 2.4 of [LS2] (also recalled in [S9]) and notice that in that setting the stable integral is not simply the sum of the integrals over the contributing conjugacy classes. Within the semiregular setting we will make certain choices in the descent that will allow us to replace sign calculations in [S3] and [S5] with the conclusion that the canonical relative transfer factor has trivial limiting behavior across the appropriate imaginary walls. This conclusion involves some lengthy arguments from [LS2] which we discuss a little further in Section 13 (see also [S9]).

We return to the setting of Section 6 and a homomorphism $\eta = \eta_{B_1,B} : T_1 \to T$ defined over $\mathbb{R}$. We will work inside M, so that T_1 lies in $M_1 = M_{H_1}$, T lies in M, and $\eta_{B_1,B} = \eta_{M_1 \cap B_1, M \cap B}$. Suppose that $\gamma_{1,0} \in T_1(\mathbb{R})$ is semiregular and that the unique root α_1 of T_1 in B_1 annihilating $\gamma_{1,0}$ is imaginary, *i.e.* is in $M_1 \cap B_1$, and is noncompact. We also assume that $\gamma_{1,0}$ lies on none of the walls outside H_1, so that $\delta_0 = \eta(\gamma_{1,0})$ is semiregular in G and is annihilated by the transport α of α_1 to T. We assume that the imaginary root α is not totally compact (we will deal with the totally compact case separately at the end), and then adjust η to assume that α itself is noncompact. We may assume that $M \cap B$ and $M_1 \cap B_1$ provide positive systems for the imaginary roots of T and T_1 that are adapted to α and α_1 respectively. Let γ_0 be the image of $\gamma_{1,0}$ under $H_1(\mathbb{R}) \to H(\mathbb{R})$. Then our assumptions ensure that $H^{\gamma_0} = M_H^{\gamma_0}$ is isomorphic to $G^{\delta_0} = M^{\delta_0}$ over $\mathbb{R}$, and thus that after descent the endoscopy will be just that of a trivial inner twist of G^{δ_0}. To compare transfer factors directly, we will pass from H^{γ_0} to $H_1^{\gamma_{1,0}}$, its inverse image in H_1. For more discussion of the set of endoscopic data obtained by descent we refer to Sections 1.4 - 1.6 of [LS2]. It is straightforward also to attach to $\xi_1 : \mathcal{H} \to {}^L H_1$ an embedding for $\mathcal{H}_{\gamma_0}$ in ${}^L H_1^{\gamma_{1,0}}$, which may modify the attached character λ_1 on $Z_1(\mathbb{R})$; we will not need the details here.

To prepare more carefully for the local information the descent theorem of [LS2] gives us about terms in the transfer factors for G, we reintroduce the quasisplit inner form G^* (and M^*) explicitly into our setup. Thus η is a composition $T_1 \to T_H \to T^* \to T$, and we may assume that the image δ_0^* of γ_0 in T^* is annihilated by a noncompact imaginary root α^* (there are no totally compact roots in the quasisplit form G^*). We write η^* for both maps $T_1 \to T^*$ and $T_H \to T^*$. Then let s_H be a Cayley transform in H^{γ_0} with respect to α_H (the image of α_1) mapping T_H to $T_H^{'}$. The torus $T_H^{'}$ also has admissible embeddings over $\mathbb{R}$ into G, as we will see explicitly. Let s^* be the standard Cayley transform with respect to α in $(G^*)^{\delta_0^*}$, mapping T^* to $T^{*'}$. Then $\eta^{*'} = s^* \circ \eta^* \circ s_H^{-1} : T_H^{'} \to T^{*'}$ is an admissible embedding in M^* over $\mathbb{R}$, and it is the one we will use in defining individual terms in the transfer factor. Now to move across to G (or, more precisely, to M), we may modify the inner twist ψ_M by an inner automorphism of M and then assume that the restriction of ψ_M to $(G^*)^{\delta_0^*}$ is an isomorphism over $\mathbb{R}$ of $(G^*)^{\delta_0^*}$ with G^{δ_0}.

Note that we can adapt the arguments of the last paragraph, keeping the setup in H and G^* but dropping the noncompactness assumption on α in G, to see that $T_H^{'}$ has admissible embeddings over $\mathbb{R}$ into G if and only if α is not totally compact ([S3], Proposition 9.3).

8. Geometric transfer factors

By the *very regular* set of $H_1(\mathbb{R}) \times G(\mathbb{R})$ we will mean the set of pairs (γ_1, δ), where γ_1 is strongly G-regular in $H_1(\mathbb{R})$ and δ is strongly regular in $G(\mathbb{R})$. The canonical transfer factor Δ of [LS1], which we will now label as the geometric transfer factor, is a function on the very regular set with the following properties:

(i) $\Delta(\gamma_1, \delta) = 0$ *unless* γ_1 *is an image of* δ,

(ii) $\Delta(\gamma_1', \delta) = \Delta(\gamma_1, \delta)$ *if* γ_1' *is stably conjugate to* γ_1,

(iii) $\Delta(\gamma_1, \delta') = \Delta(\gamma_1, \delta)$ *if* δ'*is conjugate to* δ, *and*

(iv) $\Delta(z_1\gamma_1, \delta) = \lambda_1(z_1)^{-1}\Delta(\gamma_1, \delta)$ *for* z_1 *in* $Z_1(\mathbb{R})$.

Then we may prescribe the matching of orbital integrals for $f \in \mathcal{C}(G(\mathbb{R}))$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ as

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \sum_{\delta,conj} \Delta(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

It is the relative transfer factor $\Delta(\gamma_1, \delta)/\Delta(\gamma_1', \delta') = \Delta(\gamma_1, \delta; \gamma_1', \delta')$ that is canonical, and so we need a normalization for the absolute factor. As in [LS1], we fix a pair $(\overline{\gamma}_1, \overline{\delta})$ in the very regular set, with $\overline{\gamma}_1$ an image of $\overline{\delta}$ (if none exists, there is no transfer to make, and none needed), fix $\Delta(\overline{\gamma}_1, \overline{\delta})$ arbitrarily, and then set

$$\Delta(\gamma_1, \delta) = \Delta(\overline{\gamma}_1, \overline{\delta})\Delta(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta}).$$

We say that the transfer factor has been normalized *by choice of related pair*. Any normalization can be recovered in this manner. In [S7] we will discuss normalization more systematically. The chosen normalization of the geometric transfer factor determines uniquely the dual map on tempered characters, and we will be particularly interested in those normalizations where the coefficients in the dual map, i.e. the spectral transfer factors, are simply signs.

The canonical factor $\Delta(\gamma_1, \delta; \gamma_1', \delta')$ is constructed in [LS1] as a product of five terms: Δ_I, Δ_{II}, Δ_{III_1}, Δ_{III_2} and Δ_{IV}. All terms except Δ_{III_1} are quotients of *absolute* terms $\Delta_I(\gamma_1, \delta)$, $\Delta_I(\gamma_1', \delta')$ etc. It is convenient for the purposes of this discussion, and those of [S7], now to write the product of Δ_{II}, Δ_{III_2} and Δ_{IV} as a single term Δ_{II+}. The three individual pieces Δ_I, Δ_{II+} and Δ_{III_1} depend on two choices: the toral data discussed in Section 6 (but not the Borel subgroups providing that data) and the a-data which we will discuss in the next section. Here is a list of the dependence:

Δ_I	toral data, a-data
Δ_{II+}	a-data
Δ_{III_1}	toral data

We will discuss these terms over the next several sections, and finish here with a few informal remarks. The analysis of orbital integrals we have outlined so far, and an analysis of the embeddings of L-groups that we will outline below, provide motivation for the construction of terms Δ_{II+} and Δ_{III_1}, although not for the precise nature of Δ_{III_1}. That will be taken up in [S7] and [S8]. In twisted endoscopy, the analogue of the product of Δ_{II+} and Δ_{III_1} does not factor in general. That is one reason for this separate discussion of the ordinary case.

The role of the term Δ_I, crucial for a canonical product, is less transparent. To motivate its definition, and the splitting invariant of a torus on which it is based, we may turn to the regular unipotent analysis of Section 6 in [LS1]. That topic, however, will not be discussed here. We just mention a simple but instructive application of the analysis (for p-adic groups). With Haar measures normalized suitably, the Shalika germ for a regular unipotent conjugacy class $\mathcal{U}$ in a quasisplit group $G(F)$ takes value either one or zero on a regular semisimple element γ near the identity. We take an F-splitting of G (the choice will not matter) and let $T = Cent(\gamma, G)^0$. Then we define an invariant $inv_T(\mathcal{U})$ for $\mathcal{U}$ in a straightforward manner. To detect which value we obtain for the germ for $\mathcal{U}$ at the element γ sufficiently close to the identity, we use a-data for T to construct an invariant $inv(\gamma)$ for γ. Then the value of the germ at γ equals one exactly when the product of $inv(\gamma)$ with the splitting invariant of T equals $inv_T(\mathcal{U})$. See [S6]. We will be concerned with the splitting invariant again in our discussion of spectral transfer factors.

Despite the significance of Δ_I, we avoid an explicit discussion of it here. Instead, as we have indicated, we will invoke the descent property of the canonical transfer factors from [LS2]. We will discuss that and its proof at various points along the way.

The product Δ_{II+} can be regrouped into two pieces: the product $\Delta_{II}\Delta_{IV}$ which we regard as the quotient of nonsymmetric generalized Weyl denominators for G and H_1 and a symmetrizing, or ι-shift, character Δ_{III_2}. The definition of each piece involves the use of χ-data, but the effects of changing the data are readily seen from [LS1] to cancel. Thus we may just as well use the choice that facilitates reading results of Harish Chandra. We will outine the the construction of the two pieces in the next few sections. We should mention that it is the insertion of a-data in the construction of Δ_{II} that removes the dependence of the product Δ_{II+} on χ-data. In the present discussion, once we have picked the χ-data we will group the contribution from the a-data with Δ_I and handle it by descent.

The two terms Δ_I and Δ_{II+} tell us nothing about the position of a strongly regular conjugacy class in $G(\mathbb{R})$ within its stable conjugacy class. The last term Δ_{III_1} remedies this, but only in a relative manner if G is not quasisplit. For the present discussion we just need a transformation rule for the canonical product to extract *κ-orbital integrals* (Lemma 12.1), and then we rely once again on semiregular descent to avoid a direct analysis of the term.

9. a-data, χ-data and Weyl denominators

Let T be a maximal torus in G defined over $\mathbb{R}$. Recall that the symmetric orbits of $\Gamma = \{1, \sigma\}$ in the roots of T are simply the pairs $\{\pm\alpha\}$ of imaginary roots, and the asymmetric orbits are either singletons $\{\alpha\}$ if α is real, or pairs $\{\alpha, \sigma\alpha\}$ if α is complex. A set of a-data consists of non-zero complex numbers a_α such that $a_{\sigma\alpha} = \sigma(a_\alpha)$ and $a_{-\alpha} = -\ a_\alpha$, for all roots α.

The χ-data consist of a set $\{\chi_\alpha\}$ of characters on $\mathbb{C}^\times$ if α is imaginary or complex, or on $\mathbb{R}^\times$ if α is real, such that $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}, \chi_{-\alpha} = \chi_\alpha^{-1}$ and if α is imaginary χ_α extends the sign character on $\mathbb{R}^\times$. These data are involved directly only in terms Δ_{II} and Δ_{III_2}. We may make what we will call the *based* choice given a positive system: if α is positive imaginary, set $\chi_\alpha(z) = z/|z| = (z/\overline{z})^{1/2}$; if α is negative imaginary, set $\chi_\alpha(z) = \ |z|/z = (\overline{z}/z)^{1/2}$, and otherwise set χ_α trivial.

We can now describe one contribution to the transfer factor. The term $\Delta_{II}(\gamma_1, \delta; \gamma_1', \delta')$ is a quotient $\Delta_{II}(\gamma_1, \delta)/\Delta_{II}(\gamma_1', \delta')$ where

$$\Delta_{II}(\gamma_1, \delta) = \prod_{\mathcal{O}} \chi_\alpha\left(\tfrac{\alpha(\delta)-1}{a_\alpha}\right),$$

and the product is taken over representatives α for all orbits $\mathcal{O}$ of Γ outside H_1.

We return for a moment to our discussion before endoscopy was introduced. Given a positive system for the roots of T we mean by Weyl denominator the function Δ' on $T(\mathbb{R})$ given by

$$\Delta'(\gamma) = \prod_{\alpha>0,\mathrm{real}} \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right| \prod_{\alpha>0,\mathrm{complex}} \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right| \times \prod_{\alpha>0,imaginary} (\alpha(\gamma) - 1).$$

Recall that $\left|z^{1/2} - z^{-1/2}\right|$ is to be interpreted as $|(1-z)(1-z^{-1})|^{1/2}$. If α is imaginary then

$$|\alpha(\gamma) - 1| = \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right|,$$

so that with the *based* choice of χ-data described above, we have

$$\begin{aligned}\Delta'(\gamma) &= \prod_{\alpha>0} \left|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\right| \prod_{\alpha} \chi_\alpha(\alpha(\gamma) - 1) \\ &= \left|\det_{\mathfrak{g}/\mathfrak{t}}(Ad(\gamma) - I)\right|^{1/2} \prod_{\alpha} \chi_\alpha(\alpha(\gamma) - 1),\end{aligned}$$

where the summation is over representatives α for all orbits of Γ. We may include asymmetric orbits as well as symmetric orbits here since χ_α is trivial unless α is imaginary.

For any sets of χ-data $\{\chi_\alpha\}$ and a-data $\{a_\alpha\}$, we adjust this last formula to define $\Delta'(\gamma, \{\chi_\alpha\}, \{a_\alpha\})$ as

$$\Delta'(\gamma, \{\chi_\alpha\}, \{a_\alpha\}) = \left|\det_{\mathfrak{g}/\mathfrak{t}}(Ad(\gamma) - I)\right|^{1/2} \textstyle\prod_{\mathcal{O}} \chi_\alpha(\frac{\alpha(\gamma)-1}{a_\alpha}),$$

where the summation is over representatives α for all orbits of $\mathcal{O}$ of Γ. This of course can be done for any local field of characteristic zero. Notice that the choice of representative α for an orbit has no effect:

$$\chi_{\sigma\alpha}(\tfrac{\sigma\alpha(\gamma)-1}{a_{\sigma\alpha}}) = \chi_{\sigma\alpha}(\sigma(\tfrac{\alpha(\gamma)-1}{a_\alpha})) = \chi_\alpha(\tfrac{\alpha(\gamma)-1}{a_\alpha}),$$

and the dependence is on the choice of a-data and χ-data, rather than on the choice of a positive system (or gauge) for the roots.

We could argue throughout with the *generalized Weyl denominator* $\Delta'(\gamma, \{\chi_\alpha\}, \{a_\alpha\})$ in place of $\Delta'(\gamma)$, but there is little change in the analysis on the geometric side, for all the additional notation. Instead, while proving geometric transfer, we will work with Harish Chandra's factor $\Delta'(\gamma)$ and use based χ-data in the relevant terms of the transfer factor. Recall that the combined term Δ_{II+} is, in any case, independent of the choice of χ-data. It does depend on a-data, but having chosen χ-data we will now factor off the piece depending on a-data and combine it with Δ_I.

We thus rewrite the product of the absolute terms $\Delta_I(\gamma_1, \delta)$ and $\Delta_{II}(\gamma_1, \delta)$ as a product

$$\Delta_I^*(\gamma_1, \delta)\ \Delta_{II}^*(\gamma_1, \delta).$$

The original $\Delta_I(\gamma_1, \delta)$ is given as a Tate-Nakayama pairing $< \lambda(T_{sc}), \boldsymbol{s}_T >$ which we will not review further here. For $\Delta_I^*(\gamma_1, \delta)$ we add in the denominator of $\Delta_{II}(\gamma_1, \delta)$. Thus

$$\Delta_I^*(\gamma_1, \delta) = < \lambda(T_{sc}), \boldsymbol{s}_T > \prod_\alpha \chi_\alpha(a_\alpha)^{-1}$$

and

$$\Delta_{II}^*(\gamma_1, \delta) = \prod_\alpha \chi_\alpha(\alpha(\delta) - 1)$$

where the product, in each case, is over representatives α for the orbits of Γ in the roots of T outside H_1. In each product the choice of representative for an orbit does not matter.

Remark 9.1. The term $\Delta_I^*(\gamma_1, \delta)$ is independent of the choice of a-data. To check this, we replace a_α by $a'_\alpha = a_\alpha b_\alpha$. This multiplies $< \lambda(T_{sc}), \boldsymbol{s}_T >$ by $\prod_\alpha \chi_\alpha(b_\alpha)$ by Lemma 3.2.C of [LS1]. Here the product is over representatives α for the pairs of imaginary roots $\pm\alpha$ outside H_1.

We have made $\Delta_I^*(\gamma_1, \delta)$ depend on the choice of χ-data, but because we have chosen to use based χ-data that will be of no concern. Notice that

$$\Delta_{II}^*(\gamma_1, \delta)\Delta_{IV}(\gamma_1, \delta) = \Delta'_G(\delta)/\Delta'_{H_1}(\gamma_1),$$

where the terms on the right are the usual asymmetric Weyl denominators for G and H_1 respectively.

Thus it remains to consider Δ_{III_1} and Δ_{III_2}. As already mentioned, the purely relative term Δ_{III_1} will be handled by making some careful choices in descent, choices which also allow us to deal with the toral constant $\Delta_I^*(\gamma_1, \delta)$ without further explicit calculation, and then passing to κ-orbital integrals. We will start on that in Section 12. For now, we prepare for $\Delta_{III_2}(\gamma_1, \delta)$ which is the value at γ_1 of a certain character on $T_1(\mathbb{R})$. This character is determined by comparing some embeddings of L-groups.

10. χ-data and embedding the L-group of a maximal torus

Let T be a maximal torus over $\mathbb{R}$ in G. Then following [LS1], a set of χ-data for the roots of T determines a $G^\vee$-conjugacy class of embeddings of ${}^LT$ in ${}^LG$. Because we are dealing with real groups and based χ-data, we may describe an embedding from this class very simply.

First, we realize $W = W_\mathbb{R}$ as $\{z\times\tau : z \in \mathbb{C}^\times, \tau \in \Gamma\}$,with $z_1\times\tau_1.z_2\times\tau_2 = z_1\tau_1(z_2)a_{\tau_1,\tau_2} \times \tau_1\tau_2$, $a_{\tau_1,\tau_2} = 1$ unless $\tau_1 = \tau_2 = \sigma$ and $a_{\sigma,\sigma} = -1$.Fix a pair (B, T). Then (B, T) and the pair $(\mathcal{B}, \mathcal{T})$ from $spl_{G^\vee}$ determine $T^\vee \to \mathcal{T}$ by which we embed $T^\vee$ in $G^\vee$. We also have positive systems at hand by which to specify based χ-data. So it remains to define the embedding on the elements $1 \times w$, i.e. to define a suitable homomorphism of $W_\mathbb{R}$ in ${}^LG$.
We may work inside M (since for based χ-data we have chosen χ_α to be trivial except for α imaginary), and so we will map $W_\mathbb{R}$ into ${}^LM$. Recall that a splitting of $M^\vee$ has been fixed; if $\alpha^\vee$ is a positive root relative to this splitting then $\chi_\alpha(z) = (z/\overline{z})^{1/2}$ for based χ-data. The embedding ξ constructed in Section 2.5 of [LS1], which we now denote ξ_T, has

$$\xi_T(z \times 1) = (z/\overline{z})^\iota \times (z \times 1),$$

where ι is the transport to $\mathcal{T}$ of one-half the sum of the positive roots of T in M. Here the element $(z/\overline{z})^\iota$ of $\mathcal{T}$ is defined by $\lambda^\vee((z/\overline{z})^\iota) = (z/\overline{z})^{<\iota,\lambda^\vee>}$, for each rational character $\lambda^\vee$ on $\mathcal{T}$. Also $\xi_T(1 \times \sigma) = n \times (1 \times \sigma)$, where $n \times (1 \times \sigma)$ acts on $\mathcal{T}$ as the dual of $\sigma_T = \omega(\sigma, M/T) \circ \sigma_M$.We choose n to be the element $n(\omega(\sigma, M/T))$ attached in Section 2.5 of [LS1] to $\omega(\sigma, M/T)$ using the root vectors from $spl_{M^\vee}$, so that

$$\xi_T(1 \times \sigma) = n(\omega(\sigma, M/T)) \times (1 \times \sigma).$$

It is also convenient to write $r_T(z\times\tau)$ for the element $(z/\overline{z})^\iota$ of $\mathcal{T}$, where τ is either 1 or σ: it is the term $r_p(z \times \tau)$ associated in Section 2.5 of [LS1] to based χ-data for (the gauge p associated with) the pair $(\mathcal{B}, \mathcal{T})$. If we also set $\omega(1, M/T)$ to be the identity, then we have

$$\xi_T(z \times \tau) = r_T(z \times \tau)n(\omega(\tau, M/T)) \times (z \times \tau).$$

It will be useful to know that the precise choice of $n(\omega(\sigma, M/T))$ is not necessary for this last formula to define an L-homomorphism of ${}^LT$ in ${}^LG$. Lemma 3.2 of [L3] shows that it is enough to take any element n as above in the *derived group* of $M^\vee$ instead.

11. ι-shift characters and endoscopic embeddings

First we attach data to the embeddings ξ_1 and ξ. For this, we adjust the definitions in [S4] to account for the z-extension $H_1 \to H$, following Section 4.4 of [KS]. Here we will outline a simpler version since we are dealing only with the extension $\mathbb{C}/\mathbb{R}$ and with no twisting in the endoscopy. We may assume that ξ_1 and ξ are inclusion on $H^\vee$, that $\mathfrak{s} \in \mathcal{T}$, that $spl_{H^\vee} = (\mathcal{B} \cap H^\vee, \mathcal{T}, ...)$, that $\mathcal{T} \subseteq \mathcal{T}_1$ and $spl_{H^\vee}$ is extended to a splitting for $H_1^\vee$, thus embedding ${}^LH$ naturally in ${}^LH_1$. Moreover, since we will ultimately work in M, and want compatibility with the descent data of Section 7, we will assume T compact modulo the center of G, i.e. that every root is imaginary. For w in $W_\mathbb{R}$ mapping to τ in Γ we choose $h(w)$ in $\mathcal{H}$ acting on $H^\vee$ as τ_H and mapping to w under $\mathcal{H} \to W_\mathbb{R}$. Note that the element $h(w)$ is unique up to multiplication by an element of the center of $H^\vee$. On the other hand, the element $\xi(h(w))$ in ${}^LG$ is of the form $n(w) \times w$, where $n(w)$ lies in the normalizer of $\mathcal{T}$ in $G^\vee$ and so acts as an element $\omega(\tau, G/H)$ of the Weyl group of $\mathcal{T}$. Let $n(\omega(\tau, G/H))$ be the standard element acting thus (see [LS1], Section 2.1). Then we may define $t_\xi(w)$ by

$$n(w) = t_\xi(w) n(\omega(\tau, G/H)).$$

If $h(w)$ is multiplied on the left by an element $z_H(w)$ in the center of $H^\vee$ then so is $t_\xi(w)$. The embedding ξ_1 of $\mathcal{H}$ in ${}^LH_1$ has a simpler form:

$$\xi_1(h(w)) = t_{\xi_1}(w) \times w,$$

where $t_{\xi_1}(w)$ lies in $\mathcal{T}_1$ and is central in $H_1^\vee$. Again, multiplying $h(w)$ by $z_H(w)$ does the same to $t_{\xi_1}(w)$. Thus the element

$$t_{\xi,\xi_1}(w) = t_\xi(w) t_{\xi_1}(w)^{-1}$$

of $\mathcal{T}_1$ is independent of the choice of $h(w)$, but it is not in general a cocycle.

The standard χ-data give two embeddings: the embedding $\xi_{T_H} : {}^LT_H \to {}^LH$ which extends naturally to $\xi_{T_1} : {}^LT_1 \to {}^LH_1$ and the embedding $\xi_T : {}^LT \to {}^LG$ ([LS1], Section 2.5). Let

$$a_{T_1} = t_{\xi,\xi_1}\ r_{T_1}\ r_T^{-1}.$$

First of all we observe that the map $a_{T_1} : W_{\mathbb{R}} \to \mathcal{T}_1$ is a 1-cocycle. That is immediate if ξ_1 is the identity map, and so Z_1 is trivial, since in that case $a_{T_1} = a_{T_H}$ which measures the difference between two embeddings of ${}^LT_H \cong {}^LT$ in ${}^LG$ (see Section 3.5 of [LS1]). To show that a_{T_1} is a cocycle in general, we go back to the last comment in the last section. We choose $n_H(\sigma) \in H^\vee_{der}$ in the usual way such that $u(w) = n_H(\sigma)h(w)$ acts on $\mathcal{T}_1$ as σ_{T_1} if $w = z \times \sigma$, and we set $u(z \times 1) = h(z \times 1)$. Then we observe that we still obtain L-homomorphisms after replacing ξ_{T_1} by ξ'_{T_1}, where

$$\xi'_{T_1}(t \times w) = t\ r_{T_1}(w)t_{\xi_1}(w)^{-1}\xi_1(u(w)),$$

and ξ_T by ξ'_T, where

$$\xi'_T(t \times w) = t\ r_T(w)t_\xi(w)^{-1}\xi(u(w)).$$

The needed calculation is that $t_{\xi_1}(w)^{-1}\xi_1(u(w))$ and $t_\xi(w)^{-1}\xi(u(w))$ lie in the derived groups of $H_1^\vee$ and $G^\vee$ respectively, when $w = 1 \times \sigma$. Now we use the fact that ξ'_{T_1} and ξ'_T are homomorphisms to calculate the coboundaries of $w \to r_{T_1}(w)t_{\xi_1}(w)^{-1}$ and $w \to r_T(w)t_\xi(w)^{-1}$ as

$$\xi_1(u(w_1w_2)u(w_1)^{-1}u(w_2)^{-1})$$

and

$$\xi(u(w_1w_2)u(w_1)^{-1}u(w_2)^{-1}),$$

respectively. But these are, by definition, the same element of $\mathcal{T}$, and so a_{T_1} is a cocycle.

Next we set

$$\Delta_{III_2}(\gamma_1, \delta) = < a_{T_1}, \gamma_1 >$$

if strongly G-regular γ_1 in $T_1(\mathbb{R})$ is an image of strongly regular δ in $T(\mathbb{R})$.The pairing is that of the Langlands parametrization of quasicharacters on a real torus, which we will describe more explicitly shortly. Our first remark is that this definition of $\Delta_{III_2}(\gamma_1, \delta)$ is that of [LS1] when ξ_1 is the identity map. Notice also that the associated relative factor is correct for the definition of general twisted factors in [KS]: the hypercohomology group factors and we need just track the cocycle $a_T(w)$ on p. 45 which we can regard as a cocycle with values in $\mathcal{T}_1$ when there is no twisting.

An explicit Langlands parameter for the character $\chi(\gamma_1) = < a_T, \gamma_1 >$ is a pair $(\mu, \lambda) \in (X_*(\mathcal{T}_1) \otimes \mathbb{C})^2$, where $a_{T_1}(z \times 1) = z^{\mu} \overline{z}^{\sigma_{T_1}\mu}$ and $a_{T_1}(1 \times \sigma) = e^{2\pi i \lambda}$ ([L3]). We write $\chi = \chi(\mu, \lambda)$.The pair (μ, λ) has the property that

$$\tfrac{1}{2}(\mu - \sigma_{T_1}\mu) + \lambda + \sigma_{T_1}\lambda \in X_*(\mathcal{T}_1);$$

μ is determined uniquely, whereas λ is unique only modulo

$$X_*(\mathcal{T}_1) + (1 - \sigma_{T_1}) X_*(\mathcal{T}_1) \otimes \mathbb{C}.$$

We attach a pair $(\mu^*, \lambda^*) \in (X_*(\mathcal{T}_1) \otimes \mathbb{C})^2$ to the embeddings $\xi_1 : \mathcal{H} \rightarrow {}^L H_1$ and $\xi : \mathcal{H} \rightarrow {}^L G$, as follows. We may write the element $t_{\xi,\xi_1}(z \times 1)$ as $z^{\mu^*} \overline{z}^{\upsilon^*}$, and then observe that $\upsilon^* = \sigma_{H_1}\mu^*$ and that $< \mu^*, \alpha_1^\vee > = 0$ for all roots $\alpha_1^\vee$ of $\mathcal{T}_1$ in $H_1^\vee$. Then we also have $\upsilon^* = \sigma_{T_1}\mu^*$. We specify λ^* by requiring that $t_{\xi,\xi_1}(1 \times \sigma) = e^{2\pi i \lambda^*}$. Recall from the last section that $\iota = \iota_G$ satisfies $r_T(z \times 1) = (z/\overline{z})^{\iota} = z^{\iota} \overline{z}^{\sigma_T \iota}$. Similarly, we have $\iota = \iota_1$ for r_{T_1}. On the other hand, $r_T(1 \times \sigma) = r_{T_1}(1 \times \sigma) = 1$.Comparing definitions, we conclude that

$$\mu = \mu^* + \iota_1 - \iota_G,$$

and that for λ we may take λ^*. This completes our description of $\Delta_{III_2}(\gamma_1, \delta)$. We have considered only the case needed for working with parabolic descent. The general case involves a simple modification using Lemma 3.5.A of [LS1] (to change to general χ-data $\{\chi_\alpha\}$, set $\zeta_\alpha = 1$ for α imaginary and $\zeta_\alpha = \chi_\alpha$ for α nonimaginary).

Thus we have recovered the correction characters of [S4] in the case when $H_1 = H$ (see Section 4.3 of [S4] for some explicit examples), with just a minor modification for the general case. In [S4] the correction characters were defined only for a fixed choice of toral data for chosen Cartan subgroups, and the behavior of the transfer factors under stable conjugacy *from* H required a long argument (Theorem 4.5.2 of [S4]). Using canonical transfer factors this step is no longer necessary for the geometric transfer (see Lemma 4.1.C of [LS1]). We may also avoid using the second main result of [S4], Theorem 6.1.1 on compatibility across walls of adjacent Cartan subgroups, by a direct appeal to the descent theorem of [LS2] (see, however, the comments in Section 13 below).

Remark 11.1 (descent). The pair (μ^*, λ^*) will not change in our setting for parabolic descent. It may change in semiregular descent : one positive root becomes simple after descent.

Remark 11.2 (trivial endoscopy). If H is an inner form of G, i.e the endoscopic datum $\mathfrak{s}$ is central in $G^\vee$, and we pass to a suitable extension H_1 then $\iota_1 = \iota_G$ on each Cartan subgroup of $H_1(\mathbb{R})$ and so the pair (μ^*, λ^*) determines a character $\chi(\mu^*, \lambda^*)$ on each Cartan subgroup. These characters together extend to a single character $\chi(\mu^*, \lambda^*)$ on $H_1(\mathbb{R})$ itself.

Remark 11.3 (transformation rule for transfer factors). Notice that $\Delta_{III_2}(\gamma_1, \delta)$ is the only term in the transfer factor $\Delta(\gamma_1, \delta)$ that depends directly on γ_1 rather than on the image of γ_1 under $H_1 \to H$. We have

$$\Delta(z_1\gamma_1, \delta) = \lambda_1(z_1)^{-1}\Delta(\gamma_1, \delta)$$

for $z_1 \in Z_1(\mathbb{R})$, where $Z_1 = Ker(H_1 \to H)$ ([LS1] and [KS]). Let (μ_z^*, λ_z^*) be the transport of (μ^*, λ^*) under $\mathcal{T}_1 \to Z_1^\vee$, or equivalently, under restriction to the Lie algebra of $Z_1(\mathbb{R})$. Then the character $\chi(\mu_z^*, \lambda_z^*)$ is welldefined and, by inspection of our formula above for $\chi(\gamma_1) = < a_T, \gamma_1 >$ we have

$$\chi(\mu_z^*, \lambda_z^*)(z_1) = \lambda_1(z_1)^{-1}$$

for $z_1 \in Z_1(\mathbb{R})$ (see Section 4.1 of [S4]). For the spectral transfer we will then consider those tempered irreducible representations π_1 of $H_1(\mathbb{R})$ for which

$$\pi_1(z_1\gamma_1) = \chi(-\mu_z^*, -\lambda_z^*)(z_1)\ \pi_1(\gamma_1).$$

Remark 11.4 (linear form on Lie algebra of the endoscopic group) In general, the datum μ^* attached to the embeddings $\xi_1 : \mathcal{H} \to {}^L H_1$ and $\xi : \mathcal{H} \to {}^L G$ may be identified as a linear form on the Lie algebra of any Cartan subgroup of $H_1(\mathbb{R})$, or better, as a linear form on the Lie algebra of $H_1(\mathbb{R})$. There is no harm in assuming that μ^* takes only purely imaginary values, and we will do so to avoid having to introduce *essentially tempered* representations into our discussion.

12. Beginning geometric transfer with canonical factors

To $f \in \mathcal{C}(G(\mathbb{R}))$ we attach

$$\Phi(\gamma_1, dt_1, dh) = \textstyle\sum_{\delta, conj} \Delta(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$, where $\delta, conj$ indicates summation over the conjugacy classes of strongly regular elements δ in $G(\mathbb{R})$, and measures are as before. We want to find $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ such that

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \Phi(\gamma_1, dt_1, dh)$$

and so return to the conditions (i) - (vi) in Section 3. The invariance properties (i),(ii) and (iii) will be immediate from the properties of transfer factors, at least on the strongly G-regular set. Our main concern is thus to analyze the various potential jumps. We examine $\Phi(\gamma_1)$ near each imaginary wall, including the G-walls, in a Cartan subgroup $T_1(\mathbb{R})$. As before, we choose a positive system for the imaginary roots of T_{γ_1} to define the normalizing factor Δ'_{H_1}, and then set

$$\Psi(\gamma_1) = \Delta'_{H_1}(\gamma_1)\Phi(\gamma_1).$$

Recall that we plan to use a different positive system for each noncompact wall we examine. Moreover, we will need to normalize the various $O_\delta(f, dt, dg)$, and introduce $\Delta'_G(\delta)$ for some δ in each stable conjugacy class.

Here is where we will exploit the fact that the transfer factors are not only explicit but also canonical: since it will have no effect on whole transfer factor, we are free to make preferred choices each time (i.e. wall by wall) for the data used in defining the individual terms in the transfer factor . This will yield a simple local comparison of the transfer factor with the appropriate $\Delta'_{H_1}(\gamma_1)/\Delta'_G(\delta)$ not available for the calculations in [S3, S5]. In particular, the κ-signature of a Cayley transform introduced in those calculations will now be trivial. Recall from Section 6 that κ is determined by the endoscopic datum $\mathfrak{s}$ and our choice of toral data $\eta = \eta_{B_1,B} : T_1 \rightarrow T$ over $\mathbb{R}$. We then have weights $\kappa(w) = \pm 1$ for the κ-orbital integrals $\sum_w \kappa(w)'F_f^w(\delta)$ which, as functions of γ_1, are dependent on the choice of η. Set $\delta = \eta(\gamma_1)$.

Lemma 12.1

We may rewrite

$$\Psi(\gamma_1) = \Delta'_{H_1}(\gamma_1) \textstyle\sum_{\delta, conj} \Delta(\gamma_1, \delta) O_\delta(f)$$

as a κ-orbital integral:

$$\tfrac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)} \Delta(\gamma_1, \delta) \textstyle\sum_w \kappa(w)\ 'F_f^w(\delta).$$

Proof: Gathering definitions, we see that all we need for this is an appropriate transformation rule for transfer factors:

$$\Delta(\gamma_1, \delta^w) = \Delta(\gamma_1, \delta) < inv(\delta, \delta^w), \mathbf{s}_T > = \Delta(\gamma_1, \delta)\kappa(w).$$

It is proved in greater generality as Lemma 5.1.D(i) in [KS], and is not difficult to prove directly from the definition of Δ_{III_1} in Section 3.4 of [LS1]. We recall also the caution regarding the pairing from the last paragraph of Section 6.

Then there are two steps remaining in our examination of Ψ: a local analysis of $\frac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)}\Delta(\gamma_1, \delta)$ for certain convenient choices of defining data, which will be given in the next section, and the jump behavior of κ-orbital integrals, with the same convenient choices, which is available from [S3].

13. Local properties of transfer factors

Again we fix an isomorphism $\eta_{B_1,B} : T_1 \to T$ over $\mathbb{R}$ as in Section 6, and suppose that $\gamma_1 \in T_1(\mathbb{R})$ is strongly G-regular. Set $\delta = \eta_{B_1,B}(\gamma_1)$. Use B_1, B to specify Δ'_{H_1}, Δ'_G. Then we write

$$\frac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)}\Delta(\gamma_1, \delta) = D(\gamma_1).$$

We shall see that $D(\gamma_1)$ is *almost* a constant near a semiregular element in $T_{\gamma_1}(\mathbb{R})$. The precise sense of *almost* will be evident in our first lemma. The second lemma will compare the constants on adjacent Cartan subgroups.

We start then with a single Cartan subgroup $T_1(\mathbb{R})$ for which $\eta_{B_1,B}$: $T_1 \to T$ over $\mathbb{R}$ exists. Suppose that $\gamma_{1,0} \in T_1(\mathbb{R})$. We do not need $\gamma_{1,0}$ to be semiregular for Lemma 13.1. For X sufficiently small and nonzero in the Lie algebra $\mathfrak{t}_1(\mathbb{R})$ such that $\gamma_{1,0} \exp X$ is strongly G-regular we set

$$D_{\gamma_{1,0}}(X) = D(\gamma_{1,0} \exp X).$$

We regard the term $\mu^* + \iota_1 - \iota_G$ from Section 11 as a linear form on $\mathfrak{t}_1(\mathbb{R})$. Then:

Lemma 13.1

For X as above, we have

$$D_{\gamma_{1,0}}(X) = Ae^{\mu^* + \iota_1 - \iota_G(X)},$$

where A is independent of X.

Proof: This does not require much more argument but we will take this opportunity to gather the pieces of the transfer factors in one place. Recall that the transfer factor is normalized as

$$\Delta(\gamma_1,\delta)=\Delta(\overline{\gamma}_1,\overline{\delta})\Delta(\gamma_1,\delta;\overline{\gamma}_1,\overline{\delta}),$$

where $(\overline{\gamma}_1,\overline{\delta})$ is a fixed related pair and $\Delta(\overline{\gamma}_1,\overline{\delta})$ has been chosen arbitrarily. So $\Delta(\overline{\gamma}_1,\overline{\delta})$ is our first contribution to the constant A. The (canonical) relative factor $\Delta(\gamma_1,\delta;\overline{\gamma}_1,\overline{\delta})$ is composed of five pieces, four of which are quotients. We include the denominators $\Delta_I^*(\overline{\gamma}_1,\overline{\delta})$, $\Delta_{II}^*(\overline{\gamma}_1,\overline{\delta})$, $\Delta_{III_2}(\overline{\gamma}_1,\overline{\delta})$ and $\Delta_{IV}(\overline{\gamma}_1,\overline{\delta})$ of those quotients in A. For the numerators, we can include the toral invariant $\Delta_I^*(\gamma_1,\delta)$ in A, and now use the explicit form $\gamma_1=\gamma_{1,0}\exp X$ and $\delta=\eta_{B_1,B}(\gamma_1)$ to evaluate

$$\Delta_{II}^*(\gamma_1,\delta)\Delta_{III_2}(\gamma_1,\delta)\Delta_{IV}(\gamma_1,\delta)$$

as the product of

$$\begin{aligned}&\chi(\mu^*+\iota_1-\iota_G,\lambda^*)(\gamma_{1,0}\exp X)\\ &=\chi(\mu^*+\iota_1-\iota_G,\lambda^*)(\gamma_{1,0})\ e^{\mu^*+\iota_1-\iota_G(X)}\end{aligned}$$

with

$$\textstyle\prod_{im}\ (\alpha_1(\gamma_{1,0}\exp X)-1)\prod_{r,c}\left|\alpha_1(\gamma_{1,0}\exp X)-1\right|,$$

where $\prod_{im}$ indicates the product is over positive imaginary roots α_1 outside H_1, and $\prod_{r,c}$ indicates the product over positive real or complex roots α_1 outside H_1.

The first term $\chi(\mu^*+\iota_1-\iota_G,\lambda^*)(\gamma_{1,0})$ contributes to A, the second term appears in the statement of the lemma, and the last two terms together cancel with the term $\Delta'_{H_1}(\gamma_{1,0}\exp X)/\Delta'_G(\eta_{B_1,B}(\gamma_{1,0}\exp X))$ in the definition of $D_{\gamma_{1,0}}(X)$. Thus the lemma will be proved if we show that the remaining term $\Delta_{III_1}(\gamma_1,\delta;\overline{\gamma}_1,\overline{\delta})$ in the transfer factor is constant for our choice of γ_1 and δ. But that is true because δ is $\eta_{B_1,B}(\gamma_1)$ For this, see [LS1]: in Section 3.3, the cochain $v(\sigma)$ is the same for all δ so chosen.

Turning now to adjacent Cartan subgroups $T_1(\mathbb{R})$ and $T_1'(\mathbb{R})$, we will need only to consider the setting established in the last two paragraphs of Section 7. Thus the semiregular element $\gamma_{1,0}$ is common to $T_1(\mathbb{R})$ and $T_1'(\mathbb{R})$ and is annihilated by the positive noncompact imaginary root α_1 of T_1 in H_1. The additional data are chosen so that the transport of α to T is noncompact (we are considering only the case where that is possible) and so forth. Again we have $D_{\gamma_{1,0}}(X)$ and the constant A of Lemma 13.1 which we now write as

$A(T_1, \gamma_{1,0})$. We also have the analogous term for $T_1'(\mathbb{R})$ and then the constant $A(T_1', \gamma_{1,0})$, although here $\gamma_{1,0}$ is annihilated by the *real* root α_1'.

Lemma 13.2

In the setting described above, we have $A(T_1, \gamma_{1,0}) = A(T_1', \gamma_{1,0})$.

Proof: Now $\gamma_1 = \gamma_{1,0} \exp X$, $\delta = \eta_{B_1,B}(\gamma_1)$, $\gamma_1' = \gamma_{1,0} \exp X'$, and $\delta' = \eta_{B_1',B'}(\gamma_1')$. We start by comparing $\Delta(\gamma_1, \delta)$ with $\Delta(\gamma_1', \delta')$. By the transitivity property of relative transfer factors ([LS1], Lemma 4.1.A) we can ignore the fixed related pair $(\overline{\gamma}_1, \overline{\delta})$ and write the quotient of these terms as $\Delta(\gamma_1, \delta; \gamma_1', \delta')$. We will observe in a separate lemma below that

$$\lim_{X,X' \to 0} \Delta(\gamma_1, \delta; \gamma_1', \delta') = 1.$$

So now we compare $\frac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)}$ with $\frac{\Delta'_{H_1}(\gamma_1')}{\Delta'_G(\delta')}$. First we cancel within each quotient, writing each as a product over roots outside H_1. Since the positive system of imaginary roots for T_1 is adapted to α_1, we recall our discussion in Section 2 and conclude then that

$$\lim_{X,X' \to 0} \frac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)} / \frac{\Delta'_{H_1}(\gamma_1')}{\Delta'_G(\delta')} = 1.$$

Since, by the last lemma, we may compute $A(T_1, \gamma_{1,0}) / \ A(T_1', \gamma_{1,0})$ as

$$\lim_{X,X' \to 0} D(\gamma_{1,0} \exp X) / D(\gamma_{1,0} \exp X'),$$

it follows that

$$A(T_1, \gamma_{1,0}) / \ A(T_1', \gamma_{1,0}) = 1,$$

and the lemma is proved.

Lemma 13.3

In the setting of the proof of Lemma 13.2, we have

$$\lim_{X,X'\to 0} \Delta(\gamma_1, \delta; \gamma_1', \delta') = 1.$$

Proof: Replacing (H_1, G) by $(H_1^{\gamma_{1,0}}, G^{\delta_0})$, we obtain the transfer factor $\Delta_{\gamma_{1,0}}(\gamma_1, \delta; \gamma_1', \delta')$ for a trivial inner twist, *i.e.* an isomorphism over $\mathbb{R}$. Since we are working with $H_1^{\gamma_{1,0}}$ in place of H^{γ_0}, this factor is the value at γ_1/γ_1' of a character on $H_1^{\gamma_{1,0}}(\mathbb{R})$ (see Remarks 11.1 and 11.2). Thus we have

$$\lim_{X,X'\to 0} \Delta_{\gamma_{1,0}}(\gamma_1, \delta; \gamma_1', \delta') = 1.$$

The descent theorem for transfer factors ([LS2], Theorem 1.6) states that

$$\lim_{X,X'\to 0} \Theta(\gamma_1, \delta; \gamma_1', \delta') = 1,$$

where

$$\Theta(\gamma_1, \delta; \gamma_1', \delta') = \Delta(\gamma_1, \delta; \gamma_1', \delta')/\Delta_{\gamma_{1,0}}(\gamma_1, \delta; \gamma_1', \delta'),$$

and so the lemma follows once we observe that our slight modification of the transfer factor does not affect the statement of Theorem 1.6 (see Remark 11.3).

For the proof of Lemma 13.3 we have appealed directly to the general Theorem 1.6 of [LS2], which applies to all semisimple descent in ordinary endoscopy for all local fields of characteristic zero. An argument just for the setting of Lemma 13.3 is shorter. For Δ_I or Δ_I^* we need the first comparison lemma of Section 3.3 of [LS2], and may as well proceed more or less as in [LS2] for all terms but Δ_{III_2}. That is the term which has a long argument in general. Theorem 6.1.1 of [S4] handles Δ_{III_2} for just the setting of Lemma 13.3. Alternatively, we could argue with the second comparison lemma of [LS2] and avoid some of the case-by-case analysis used in the proof of Theorem 6.1.1.

14. Statement and proof of transfer

We pause for one last and elementary step: fitting together parabolic descent assertions for endoscopic data, transfer factors, and orbital integrals. We return then to the setting of Section 7. Thus we have $\eta = \eta_{B_1,B} : T_1 \to T$ defined over $\mathbb{R}$, and $M_1 = Cent(S_{T_1}, H_1)$ is endoscopic for $M = Cent(S_T, G)$. Let P be a parabolic subgroup of G defined over $\mathbb{R}$ and containing M as Levi subgroup, and let N be its unipotent radical. Then to $f \in \mathcal{C}(G(\mathbb{R}))$ we attach $f^{(P)} \in \mathcal{C}(M(\mathbb{R}))$, following [HC2]. Similarly, but not needed yet, we have for $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ and parabolic subgroup P_1 of H_1, defined over $\mathbb{R}$ and with M_1 as Levi subgroup, the function $f_1^{(P_1)} \in \mathcal{C}(M_{H_1}(\mathbb{R}), \lambda_1)$. Measures are normalized in the definition of $f^{(P)}$ so that for given dm, dg we have

$$O_\delta(f, dt, dg) = \left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\delta) - I\right|^{-1/2} O_\delta(f^{(P)}, dt, dm)$$

for all δ in $M(\mathbb{R})$ that are strongly regular in G.

Let δ be strongly regular in G and lie in $M(\mathbb{R})$. Let $T' = Cent(\delta, G)$ and $M' = Cent(S_{T'}, G)$. Then the Weyl group quotient $\Omega(M', T')/\Omega_{\mathbb{R}}(M', T')$ provides a complete and irredundant set of representatives for the conjugacy classes in the stable conjugacy class of δ, whether in G or in M (or in any Levi group containing M'). Thus the summations in the statement of Lemma 14.2 below are the same if strongly G-regular γ_1 is an image within M, i.e within the setting of endoscopy for M. To prove the lemma, it remains then to check that transfer factors match up term by term if normalized appropriately.

In Section 8 we have normalized transfer factors by the choice of a related pair. Thus $(\overline{\gamma}_1, \overline{\delta})$, with $\overline{\gamma}_1$ strongly G-regular in $H_1(\mathbb{R})$ an image of $\overline{\delta}$ in $G(\mathbb{R})$, has been fixed and $\Delta(\overline{\gamma}_1, \overline{\delta})$ chosen arbitrarily. Suppose in M we choose the related pair $(\overline{\gamma}_1^M, \overline{\delta}^M)$. It shortens the discussion (we avoid taking limits) if we assume $\overline{\gamma}_1^M$ is strongly G-regular, rather than just strongly M-regular, and we do so. Then $\overline{\delta}^M$ is strongly regular in G so that $\det_{\mathfrak{g}/\mathfrak{m}}(Ad(\overline{\delta}^M) - I)$ is nonzero. Moreover, the number $\Delta(\overline{\gamma}_1^M, \overline{\delta}^M)$ is welldefined and uniquely determined by the normalization for G. We say that *the transfer factors* Δ_M *and* Δ *are normalized compatibly* if $\Delta_M(\overline{\gamma}_1^M, \overline{\delta}^M)$ is chosen so that

$$\Delta_M(\overline{\gamma}_1^M, \overline{\delta}^M) = \left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\overline{\delta}^M) - I\right|^{-1/2} \Delta(\overline{\gamma}_1^M, \overline{\delta}^M).$$

Lemma 14.1

If the transfer factors Δ_M and Δ are normalized compatibly then

$$\Delta_M(\gamma_1, \delta) = \left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\delta) - I\right|^{-1/2} \Delta(\gamma_1, \delta)$$

if strongly G-regular γ_1 is an image of δ within M.

Proof: Suppose also strongly G-regular γ_1' is an image of δ' within M. Then by transitivity of the relative transfer factor it is enough to check that $\Delta_M(\gamma_1, \delta; \gamma_1', \delta')$ coincides with $\Delta(\gamma_1, \delta; \gamma_1', \delta')$ times

$$\left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\delta) - I\right|^{-1/2} \left|\det_{\mathfrak{g}/\mathfrak{m}} Ad(\delta') - I\right|^{1/2}.$$

We return to the definitions of the terms Δ_I, Δ_{II} etc. in [LS1]. First we dispose of Δ_{IV} immediately: Δ_{IV} for M is simply the last displayed term times Δ_{IV} for G. For the remaining terms, we have to show that the choices we have made yield the same term for both M and G. In particular, we have chosen χ-data and a-data for G to be trivial on asymmetric orbits. But then the terms Δ_I, Δ_{II} and Δ_{III_2} will have no contributions from orbits outside M, and are then the same for G and M; see Section 11 regarding Δ_{III_2}. It remains to check that calculating within M yields the same terms Δ_I and Δ_{III_1}. For the term Δ_I, we may replace the term $\lambda(T_{sc})$ in Section 3.1 of [LS1] by its image $\lambda(T)$ in $H^1(\Gamma, T)$ (... here T denotes the maximal torus in M containing whichever of δ, δ' we are considering) and pair with $\mathfrak{s}'$ from Section 6, and then we see the terms may be constructed the same way in G and M. For the relative term Δ_{III_1}, we may replace the inner twist ψ for G by ψ_M without harm, and then argue as in Lemma 3.1.A of [LS2] to complete the proof.

We have now established the following:

Lemma 14.2

With P and $f^{(P)}$ as above, we have

$$\sum_{\delta,conj,M} \Delta_M(\gamma_1,\delta)O_\delta(f^{(P)},dt,dm)$$
$$= \sum_{\delta,conj,G} \Delta(\gamma_1,\delta)O_\delta(f,dt,dg)$$

for all strongly G-regular γ_1 in $M_1(\mathbb{R})$, provided the transfer factors Δ_M and Δ are normalized compatibly.

We may now complete the geometric transfer:

Theorem 14.3

Let $(H,\mathcal{H},\mathfrak{s},\xi)$ be a set of endoscopic data for G, and (H_1,ξ_1) be a z-pair for H with attached character λ_1 on the central subgroup $Z_1(\mathbb{R})$, where $Z_1 = Ker(H_1 \to H)$. Let Δ be the attached geometric transfer factor, normalized by the choice of related pair. Then for each $f \in \mathcal{C}(G(\mathbb{R}))$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}),\lambda_1)$ such that

$$SO_{\gamma_1}(f_1,dt_1,dh_1) = \sum_{\delta,conj} \Delta(\gamma_1,\delta)O_\delta(f,dt,dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

Proof: We have defined Φ on the strongly G-regular elements γ_1 of $H_1(\mathbb{R})$ by

$$\Phi(\gamma_1,dt_1,dh) = \sum_{\delta,conj} \Delta(\gamma_1,\delta)O_\delta(f,dt,dg).$$

Here we may as well fix dg and dh. The choice of dt is arbitrary if γ_1 is not an image of δ since $\Delta(\gamma_1,\delta) = 0$ in that case. If γ_1 is an image of δ then dt is to be obtained from given dt_1 by transport.

Our first step is to extend Φ to all G-regular elements in $H_1(\mathbb{R})$. Suppose γ_1 lies in the Cartan subgroup $T_1(\mathbb{R})$ and is G-regular. If γ_1 is not an image there is nothing to do: Φ is zero on all G-regular elements of $T_1(\mathbb{R})$. We should note at this point that the notion of *image* is defined for any semisimple element in Section 1.2 of [LS2]. If γ_1 is an image of δ then δ is regular semisimple and then we can extend Φ to γ_1 by smoothness of each of the terms on the right (see Section 4.3 of [LS1] for details).

Next we fix T_1 and consider images $\gamma_{1,0}$ on walls of $T_1(\mathbb{R})$ outside H_1. There is no harm in working with the normalized Ψ in place of Φ (see Section 12). Thus $\gamma_{1,0}$ is regular in H_1 but any element δ_0 of which it is an image is singular in G. For elements on the *imaginary* walls outside H_1 we will proceed one wall at a time. For an element $\gamma_{1,0}$ on real or complex walls outside H_1 we observe from the statement of parabolic descent in Lemma 14.1 that Φ extends smoothly in a neighborhood of $\gamma_{1,0}$. We can extend this observation to the real or complex walls inside H_1 as long as we replace Φ by Ψ. Now we may argue by Harish Chandra's principle that to show that Ψ extends to a Schwartz function on $T_1(\mathbb{R})_{I-reg}$ (as needed in (iv) of Theorem 4.1) it is enough to show that the jump of at each G-semiregular element $\gamma_{1,0}$ on an imaginary wall outside H_1 is zero.

There are two cases to consider. As usual, let $\eta_{B_1,B} : T_1 \to T$ be defined over $\mathbb{R}$. Suppose $\alpha_1(\gamma_{1,0}) = 1$, where α_1 is a character on T_1 but not a root of H_1, and the transport α of α_1 to T is an imaginary root in G. Set $\delta_0 = \eta_{B_1,B}(\gamma_{1,0})$. Then the G-semiregularity assumption is simply that δ_0 is semiregular, i.e. that $\alpha(\delta_0) = 1$ determines the root α uniquely up to sign. The first case is that α is totally compact in G. Then all integrals $'F_f^w$ appearing in Ψ (Lemma 12.1), and their derivatives, have zero jump across δ_0. This together with Lemma 13.1 implies that Ψ and its derivatives have zero jump across $\gamma_{1,0}$.

For the second case, if α is not totally compact we can adjust $\eta_{B_1,B}$ so that α itself is noncompact. We now reexamine the last two paragraphs of Section 2. There we considered stable orbital integrals, i.e the case κ trivial, and we had two cases: $d(\alpha) = 1$ or $d(\alpha) = 2$. In the first case only, integrals were paired and each integral in a pair contributed the same jump. Now we argue that because α is noncompact and outside H_1, we have $\kappa(w_\alpha) = -1$ so that $d(\alpha) = 1$ must be true in our present setting, and that the jumps for the integrals in each pair are now opposite in sign. Thus again we conclude that Ψ and its derivatives have zero jump across $\gamma_{1,0}$. Notice in these arguments, and again below, the term $e^{\iota_1-\iota_G}$ in Lemma 13.1 is used to transform the map on differential operators $D \to \widehat{D}$ for H_1 to that for G, the term e^{μ^*} being harmless since $<\mu^*, \beta^\vee> = 0$ for each root β of T_1 in H_1.

We have now dealt with (i) - (iv) in the characterization theorem, and come to the semiregular analysis for imaginary walls *inside* H_1. Since (v) is vacuous for a quasi-split group, only (vi) remains. There are two cases: either the root α in G is totally compact or it is not. By the comment of the last paragraph of Section 7, if α is totally compact then the right side of the formula in (vi) is zero. So also is the left since we have only compact walls to cross. For the second case we may return to the setting of Section 7 and Lemma 13.2, and then observe that the jump formula for κ-orbital integrals in Lemma 4.4(ii) of [S3] takes exactly the form we need to combine with Lemmas 13.1 and 13.2 to obtain (vi). For this observation, we note that the κ-signature of the Cayley transform in G, as defined in [S3], is trivial since the transform is now chosen in the descent group G^{δ_0} and we have $\kappa(w_\alpha) = \kappa(\alpha^\vee) = 1$. The proof of Theorem 14.3 is then easily completed.

It is convenient to have a separate statement for parabolic descent.

Lemma 14.4

Suppose that M is a cuspidal Levi group in G, that $(M_H, \mathcal{H}_M, \mathfrak{s}_M, \xi_M)$ and $(M_{H_1}, \xi_{1,M})$ are data for M attached to $(H, \mathcal{H}, \mathfrak{s}, \xi)$ and (H_1, ξ_1) by descent. Suppose also that $f \in \mathcal{C}(G(\mathbb{R}))$, $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ and

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \textstyle\sum_{\delta, conj} \Delta(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. Then $f^{(P)} \in \mathcal{C}(M(\mathbb{R}))$ and $f_1^{(P_1)} \in \mathcal{C}(M_{H_1}(\mathbb{R}), \lambda_1)$ have the same property relative to the descent data for M, that is,

$$SO_{\gamma_1}(f_1^{(P_1)}, dt_1, dh_{M,1}) = \textstyle\sum_{\delta, conj} \Delta_M(\gamma_1, \delta) O_\delta(f^{(P)}, dt, dm)$$

for all strongly M-regular γ_1 in $M_{H_1}(\mathbb{R})$, provided that the transfer factor Δ_M is normalized compatibly.

Proof: Apply Lemma 14.2 to each side of the two equations for strongly G-*regular* γ_1 *in* $M_{H_1}(\mathbb{R})$, and note Remark 11.2 for the comparison on the left. Then extend smoothly to *strongly* M*-regular* γ_1.

15. Dual transfer map

By the space of stable tempered distributions we will mean the weak closure of the space generated by the stable orbital integrals $f \to SO_\gamma(f)$, for γ strongly regular, in the dual of the space $\mathcal{C}(G(\mathbb{R}), \lambda_0)$ of Section 2, although we avoid a more systematic discussion of this space here. Let Θ_1 be a stable tempered distribution on $H_1(\mathbb{R})$. If $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ is attached to $f \in \mathcal{C}(G(\mathbb{R}))$ by Theorem 14.3 then we define the *transfer* Θ of Θ_1 to $G(\mathbb{R})$ by $\Theta(f) = \Theta_1(f_1)$. Our interest is in invariant eigendistributions. Notice that an invariant eigendistribution is stable in the sense above if and only if it is represented by a stably invariant function on the regular semisimple set, and that the transfer Θ is a welldefined tempered invariant distribution on $G(\mathbb{R})$ (see [S2]).

Continuing with the transfer Θ of a stable tempered eigendistribution Θ_1 on $H_1(\mathbb{R})$, suppose that $z_1\Theta_1 = \chi_1(z_1)\Theta_1$ for z_1 in the center $\mathfrak{Z}_1$ of the enveloping algebra of $\mathfrak{h}_1$. We will describe shortly a homomorphism $z \to z_1$ of the center $\mathfrak{Z}$ of the enveloping algebra of $\mathfrak{g}$ into $\mathfrak{Z}_1$ and check that if f_1 is attached to f then $z_1 f_1$ is attached to zf. Then

$$z\Theta(f) = \Theta(zf) = \Theta_1(z_1 f_1) = z_1\Theta_1(f_1) = \chi_1(z_1)\Theta_1(f_1) = \chi(z)\Theta(f),$$

where χ is defined by $\chi(z) = \chi_1(z_1)$ for $z \in \mathfrak{Z}$. We will also need explicit information about the dual map $\chi_1 \to \chi$ on infinitesimal characters.

To define $\mathfrak{Z} \to \mathfrak{Z}_1$, choose any toral data $T_1 \to T$ for H_1 and G. We use the Harish Chandra isomorphism $\boldsymbol{\gamma}$ to identify $\mathfrak{Z}$ with the Weyl invariants in the symmetric algebra $\mathcal{S}$ on the Lie algebra $\mathfrak{t}$ of $T(\mathbb{R})$. Because the isomorphism $T_1/Z_1 \to T$ transports the Weyl group in H_1 into that of G, we have an embedding of the Weyl invariants in $\mathcal{S}$ into the Weyl invariants in $\mathcal{S}_1$. Recall the linear form μ^* on $\mathfrak{t}_1$ from Section 11. The isomorphism I_{μ^*} of $\mathcal{S}_1$ defined on $\mathfrak{t}_1$ by $X \to X + \mu^*(X)I$ preserves the Weyl invariants because μ^* is perpendicular to the roots of H_1. Then $\boldsymbol{\gamma}_1^{-1} \circ I_{\mu^*} \circ \boldsymbol{\gamma}$ is the (injective) homomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}_1$ that we will denote by $z \to z_1$. It is independent of the choice of toral data. It is then easy to describe $\chi_1 \to \chi$ in terms of linear forms. Recall that Θ_1 belongs to the dual of $\mathcal{C}(H_1(\mathbb{R}), \lambda_1)$. Thus if we write χ_1 as $\mu_1 \circ \boldsymbol{\gamma}_1$, where $\mu_1 \in \mathfrak{t}_1^*$ is extended to $\mathcal{S}_1$ as usual, then the restriction of μ_1 to the Lie algebra of $H_1(\mathbb{R})$ must be the negative of the restriction of μ^* (see Remark 11.3). Thus $\mu = \mu_1 + \mu^*$ defines a linear form on $\mathfrak{t}$, and

$$\chi(z) = \chi_1(z_1) = \chi_1(\boldsymbol{\gamma}_1^{-1}(I_{\mu^*}(\boldsymbol{\gamma}(z)))) = \mu_1(I_{\mu^*}(\boldsymbol{\gamma}(z)))) = (\mu_1 + \mu^*)(\boldsymbol{\gamma}(z)).$$

Thus $\chi = \mu \circ \boldsymbol{\gamma}$, and so we see that on the spectral side μ^* serves as a shift in infinitesimal character. Recall that on the geometric side μ^* contributed to the symmetrizing characters for quotients of Weyl denominators.

Lemma 15.1

Let $f_1 \in C(H_1(\mathbb{R}), \lambda_1)$ be attached to $f \in C(G(\mathbb{R}))$ by Theorem 14.3. Then, with the map $z \to z_1$ as defined above, we have that $z_1 f_1$ is attached to zf, for all z in the center of the universal enveloping algebra of G.

Corollary 15.2

If Θ_1 is a stable tempered eigendistribution on $H_1(\mathbb{R})$ with infinitesimal character μ_1 then Θ is a tempered invariant eigendistribution on $G(\mathbb{R})$ with infinitesimal character $\mu = \mu_1 + \mu^*$.

Proof of lemma: As mentioned earlier, here is where we make use of Harish Chandra's differential equations. Let $z \in \mathfrak{Z}$. Then, returning to the setting and notation of Section 3, we write the equation for z as

$${}'F_{zf}^T = \widehat{\boldsymbol{\gamma}(z)}\ \ {}'F_f^T.$$

Since $\boldsymbol{\gamma}(z)$ is invariant under the Weyl group we see easily that this equation holds with ${}'F_f^T$ replaced by ${}'F_f^w$, for each w in the imaginary Weyl group. We may pick any toral data. It is easiest to start with the expression

$$\tfrac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)} \Delta(\gamma_1, \delta) \textstyle\sum_w \kappa(w)\ {}'F_f^w(\delta)$$

from Lemma 12.1. Here δ has been chosen specifically to be the image of γ_1 under $T_1 \to T$. By Theorem 14.3, we now know that the expression coincides with normalized stable orbital integrals of f_1 which we may write as

$$\textstyle\sum_{w_1} {}'F_{f_1}^{w_1}(\gamma_1).$$

Replace f by zf in the first expression. Then to prove the lemma we need to show that if we move the operator $\widehat{\boldsymbol{\gamma}(z)}$ to the left of the function

$$\frac{\Delta'_{H_1}(\gamma_1)}{\Delta'_G(\delta)}\Delta(\gamma_1, \delta)$$

then we must replace it by the operator $\widehat{\boldsymbol{\gamma}_1(z_1)}$. Lemma 13.1 makes this a routine calculation, and so the lemma follows.

If we combine geometric transfer with the Weyl integration formula then, regarding Θ_1 and Θ as functions on the regular semisimple sets, we obtain Θ explicitly in terms of Θ_1 on each shared Cartan subgroup of $G(\mathbb{R})$. We exploit this, for example, to identify discrete series characters (see [S5] and [S7]).

We return to the setting of Section 4. Thus let π_1 be a tempered irreducible representation of $H_1(\mathbb{R})$ that transforms under $Z_1(\mathbb{R})$ according to the character λ_1. Then we apply Corollary 15.2 to $\Theta_1 = St\text{-}Tr\ \pi_1$ to conclude that Θ, defined by $\Theta(f) = St\text{-}Tr\ \pi_1(f_1)$, is a tempered invariant eigendistribution. Theorem 4.1.1 of [S5] now shows that there are well-defined coefficients $\Delta(\pi_1, \pi) = \pm C$, where C is a constant depending only on the normalization of the geometric transfer factors, such that

$$\Theta(f) = \textstyle\sum_\pi \Delta(\pi_1, \pi)\ Tr\ \pi(f),$$

where the summation is over tempered irreducible π in the L-packet attached to that of π_1 by the pair of embeddings $\xi_1 : \mathcal{H} \to {}^LH_1$ and $\xi : \mathcal{H} \to {}^LG$. We set $\Delta(\pi_1, \pi) = 0$ for all other tempered irreducible representations of $G(\mathbb{R})$.

Finally, we remark that if we start with $f \in C_c^\infty(G(\mathbb{R}))$ in Theorem 14.3 then an examination of the support of

$$\textstyle\sum_{\delta, conj} \Delta_{geom}(\gamma_1, \delta) O_\delta(f, dt, dg)$$

shows that we may apply a minor variant of Theorem 6.2.1of [B] to conclude that we can find $f_1 \in C_c^\infty(H_1(\mathbb{R}), \lambda_1)$ so that this expression coincides with

$$SO_{\gamma_1}(f_1, dt_1, dh_1)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$, *i.e.* geometric transfer is also true for smooth functions of compact support. On the dual side, Corollary 15.2 remains true, *i.e.* a stable eigendistribution Θ on $H_1(\mathbb{R})$ with infinitesimal character μ_1 transfers to an invariant eigendistribution Θ on $G(\mathbb{R})$ with infinitesimal character $\mu = \mu_1 + \mu^*$, and again, in terms of functions on the regular semisimple set, we may describe Θ explicitly in terms of Θ_1. At that point we must turn to A-packets and the work of [ABV], where we find a definition for a generalization of $St\text{-}Tr\ \pi_1$. First, however, it is instructive to look for an explicit formula for $\Delta(\pi_1, \pi)$ in the tempered case.

16. Conclusion

Now write the geometric transfer factor $\Delta(\gamma_1, \delta)$ as $\Delta_{geom}(\gamma_1, \delta)$. In [S7] we will define tempered spectral transfer factors $\Delta(\pi_1, \pi) = \Delta_{spec}(\pi_1, \pi)$ in an analogous manner, along with a canonical compatibility factor for normalizations of Δ_{geom} and Δ_{spec}. The definition of the spectral factors is much simpler: the product $\Delta_{II+} = \Delta_{II}\Delta_{III_2}\Delta_{IV}$, involving generalized Weyl denominators and the symmetrizing character, is replaced by a single term, a fourth root of unity. We may choose compatible normalizations so that $\Delta_{spec}(\pi_1, \pi)$ is simply a sign, although we do not insist on this for the transfer theorem. Finally we will verify $\Delta_{spec}(\pi_1, \pi)$ may replace the implicitly defined coefficients in the proof of Theorem 4.1.1 of [S5]. The following then summarizes tempered endoscopic transfer for the group G.

Theorem 16.1 (see [S7])

Let $(H, \mathcal{H}, \mathfrak{s}, \xi)$ be a set of endoscopic data for G, and (H_1, ξ_1) be a z-pair for H with attached character λ_1 on the central subgroup $Z_1(\mathbb{R})$, where $Z_1 = Ker(H_1 \to H)$. Let Δ_{geom} and Δ_{spec} *be transfer factors attached to this endoscopic data and z-pair, with compatible normalization. Then for each $f \in \mathcal{C}(G(\mathbb{R}))$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ such that*

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \sum_{\delta, conj} \Delta_{geom}(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. Moreover, there is a dual transfer of stable tempered characters given by

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi)\ Tr\ \pi(f)$$

for all tempered irreducible representations π_1 of $H_1(\mathbb{R})$ transforming under $Z_1(\mathbb{R})$ according to λ_1, and, conversely, if $f \in \mathcal{C}(G(\mathbb{R}))$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ satisfy

$$St\text{-}Tr\ \pi_1(f_1) = \sum_{\pi, temp} \Delta_{spec}(\pi_1, \pi)\ Tr\ \pi(f)$$

for all tempered irreducible representations π_1 on $H_1(\mathbb{R})$ transforming under $Z_1(\mathbb{R})$ according to λ_1 then

$$SO_{\gamma_1}(f_1, dt_1, dh_1) = \sum_{\delta, conj} \Delta_{geom}(\gamma_1, \delta) O_\delta(f, dt, dg)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

Here measures dh_1, dg and dt_1 have been chosen arbitrarily, but dt is related to dt_1 by transport (Section 2).

Notice that the converse matching statement follows easily from Theorem 4.1 along with the geometric matching, Theorem 3.1: given that f and f_1 match spectrally, use Theorem 3.1 to pick f_2 in $\mathcal{C}(H_1(\mathbb{R}), \lambda_1)$ so that f and f_2 have matching orbital integrals. Then by dual transfer for f and f_2, the functions f_1 and f_2 agree on stable tempered characters and hence, by Theorem 4.1, have same stable orbital integrals. Thus f and f_1 have matching orbital integrals.

We finish with a remark on the local form around the identity for the geometric transfer factor. The result is a little surprising after the arguments for Lemma 13.1. Its proof is quite simple but we cannot give it without a digression into spectral transfer factors. This result applies, for example, to a straightforward generalization of the Whittaker normalization introduced in [KS] for the geometric factors. The ε below is then the epsilon factor defined there. That factor accounts for the fact that maximally split tori in an endoscopic group H_1 need not be maximal among split tori in a quasisplit form of G, as of course happens in the familiar example of a compact torus and $SL(2)$.

Lemma 16.2 [S7]

Suppose that Δ_{geom} and Δ_{spec} are normalized compatibly and that

$$\Delta_{spec}(\pi_1, \pi) = \pm 1$$

for some, and hence every, G-regular related pair (π_1, π). Then if we remove the term Δ_{IV} from Δ_{geom} we obtain

$$\Delta_{geom}(\gamma_1, \delta) = \pm\ \varepsilon e^{\mu^*(X)}$$

for all strongly G-regular related pairs (γ_1, δ) with $\gamma_1 = \exp X$, where X is sufficiently close to the origin in the Lie algebra of $H_1(\mathbb{R})$ and ε is a constant fourth root of unity.

Recall that μ^* was defined in Section 11 as a linear form on the Lie algebra of $H_1(\mathbb{R})$ specified by L-group embeddings, and that in Section 15 we saw that it provides a shift of infinitesimal character in passage from H_1 to G.

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