SOME CHARACTER RELATIONS FOR

REAL REDUCTIVE ALGEBRAIC GROUPS

DIANA FROST SHELSTAD

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§1. Introduction

Suppose, for example, that G and H are simply-connected semisimple algebraic groups defined over $\mathbb{R}$. Suppose also that H is obtained from G by an inner twist in the sense that there exists an isomorphism $\mu : G \longrightarrow H$ defined over $\mathbb{C}$, for which the automorphism $\sigma(\mu)\mu^{-1}$ of H is inner (here σ denotes the nontrivial element of the Galois group of $\mathbb{C}/\mathbb{R}$, acting on μ in the usual manner). There is then a formal correspondence between the representations of $G(\mathbb{R})$ which appear in the Plancherel formula for $G(\mathbb{R})$ ([11]) and those of $H(\mathbb{R})$ which appear in the Plancherel formula for $H(\mathbb{R})$. Our purpose is to show that this correspondence is reflected in certain relations among the characters associated to these representations; these relations are analogues of those obtained in [15, §15].

We shall adopt the viewpoint of [18] and consider a slightly more general problem. Thus we shall suppose that G and H are connected reductive algebraic groups defined over $\mathbb{R}$ and that $\mu : G \longrightarrow H$ is an isomorphism over $\mathbb{C}$ for which $\sigma(\mu)\mu^{-1}$ is inner.

The group $G(\mathbb{R})$ of $\mathbb{R}$-rational points on G is a real reductive Lie group. We denote by $\Pi(G)$ the set of all infinitesimal equivalence classes of irreducible Banach representations of $G(\mathbb{R})$ ([18]). According to [18] $\Pi(G)$ is almost parametrized by a certain space $\Phi(G)$ in the following sense. Namely, to each element φ of $\Phi(G)$ there is associated a nonempty finite subset $\Pi_{\varphi}(G)$ of $\Pi(G)$; if φ and φ' are distinct elements of $\Phi(G)$ then $\Pi_{\varphi}(G)$ and $\Pi_{\varphi'}(G)$

are disjoint and moreover $\Pi(G)$ is exhausted by the collection of all subsets $\Pi_{\varphi}(G)$.

We denote by $G^{\wedge}$ the canonical form of the associate group for G. We may then define $\Phi(G)$ as a collection of equivalence classes of homomorphisms of the Weil group of $\mathbb{C}/\mathbb{R}$ into $G^{\wedge}$; the space $\Phi(H)$ is defined in the same way. The isomorphism $\mu : G \longrightarrow H$ determines a canonical isomorphism $\mu^{\wedge} : G^{\wedge} \longrightarrow H^{\wedge}$ which then determines a canonical bijection, also denoted $\mu^{\wedge}$, between a subspace of $\Phi(G)$ and a subspace of $\Phi(H)$. In particular, if H is quasisplit then $\mu^{\wedge}$ is a canonical embedding of $\Phi(G)$ in $\Phi(H)$.

For our purposes it will be convenient to make a noncanonical definition of $\Phi(G)$ and $\Phi(H)$ (cf. §2). Thus we shall realize $\Phi(G)$ and $\Phi(H)$ as subspaces of some large space $\Phi(G_1)$; $\mu^{\wedge}$ will be a bijection on $\Phi(G_1)$ which embeds the (nonempty) subspace $\Phi_{\mu}(G) = \Phi(G) \cap \mu^{\wedge^{-1}}(\Phi(H))$ of $\Phi(G)$ in $\Phi(H)$. To each element φ of $\Phi_{\mu}(G)$ there is then associated the subsets $\Pi_{\varphi}(G)$ of $\Pi(G)$ and $\Pi_{\mu^{\wedge}(\varphi)}$ of $\Pi(H)$.

We shall consider only those elements of $\Phi(G)$ which are essentially tempered in the sense of [18]. We denote by $\Pi_{o}(G)$ the inverse image of the set of all essentially tempered elements in $\Phi(G)$ under the natural projection of $\Pi(G)$ onto $\Phi(G)$. If G is semisimple and simply-connected then $\Pi_{o}(G)$ consists of those classes in $\Pi(G)$ which appear in the Plancherel formula for $G(\mathbb{R})$.

To each essentially tempered element φ in $\Phi(G)$ we associate a character $\chi_{\varphi}^{(G)}$; $\chi_{\varphi}^{(G)}$ is a locally L^1 class function on $G(\mathbb{R})$ which vanishes outside the set of regular elements (cf. Lemmas 3.1., 4.1.). By construction

$$\chi_{\varphi}^{(G)} = \sum_{\pi \varepsilon \Pi_{\varphi}(G)} m(\pi)\chi(\pi)$$

where $\chi(\pi)$ is the character of the infinitesimal equivalence class π and $m(\pi)$ is a positive integer. When φ is discrete (cf. §2) the integers $m(\pi)$ are all equal to 1; we expect this to be true in general (cf. footnote p. 20).

Suppose now that φ is an essentially tempered element which belongs to $\Phi_{\mu}(G)$. We remark that $\mu^{\wedge}(\varphi)$ is then also essentially tempered (in the space $\Phi(H)$). Therefore we have defined the characters $\chi_{\varphi}^{(G)}$ and $\chi_{\mu^{\wedge}(\varphi)}^{(H)}$. Our purpose is now to prove some relations between $\chi_{\varphi}^{(G)}$ and $\chi_{\mu^{\wedge}(\varphi)}^{(H)}$. We proceed as follows.

Suppose that $T(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$. We assume for the present that H is quasisplit. The isomorphism μ of G with H then determines a family of isomorphisms of $T(\mathbb{R})$ with various Cartan subgroups of $H(\mathbb{R})$. Let μ_T and μ'_T be two such isomorphisms. Then we shall prove that

$$\chi_{\mu^{\wedge}(\varphi)}^{(H)} \circ \mu'_T = \chi_{\mu^{\wedge}(\varphi)}^{(H)} \circ \mu_T$$

(Theorem 7.1.). We are then able to show that

$$\chi_{\varphi}^{(G)} = \varepsilon(G,H) \quad \chi^{(H)}_{\mu^{\wedge}(\varphi)} \circ \mu_T$$

where $\varepsilon(G,H)$ is a constant, equal to ± 1 and determined by G and H alone (Theorem 7.2.). If H is not quasisplit then we obtain the same relations whenever μ_T is well-defined (cf. §8); in particular, the relations are valid and nontrivial on at least one conjugacy class of Cartan subgroups.

Hence we obtain a set of relations between the characters associated to the classes in $\Pi_o(G)$ and those associated to the classes in $\Pi_o(H)$. The calculations in [12] indicate that we cannot expect such relations for classes outside $\Pi_o(G)$; for example Theorem 7.1. may fail.

We remark that our results are consequences of Harish-Chandra's theory of the discrete series ([8], [10]) and a characterization of the invariant eigendistributions on a semisimple Lie group ([14], [21]).

§2. <u>The spaces</u> $\Phi(G)$ <u>and</u> $\Phi_o(G)$

We start with an outline of some constructions in [18]. As before, G is a connected reductive algebraic group defined over $\mathbb{R}$. We choose a quasisplit group G_1 over $\mathbb{R}$ and an isomorphism $\lambda : G \longrightarrow G_1$ defined over $\mathbb{C}$ such that $\sigma(\lambda)\lambda^{-1}$ is inner (cf. [21]). Here again σ is the nontrivial element in the Galois group $\mathcal{G}(\mathbb{C}/\mathbb{R})$ of $\mathbb{C}/\mathbb{R}$.

We denote by $W_{\mathbb{C}/\mathbb{R}}$ the Weil group of $\mathbb{C}/\mathbb{R}$; $W_{\mathbb{C}/\mathbb{R}}$ is a nonsplit extension of $\mathbb{C}^x$ by $\mathcal{G}(\mathbb{C}/\mathbb{R})$. We choose an associate group

$G_1^\wedge$ for G_1; $G_1^\wedge$ is a split extension of a connected complex algebraic group $G_1^{\hat{o}}$, ([17]), by $W_{\mathbb{C}/\mathbb{R}}$. We choose an equivalence class η of distinguished splittings ([18, Section 2]) for this extension. The pairs $(G_1^\wedge, \eta)$ and $(W_{\mathbb{C}/\mathbb{R}}, \mathrm{id})$, id being the identity homomorphism on $W_{\mathbb{C}/\mathbb{R}}$, are then objects in the category $\mathcal{G}^\wedge(\mathbb{R})$ of [18, p. 11]. Following [18], we define

$$\Phi(G_1) = \mathrm{Hom}_{\mathcal{G}^\wedge(\mathbb{R})}((W_{\mathbb{C}/\mathbb{R}}, \mathrm{id}), (G_1^\wedge, \eta)) .$$

An element of $\Phi(G_1)$ is then an equivalence class of L-homomorphisms ([18, p. 5]) for $W_{\mathbb{C}/\mathbb{R}}$ into $G_1^\wedge$; two such homomorphisms φ and φ' belong to the same equivalence class if and only if $\varphi' = \mathrm{ad}\, g \circ \varphi$ for some $g \in G_1^{\hat{o}}$.

A subgroup $P^\wedge$ of $G_1^\wedge$ is called parabolic if $P^\wedge \cap G_1^{\hat{o}}$ is a parabolic subgroup of $G_1^{\hat{o}}$ and the natural homomorphism $P^\wedge \to W_{\mathbb{C}/\mathbb{R}}$ is surjective; we denote by $\mathcal{P}^\wedge(G_1)$ the set of equivalence classes of parabolic subgroups of $G_1^\wedge$ with respect to conjugacy under $G_1^{\hat{o}}$. Let $\mathcal{P}(G_1)$ denote the set of equivalence classes of parabolic subgroups of G_1 which are defined over $\mathbb{R}$, with respect to conjugacy under $G_1(\mathbb{R})$ (or $G_1(\mathbb{C})$, [1, 4.15.]). Then there is a natural bijection between $\mathcal{P}^\wedge(G_1)$ and $\mathcal{P}(G_1)$ ([18, p. 9]). On the other hand, if $\mathcal{P}(G)$ is the set of equivalence classes of parabolic subgroups of G which are defined over $\mathbb{R}$ (with respect to conjugacy under $G(\mathbb{R})$) then the map $\lambda : G \longrightarrow G_1$ determines an embedding of $\mathcal{P}(G)$ in $\mathcal{P}(G_1)$, ([18, p. 9]). Thus, having fixed λ, we may identify $\mathcal{P}(G)$ with a subset of $\mathcal{P}^\wedge(G_1)$; we denote this subset by $\mathcal{P}^\wedge(G)$.

To each $\varphi \in \Phi(G_1)$ we associate a subset $\rho(\varphi)$ of $\rho(G_1)$ in the following way. If $P^\wedge$ is a parabolic subgroup of $G_1^\wedge$ then the conjugacy class of $P^\wedge$ belongs to $\rho(\varphi)$ if and only if $P^\wedge$ contains the image of some L-homomorphism in the class φ. Keeping in mind the various choices and identifications we have made, we define

$$\Phi(G) = \{\varphi \in \Phi(G_1) : \rho(\varphi) \subseteq \rho^\wedge(G)\}.$$

The set $\rho^\wedge(G_1)$ may be partially ordered in the usual manner. If $\rho_1 \in p^\wedge(G)$ and $\rho_2 \geq \rho_1$ then it can be shown that $\rho_2 \in \rho^\wedge(G)$, [18, Lemma 2.2.]; in particular, $\rho^\wedge(G)$ contains $\{G_1^\wedge\}$. We define

$$\Phi_d(G) = \{\varphi \in \Phi(G); \rho(\varphi) = \{G_1^\wedge\}\}$$

and call a class φ in $\Phi(G)$ discrete if and only if it belongs to $\Phi_d(G)$.

The construction in [18, Section 3] shows that $\Phi_d(G)$ is nonempty if and only if G contains a maximal torus T defined over $\mathbb{R}$, for which $T_{der} = T \cap G_{der}$ is anisotropic over $\mathbb{R}$; here G_{der} denotes the derived group of G. Moreover for each such torus T there is a natural bijection between $\Phi_d(G)$ and the set of orbits in the group of quasicharacters on $T(\mathbb{R})$ under the action of the Weyl group $N_{G(\mathbb{C})}(T(\mathbb{C}))/T(\mathbb{C})$, as described in §3. We denote the image of φ under this bijection by $X_\varphi(G,T)$. We remark that T is determined up to conjugacy under $G(\mathbb{R})$ (since $T \cap G_{der}(\mathbb{R})$ is a compact Cartan subgroup of $G_{der}(\mathbb{R})$) and if $T' = g^{-1}Tg$ where $g \in G(\mathbb{R})$ then $X_\varphi(G,T') = \{\Lambda \circ \mathrm{ad}\, g : \Lambda \in X_\varphi(G,T)\}$, ([18, p. 52]).

Suppose now that φ is any class in $\Phi(G)$. Let $\wp \in \rho^{\wedge}(G)$ be a minimal element in $\rho(\varphi)$; we choose a group $P^{\wedge} \in \wp$. Let $N^{\wedge}$ be the unipotent radical of $P^{\wedge} \cap G_1^{\hat{o}}$ and $M^{\wedge} = P^{\wedge}/N^{\wedge}$. We choose a parabolic subgroup P_1 of G_1 defined over $\mathbb{R}$, such that the class of P_1 in $\rho(G_1)$ corresponds to $\wp$ under our identifications. Finally, we choose a Levi subgroup M_1 of P_1, defined over $\mathbb{R}$. Then M_1 is quasisplit over $\mathbb{R}$ and $M^{\wedge}$ is an associate group for M_1 ([18, Lemma 2.4.]) and moreover the chosen equivalence class η of distinguished splittings for $G_1^{\wedge}$ determines a unique equivalence class η_M of distinguished splittings for $M^{\wedge}$. The pair $(M^{\wedge}, \eta_M)$ then defines a space $\Phi(M_1)$.

Suppose that T_1 is a maximal torus in M_1, defined over $\mathbb{R}$ and such that $T_1 \cap (M_1)_{der}$ is anisotropic over $\mathbb{R}$; the existence of T_1 is ensured by the definition of P_1. We choose (cf. Lemma 8.1.) an element x_1 of $G_1(\mathbb{C})$ and a maximal torus T in G defined over $\mathbb{R}$, such that $\operatorname{ad} x_1 \circ \lambda : T \longrightarrow T_1$ and the restriction of $\operatorname{ad} x_1 \circ \lambda$ is defined over $\mathbb{R}$. We set $\lambda_T = \operatorname{ad} x_1 \circ \lambda$, $P = \lambda_T^{-1}(P_1)$ and $M = \lambda_T^{-1}(M_1)$.* Then the parabolic subgroup P of G is defined over $\mathbb{R}$ and M is a Levi subgroup of P also defined over $\mathbb{R}$. We note that $\lambda_T : M \longrightarrow M_1$ is such that $\sigma(\lambda_T)\lambda_T^{-1}$ is an inner automorphism of M_1, for $\sigma \in \mathcal{G}(\mathbb{C}/\mathbb{R})$. The pair $(\Phi(M_1), \lambda_T)$ then defines a space $\Phi(M)$, as before. By definition, the class of the subgroup P in $\rho(G)$ corresponds to the chosen element $\wp$ of $\rho^{\wedge}(G)$ under our embedding of $\rho(G)$ in $\rho^{\wedge}(G_1)$. It then follows that φ may be regarded as an element of $\Phi_d(M)$.

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* We will usually write P_φ, M_φ, T_φ or P_φ^G, M_φ^G, T_φ^G for P, M, T respectively.

Therefore there is associated to φ an orbit $X_\varphi(M, T)$ in the group of quasicharacters on $T(\mathbb{R})$ under the action of the appropriate complex Weyl group. Let $\Lambda_1, \Lambda_2 \in X_\varphi(M, T)$. Then $\Lambda_1 = \Lambda_2$ on the center $Z_M(\mathbb{R})$ of $M(\mathbb{R})$. We may thus denote by Λ_φ the restriction to $Z_M(\mathbb{R})$ of any element in $X_\varphi(M, T)$; we define $|\Lambda_\varphi|$ by

$$|\Lambda_\varphi|(z) = |\Lambda_\varphi(z)|, \quad z \in Z_M(\mathbb{R}) .$$

The results of [18, pp. 78-81] show that φ determines T up to conjugacy under an element x of $G(\mathbb{C})$ for which the restriction of $\operatorname{ad} x$ to T is defined over $\mathbb{R}$. Furthermore, if $T' = xTx^{-1}$ and $M' = xMx^{-1}$ where x is such an element then the corresponding quasicharacter Λ'_φ on $Z_{M'}(\mathbb{R})$ is $\Lambda_\varphi \circ \operatorname{ad} x^{-1}$. Thus we may define a space $\Phi_o(G)$ as follows.

Let $X(P)$ be the group of rational characters on P and $D(P)$ be the group of elements in $X(P) \otimes \mathbb{R}$ which are invariant under the action of $\mathfrak{G}(\mathbb{C}/\mathbb{R})$; we define $D(G)$ similarly and embed $D(G)$ in $D(P)$ in the usual way. The quasicharacter $|\Lambda_\varphi|$ on $Z_M(\mathbb{R})$ then extends uniquely to an element of $D(P)$ which we also denote by $|\Lambda_\varphi|$. We call the class φ essentially tempered if and only if $|\Lambda_\varphi| \in D(G)$. The space $\Phi_o(G)$ then consists of all essentially tempered classes in $\Phi(G)$. In particular, $\Phi_o(G)$ contains $\Phi_d(G)$. We remark that if G is semisimple then $\varphi \in \Phi_o(G)$ if and only if Λ_φ is a character on $Z_M(\mathbb{R})$.

We recall now that H is also a connected reductive group defined over $\mathbb{R}$ and that $\mu : G \longrightarrow H$ is an isomorphism over $\mathbb{C}$ such that

$\sigma(\mu)\mu^{-1}$ is inner for $\sigma \in \mathcal{G}(\mathbb{C}/\mathbb{R})$.

We fix an isomorphism $\lambda' : H \longrightarrow G_1$ over $\mathbb{C}$ such that $\sigma(\lambda')\lambda'^{-1}$ is inner for $\sigma \in \mathcal{G}(\mathbb{C}/\mathbb{R})$ (for example, we may set $\lambda' = \lambda \circ \mu^{-1}$). Then replacing (G, λ) by (H, λ') we define the corresponding subspaces $\Phi(H)$, $\Phi_d(H)$ and $\Phi_o(H)$ of $\Phi(G_1)$. We remark that $\Phi_d(G) = \Phi_d(H)$.

The automorphism $\lambda' \circ \mu \circ \lambda^{-1}$ of G_1 determines canonically an automorphism of $G_1^\wedge$ ([17]) which we denote by $\mu^\wedge$ since both λ' and λ are fixed. We also denote by $\mu^\wedge$ the corresponding bijection on the space $\Phi(G_1)$. We then define

$$\Phi_\mu(G) = \Phi(G) \cap \mu^{\wedge -1}(\Phi(H))$$

Suppose now that $\varphi \in \Phi_\mu(G)$. Then we have defined subgroups P_φ^G, M_φ^G and T_φ^G of G and subgroups $P^H_{\mu^\wedge(\varphi)}$, $M^H_{\mu^\wedge(\varphi)}$ and $T^H_{\mu^\wedge(\varphi)}$ of H. Following through the construction of these groups, we see that there exists an element x of $H(\mathbb{C})$ such that

$$\mathrm{ad}\, x \circ \mu : T_\varphi^G \longrightarrow T^H_{\mu^\wedge(\varphi)}$$

and the restriction of $\mathrm{ad}\, x \circ \mu$ to T_φ^G is defined over $\mathbb{R}$. This implies in particular that $\Phi_d(G) = \Phi_d(H)$ is invariant under $\mu^\wedge$.

We write μ_φ for $\mathrm{ad}\, x \circ \mu$. Then

$$\mu_\varphi : P_\varphi^G \longrightarrow P^H_{\mu^\wedge(\varphi)}$$

$$\mu_\varphi : M_\varphi^G \longrightarrow M^H_{\mu^\wedge(\varphi)}$$

and $\sigma(\mu_\varphi)\mu_\varphi^{-1}$ is an inner automorphism of $M^H_{\mu^\wedge(\varphi)}$. Moreover

$$X_{\mu^\wedge(\varphi)}(M^H_{\mu^\wedge(\varphi)},\ T^H_{\mu^\wedge(\varphi)}) = \{\Lambda \circ \mu^{-1}_{\varphi} : \Lambda \in X_{\varphi}(M^G_{\varphi},\ T^G_{\varphi})\}$$

In particular, $\varphi \in \Phi_o(G)$ if and only if $\mu^\wedge(\varphi) \in \Phi_o(H)$.

Finally, in §8 we will see that $\Phi_\mu(G)$ contains a nonempty family of essentially tempered classes.

§3. The characters $\chi_\varphi^{(G)}$, φ discrete

We now describe the representations which appear in $\Pi_\varphi(G)$ when φ is discrete. We then define a character $\chi_\varphi^{(G)}$ and compute $\chi_\varphi^{(G)}$ in terms of some known distributions.

Thus let Z_G be the center of G and G_{der} be the derived group of G. We denote by $G^o_{der}(\mathbb{R})$ the connected component of the identity in $G_{der}(\mathbb{R})$.

We fix a maximal compact subgroup K in $G(\mathbb{R})$.

Suppose that π is a quasisimple irreducible representation of $G(\mathbb{R})$ on a Banach space V. Following [18], we say that π is square-integrable modulo $Z_G(\mathbb{R})$ if π is of the form $\zeta \otimes \pi'$ where ζ is a quasicharacter on $Z_G(\mathbb{R})$ and π' is a (quasisimple irreducible) representation of $G(\mathbb{R})$ on a Banach space V', which satisfies

$$\int_{Z_G(\mathbb{R})\backslash G(\mathbb{R})} |f(\pi'(g)v')|^2\, d\bar{g} < \infty$$

for each K-finite vector $v' \in V'$ and K-finite linear form f on V'. The restriction to $G^o_{der}(\mathbb{R})$ of a quasisimple irreducible representation of $G(\mathbb{R})$ is infinitesimally equivalent to a finite direct sum of quasisimple irreducible representations of $G^o_{der}(\mathbb{R})$, [18, Lemma 3.5.]; if π is square-integrable modulo $Z_G(\mathbb{R})$ then each of these representations is infinitesimally equivalent to an irreducible unitary representation of $G^o_{der}(\mathbb{R})$ which is square-integrable in the usual sense. It follows then that if $G(\mathbb{R})$ has a quasisimple irreducible representation which is square-integrable modulo $Z_G(\mathbb{R})$ then $G^o_{der}(\mathbb{R})$ has a compact Cartan subgroup, [10, Theorem 13]. The following argument (together with [10, Theorem 13]) will show that the converse is also true.

Suppose now that $T^{(o)}_{der}(\mathbb{R})$ is a compact Cartan subgroup of $G^o_{der}(\mathbb{R})$. Then $T(\mathbb{R}) = Z_G(\mathbb{R})T^{(o)}_{der}(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$, ([18]). We note that a Cartan subgroup of $G(\mathbb{R})$ is the group of $\mathbb{R}$-rational points on some maximal torus in G, defined over $\mathbb{R}$ (cf. Lemma 5.1.); this explains our notation and also what we will mean by $T(\mathbb{C})$.

We obtain the representations of $G(\mathbb{R})$ which are square-integrable modulo $Z_G(\mathbb{R})$ in the following manner.

Let $G^\sim$ be the simply-connected covering group of G_{der} and p be the natural projection of $G^\sim$ onto G_{der}. Since $G^\sim(\mathbb{R})$ is connected ([2, 4.7.]) and has the same dimension as $G_{der}(\mathbb{R})$, the image of $G^\sim(\mathbb{R})$ under p is $G^o_{der}(\mathbb{R})$. The kernel D of the restriction of p to $G^\sim(\mathbb{R})$ is finite and central; we identify $G^o_{der}(\mathbb{R})$ with $G^\sim(\mathbb{R})/D$ and $T^{(o)}_{der}(\mathbb{R})$ with $T^\sim(\mathbb{R})/D$, where $T^\sim(\mathbb{R}) = p^{-1}(T^{(o)}_{der}(\mathbb{R}))$ is a compact Cartan subgroup of $G^\sim(\mathbb{R})$. We denote by $t_{der}(\mathbb{R})$ the Lie algebra of $T^{(o)}_{der}(\mathbb{R})$ (or $T^\sim(\mathbb{R})$) and by $g_{der}(\mathbb{R})$ the Lie algebra of $G_{der}(\mathbb{R})$ (or $G^{\sim}(\mathbb{R})$); as usual, $t_{der}(\mathbb{C}) = t_{der}(\mathbb{R}) \otimes \mathbb{C}$ and $g_{der}(\mathbb{C}) = g_{der}(\mathbb{R}) \otimes \mathbb{C}$.

Suppose now that Λ is a quasicharacter on $T(\mathbb{R})$. Then the restriction restriction of Λ to $T^{(o)}_{der}(\mathbb{R})$ determines a unique element λ in the dual $t^{*}_{der}(\mathbb{C})$ of $t_{der}(\mathbb{C})$ such that

$$\lambda(\exp H) = e^{\lambda(H)}, \quad H \in t_{der}(\mathbb{C}) .$$

We choose a system P of positive roots for $t_{der}(\mathbb{C})$ in $g_{der}(\mathbb{C})$ such that λ lies in the closure of the dominant Weyl chamber for P. If $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha$ then the element $\lambda + \delta$ of $t^{*}_{der}(\mathbb{C})$ is regular in the usual sense. Since $G^{\sim}(\mathbb{C})$ is simply-connected $\lambda + \delta$ determines a (regular) character $\Lambda^{\sim}$ on $T^{\sim}(\mathbb{R})$ which satisfies

$$\Lambda^{\sim}(\exp H) = e^{(\lambda+\delta)(H)}, \quad H \in t_{der}(\mathbb{R}) .$$

Harish Chandra's theory of the discrete series ([8], [10]) then associates to $\Lambda^{\sim}$ a square-integrable irreducible unitary representation $\pi^{\sim}(\Lambda, \delta)$ of $G^{\sim}(\mathbb{R})$ or, more precisely, an equivalence class of such representations of which we choose one. The character formula, ([8]), for $\pi^{\sim}(\Lambda, \delta)$ on $T^{\sim}(\mathbb{R})$ shows that the restriction of $\pi^{\sim}(\Lambda, \delta)$ to D is trivial. Therefore $\pi^{\sim}(\Lambda, \delta)$ determines a (square-integrable irreducible unitary) representation of $G^{o}_{der}(\mathbb{R})$ which we denote by $\pi_1(\Lambda, \delta)$.

Let $\Omega(\mathbb{C}) = N_{G(\mathbb{C})}(T(\mathbb{C}))/T(\mathbb{C})$, $\Omega(\mathbb{R}) = N_{G(\mathbb{R})}(T(\mathbb{R}))/T(\mathbb{R})$ and $\Omega_o(\mathbb{R}) = N_{T^{o}_{der}(\mathbb{R})}(T^{(o)}_{der}(\mathbb{R}))/T^{(o)}_{der}(\mathbb{R})$. We identify $\Omega_o(\mathbb{R})$ with the (normal) subgroup $N_{G^o_{der}(\mathbb{R})}(T(\mathbb{R}))/T(\mathbb{R})$ of $\Omega(\mathbb{R})$ (cf. p. 11) and $\Omega(\mathbb{R})$ with a subgroup of $\Omega(\mathbb{C})$. We also identify $\Omega(\mathbb{C})$ with the Weyl group of $t_{der}(\mathbb{C})$ in $g_{der}(\mathbb{C})$. There is a unique bijection $\omega \longrightarrow \omega^\sim$ of $\Omega(\mathbb{C})$ with $\Omega^\sim(\mathbb{C}) = N_{G^\sim(\mathbb{C})}(T^\sim(\mathbb{C}))/T^\sim(\mathbb{C})$ which satisfies

$$p\omega^\sim(t) = \omega p(t), \quad t \in T^\sim(\mathbb{C}), \quad \omega \in \Omega(\mathbb{C});$$

under this bijection $\Omega_o(\mathbb{R})$ corresponds to $\Omega^\sim(\mathbb{R}) = N_{G^\sim(\mathbb{R})}(T^\sim(\mathbb{R}))/T^\sim(\mathbb{R})$.

Let $T(\mathbb{R})^\wedge$ be the group of quasicharacters on $T(\mathbb{R})$. Then since the elements of $\Omega(\mathbb{C})$ normalize $T(\mathbb{R})$ (cf. Lemma 5.2.) there is a (unique) action of $\Omega(\mathbb{C})$ on $T(\mathbb{R})^\wedge$ which satisfies

$$\omega\Lambda(\omega t) = \Lambda(t), \quad \omega \in \Omega(\mathbb{C}), \quad \Lambda \in T(\mathbb{R})^\wedge, \quad t \in T(\mathbb{R}) .$$

Harish Chandra's theorems ([8], [10]) show that if $\Lambda, \Lambda' \in T(\mathbb{R})^\wedge$ then $\pi_1(\Lambda, \delta)$ is (infinitesimally) equivalent to $\pi_1(\Lambda', \delta')$ if and only if there exists $\omega \in \Omega_o(\mathbb{R})$ such that $\Lambda' = \omega\Lambda$ and $\delta' = \omega\delta$; furthermore every square-integrable irreducible unitary representation of $G^o_{der}(\mathbb{R})$ is equivalent to some representation $\pi(\Lambda, \delta)$. Finally, if $\omega \in \Omega_o(\mathbb{R})\backslash\Omega(\mathbb{R})$ and $x \in G(\mathbb{R})$ is a representative for ω then $(\pi_1(\Lambda, \delta))^x$ is equivalent to $\pi_1(\omega\Lambda, \omega\delta)$, where we define

$$(\pi_1(\Lambda, \delta))^x(G) = \pi_1(\Lambda, \delta)(x^{-1}gx), \quad g \in G^o_{der}(\mathbb{R}) .$$

If now $\Lambda^{(o)}$ is the restriction of Λ to $Z_G(\mathbb{R})$ then $\Lambda^{(o)} \otimes \pi_1(\Lambda, \delta)$ is a representation of $G_o(\mathbb{R}) = Z_G(\mathbb{R})\, G^o_{der}(\mathbb{R})$; the representation

$$\pi(\Lambda, \delta) = \mathrm{Ind}(\Lambda^{(o)} \otimes \pi_1(\Lambda, \delta);\ G_o(\mathbb{R}),\ G(\mathbb{R}))$$

of $G(\mathbb{R})$ is square-integrable modulo $Z_G(\mathbb{R})$. We note that natural correspondence between $\Omega_o(\mathbb{R})\backslash\Omega(\mathbb{R})$ and $G(\mathbb{R})/G_o(\mathbb{R})$ is actually a bijection since $T^o_{der}(\mathbb{R})$ is compact. It then follows from our remarks above and a well-known result ([23, I, p. 424]) that $\pi(\Lambda, \delta)$ is irreducible. If $\pi(\Lambda', \delta')$ is another such representation then $\pi(\Lambda', \delta')$ is (infinitesimally) equivalent to $\pi(\Lambda, \delta)$ if and only if there exists $\omega \in \Omega(\mathbb{R})$ such that $\Lambda' = \omega\Lambda$ and $\delta' = \omega\delta$, and furthermore, every quasisimple irreducible representation of $G(\mathbb{R})$ which is square-integrable modulo $Z_G(\mathbb{R})$, is infinitesimally equivalent to some representation $\pi(\Lambda, \delta)$ (cf. [18]).

The classes in $\Pi(G)$ which contain a representation square-integrable modulo $Z_G(\mathbb{R})$ are "parametrized" by $\Phi_d(G)$; that is, they are those classes which appear in the sets $\Pi_\varphi(G)$ when φ belongs to $\Phi_d(G)$. The exact correspondence is as follows. If $\varphi \in \Phi_d(G)$ then we have associated to φ an orbit X_φ in $T(\mathbb{R})^\wedge$ under the action of $\Omega(\mathbb{C})$; $\Pi_\varphi(G)$ is then the set of those classes in $\Pi(G)$ which contain a representation $\pi(\Lambda, \delta)$, $\Lambda \in X_\varphi$. Suppose now that P is a fixed system of positive roots for $t_{der}(\mathbb{C})$ in $g_{der}(\mathbb{C})$. We define $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha$ and choose $\Lambda_\varphi \in X_\varphi$ such that the corresponding element of $t^*_{der}(\mathbb{C})$ (p. 12) belongs to the closure of the dominant Weyl chamber for P. Then $\Pi_\varphi(G)$ consists of the infinitesimal equivalence classes of the representations $\pi(\omega\Lambda_\varphi, \omega\delta)$, $\omega \in \Omega(\mathbb{R})\backslash\Omega(\mathbb{C})$; we recall that two such representations are not (infinitesimally) equivalent.

The usual notion of character ([3]) carries over to representations of $G(\mathbb{R})$. Thus, let π be a strongly continuous representation of $G(\mathbb{R})$ on a Hilbert space. Suppose that for each $f \in C_c^\infty(G(\mathbb{R}))$ the operator $\pi(f) = \int_{G(\mathbb{R})} f(g)\pi(g)dg$ is of trace class and that the mapping $\chi(\pi) = f \longrightarrow \text{trace } \pi(f)$ is a distribution on $G(\mathbb{R})$. Then $\chi(\pi)$ is called the character of π. If π is a quasisimple irreducible representation of $G(\mathbb{R})$ with character $\chi(\pi)$ and π', also quasisimple and irreducible, is infinitesimally equivalent to π then π' also has character $\chi(\pi)$; we may thus define $\chi(\pi)$ to be the character of the infinitesimal equivalence class of π.

We now determine the characters of the representations $\pi(\omega\Lambda_\varphi, \omega\delta)$, $\omega \in \Omega(\mathbb{R})\backslash\Omega(\mathbb{C})$. To begin, the representation $\pi^\sim(\omega\Lambda_\varphi, \omega\delta)$ of $G^\sim(\mathbb{R})$ certainly has a well-defined character which we write as

$$(-1)^{q_G} \det\omega^\sim \, \Theta_{\omega^\sim\Lambda_\varphi^\sim}$$

where $\omega^\sim$ and $\Lambda_\varphi^\sim$ are as defined on p. 12 and $2q_G$ is the dimension of the space $G^\sim(\mathbb{R})/K^\sim$, $K^\sim$ being a maximal compact subgroup in $G^\sim(\mathbb{R})$. This definition presumes the choice of a system of positive roots for $t_{der}(\mathbb{C})$ in $g_{der}(\mathbb{C})$; we use the system P. The distribution $\Theta_{\omega^\sim\Lambda_\varphi^\sim}$ is that associated to the regular character $\omega^\sim\Lambda_\varphi^\sim$ in [8] (we may suppose that P is the system chosen in [8]). We identify $\Theta_{\omega^\sim\Lambda_\varphi^\sim}$, in the usual way, as a locally L^1 function on $G^\sim(\mathbb{R})$ which vanishes outside the set $G^{\sim}(\mathbb{R})_{reg}$ of regular elements in $G^{\sim}(\mathbb{R})$, ([7]). We recall that $\Theta_{\omega^{\sim}\Lambda^{\sim}_{\varphi}}$ is analytic on $G^{\sim}(\mathbb{R})_{reg}$ and that

$$\Theta_{\omega^{\sim}\Lambda^{\sim}_{\varphi}}(dg) = \Theta_{\Lambda^{\sim}_{\varphi}}(g), \qquad d \in D,\ g \in G^{\sim}(\mathbb{R}) .$$

It then follows that the representation $\pi_1(\omega\Lambda_{\varphi}, \omega\delta)$ of $G_o(\mathbb{R})$ has a well-defined character given by the (locally L^1) function $(-1)^{q_G} \det\omega\ \Theta_{(\omega\Lambda_{\varphi})}$ where

$$\Theta_{(\omega\Lambda_{\varphi})} = \Theta_{\omega^{\sim}\Lambda^{\sim}_{\varphi}} \circ p^{-1}$$

and p is, as before, the projection of $G^{\sim}(\mathbb{R})$ onto $G^{o}_{der}(\mathbb{R})$. We extend $\Theta_{(\omega\Lambda_{\varphi})}$ to a function on $G_o(\mathbb{R})$ by defining

$$\Theta_{(\omega\Lambda_{\varphi})}(zg) = \Lambda_{\varphi}(z)\ \Theta_{(\omega\Lambda_{\varphi})}(g)$$

for $z \in Z_G(\mathbb{R})$, $g \in G^{o}_{der}(\mathbb{R})$; $(-1)^{q_G} \det\omega\ \Theta_{(\omega\Lambda_{\varphi})}$ is then the character of the representation $\Lambda^{(o)}_{\varphi} \otimes \pi_1(\omega\Lambda_{\varphi}, \omega\delta)$.

From a well-known result (eg. [19]) it now follows that $\pi(\omega\Lambda_{\varphi},\omega\delta)$ has a well-defined character given by the function $\chi(\pi(\omega\Lambda_{\varphi},\omega\delta))$

$$= \begin{cases} \displaystyle\sum_{x\in G(\mathbb{R})/G_o(\mathbb{R})} \chi(\Lambda^{(o)}_{\varphi} \otimes (\pi_1(\omega\Lambda_{\varphi}, \omega\delta))^{x}) & \text{on } G_o(\mathbb{R}) \\ 0 & \text{outside } G_o(\mathbb{R}). \end{cases}$$

But, by our earlier remarks,

$$\sum_{x \in G(\mathbb{R})/G_o(\mathbb{R})} \chi(\Lambda_\varphi^{(o)} \otimes (\pi_1(\omega\Lambda_\varphi, \omega\delta))^x)$$

$$= \sum_{s \in \Omega_o(\mathbb{R})\backslash\Omega(\mathbb{R})} \chi(\Lambda_\varphi^{(o)} \otimes \pi_1(s\omega\Lambda_\varphi, s\omega\delta))$$

$$= (-1)^{q_G} \sum_{s \in \Omega_o(\mathbb{R})\backslash\Omega(\mathbb{R})} (\det s\omega)\, \Theta_{(s\omega\Lambda_\varphi)} .$$

Thus, if we define

$$\chi_\varphi^{(G)} = \sum_{\pi \in \Pi_\varphi(G)} \chi(\pi)$$

then we conclude that

Lemma 3.1.

If $\varphi \in \Phi_d(G)$ *then*

$$\chi_\varphi^{(G)} = \begin{cases} (-1)^{q_G} \displaystyle\sum_{\omega \in \Omega_0(\mathbb{R})\backslash\Omega(\mathbb{C})} \det\omega \;\; \Theta_{(\omega\Lambda_\varphi)} & \underline{\text{on}}\ G_o(\mathbb{R}) \\ 0 \;\; \underline{\text{outside}}\ G_o(\mathbb{R}) . & \end{cases}$$

We note, for future reference, that $\chi_\varphi^{(G)}$ is analytic on the set of regular elements in $G(\mathbb{R})$. We also observe that on $G^o_{der}(\mathbb{R})$

$$\sum_{\omega \in \Omega_o(\mathbb{R})\backslash\Omega(\mathbb{C})} \det\omega\, \Theta_{(\omega\Lambda_\varphi)} = \sum_{\omega \in \Omega^\sim(\mathbb{R})\backslash\Omega^\sim(\mathbb{C})} \det\omega^\sim\, \Theta_{\omega^\sim\Lambda_\varphi^\sim} \circ p^{-1}.$$

§4. The characters $\chi_\varphi^{(G)}$ in general.

We next describe $\Pi_\varphi(G)$, and define $\chi_\varphi^{(G)}$, for an arbitrary class φ in $\Phi_o(G)$. We first note some properties of induced representations.

Thus, suppose that P is a parabolic subgroup of G defined over $\mathbb{R}$ and that M is a Levi subgroup of P, also defined over $\mathbb{R}$. We recall that if N is the unipotent radical of P then $P(\mathbb{R}) = M(\mathbb{R})\, N(\mathbb{R})$.

Let π be a quasisimple irreducible representation of $M(\mathbb{R})$ on a Hilbert space $\mathcal{H}_o$, or a finite direct sum of such representations. Then π extends trivially to a representation π^P of $P(\mathbb{R})$. Let $n(\mathbb{R})$ be the Lie algebra of $N(\mathbb{R})$ and define

$$\delta_P(p) = \left|\det\left(\mathrm{Ad}\,p/n(\mathbb{R})\right)\right|, \quad p \in P(\mathbb{R}) .$$

Finally, let

$$\pi^G = \mathrm{Ind}(\delta_P^{\frac{1}{2}}\, \pi^P ;\ P(\mathbb{R}),\ G(\mathbb{R})) .$$

Then the representation π^G has a finite composition series (cf. [18, pp. 64-65]) and the infinitesimal equivalence class of each irreducible subquotient of π^G belongs to $\Pi(G)$.

Let $\mathcal{H}$ be the set of all functions $f : G(\mathbb{R}) \longrightarrow \mathcal{H}_o$ which are Borel-measurable and which satisfy

$$f(pg) = \delta_P(p)^{\frac{1}{2}}\, \pi_P(p) f(g), \quad p \in P(\mathbb{R}),\ g \in G(\mathbb{R})$$

and

$$\int_K \|f(k)\|^2 \, dk < \infty$$

where K is some fixed maximal compact subgroup of $G(\mathbb{R})$. Then since $G(\mathbb{R}) = K\,P(\mathbb{R})$ each function in $\mathcal{H}$ is determined by its restriction to K ; we thus regard $\mathcal{H}$ as a closed subspace of $L^2(K)$. We recall that we may realize π^G as the right regular representation of $G(\mathbb{R})$ on $\mathcal{H}$.

Suppose that δ is an equivalence class of irreducible unitary representations of K. Then it follows from a well-known result ([23, I, p. 250]) that δ occurs in the restriction to K of π^G (and each of its irreducible subquotients) with multiplicity not greater than the degree of δ. A standard argument (cf. [3]) now shows that π^G and its irreducible subquotients each have a well-defined character. Furthermore the character of π^G is equal to the sum of the characters of its irreducible subquotients.

Suppose now that $\varphi \in \Phi_o(G)$. In §2 we have associated to φ a parabolic subgroup P_φ defined over $\mathbb{R}$ and a Levi subgroup M_φ of P_φ, also defined over $\mathbb{R}$. By construction $(M_\varphi)^o_{der}(\mathbb{R})$ has a compact Cartan subgroup and φ may be identified with an element of $\Phi_d(M_\varphi)$; we thus have associated a subset $\Pi_\varphi(M_\varphi)$ of $\Pi(M_\varphi)$. By definition ([18, p. 82]) $\Pi_\varphi(G)$ consists of the infinitesimal equivalence classes of the irreducible subquotients of the representations π^G, where π is a representation of $M(\mathbb{R})$ whose infinitesimal equivalence class belongs to $\Pi_\varphi(M_\varphi)$.

Suppose that $\pi_1, \ldots, \pi_n$ form a maximal set of representations of $M(\mathbb{R})$ whose infinitesimal equivalence classes are distinct and appear in $\Pi_\varphi(M_\varphi)$. We define $\pi_\varphi^{P_\varphi} = \pi_1^{P_\varphi} \oplus \ldots \oplus \pi_n^{P_\varphi}$. Then $\pi_\varphi^G = \mathrm{Ind}(\pi_\varphi^{P_\varphi}: P_\varphi(\mathbb{R}), G(\mathbb{R}))$ is equivalent to $\oplus_i \mathrm{Ind}(\pi_i^{P_\varphi}; P_\varphi(\mathbb{R}), G(\mathbb{R}))$ and so $\Pi_\varphi(G)$ is the set of infinitesimal equivalence classes of the irreducible constituents of π_φ^G.

Moreover if we set

$$\chi_\varphi^{(G)} = \sum_{\pi \in \Pi_\varphi(G)} m(\pi)\chi(\pi)$$

where $m(\pi)$ is the multiplicity of π in π_φ^G (*), then $\chi_\varphi^{(G)}$ is well-defined and is just the character of π_φ^G. We now compute $\chi(\pi_\varphi^G)$ in terms of $\chi(\pi_\varphi^{M_\varphi})$ using a well-known method ([13], [19]).

Thus we choose Haar measures dm, dn on $M_\varphi(\mathbb{R})$ and $N_\varphi(\mathbb{R})$ respectively and normalize left Haar measure on $P_\varphi(\mathbb{R})$ so that(**)

$$\int_{P_\varphi(\mathbb{R})} f(p)\, dp = \int_{M_\varphi(\mathbb{R}) \times N_\varphi(\mathbb{R})} f(mn)\, dm\,dn, \quad f \in C_c(P_\varphi(\mathbb{R})) .$$

We then normalize Haar measure on $G(\mathbb{R})$ so that(**)

$$\int_{G(\mathbb{R})} f(g)\, dg = \int_{M_\varphi(\mathbb{R}) \times N_\varphi(\mathbb{R}) \times K} f(mnk)\, dm\,dn\,dk, \quad f \in C_c(G(\mathbb{R})) .$$

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* We are unable to improve on this definition at present. We would expect that each $m(\pi) = 1$; this is so if $M_\varphi^G = G$ (so that φ is discrete) or if G is semisimple and simply-connected and P is minimal among the parabolic subgroups defined over $\mathbb{R}$ ([16]).

** These formulas are proved in the usual way (cf. [23]); we omit the details.

We realize π_{φ}^{G} on a space $\mathcal{H}$ as before. Then, for $f \in C_c^{\infty}(G(\mathbb{R}))$, $F \in \mathcal{H}$, and $k_o \in K$, we have

$$\pi_{\varphi}^{G}(f)F(k_o) = \int_{G(\mathbb{R})} f(g)(\pi_{\varphi}^{G}(g)F)(k_o)\, dg$$

$$= \int_{M_{\varphi}(\mathbb{R}) \times N_{\varphi}(\mathbb{R}) \times K} f(k_o^{-1}mnk)F(mnk)\, dmdndk$$

$$= \int_K \int_{M_{\varphi}(\mathbb{R}) \times N_{\varphi}(\mathbb{R})} f(k_o^{-1}mnk)\delta_P^{\frac{1}{2}}(m)\pi_{\varphi}^{P}(m)\, dmdn\; F(k)dk$$

But the restriction of π_{φ}^{P} to $M_{\varphi}(\mathbb{R})$ is the representation $\pi_1 \oplus \ldots \oplus \pi_n$ which has character $\chi_{\varphi}^{(M_{\varphi})}$ so that the operator

$$A_{f,\varphi}(k_o, k) = \int_{M_{\varphi}(\mathbb{R}) \times N_{\varphi}(\mathbb{R})} f(k_o^{-1}mnk)\delta_{P_{\varphi}}^{\frac{1}{2}}(m)\pi_{\varphi}^{P}(m)\, dmdn$$

acting on the representation space of π_{φ}^{P}, is of trace class and trace $A_{f,\varphi}(k_o, k)$

$$= \int_{M_{\varphi}(\mathbb{R}) \times N_{\varphi}(\mathbb{R})} f(k_o^{-1}mnk)\delta_{P_{\varphi}}^{\frac{1}{2}}(m)\chi_{\varphi}^{(M_{\varphi})}(m)dmdn$$

We recall that $\chi_{\varphi}^{(M_{\varphi})}$ is an analytic function on the set of regular elements in $M_{\varphi}(\mathbb{R})$. Therefore the map

$$(k_o, k) \longrightarrow \text{trace } A_{f,\varphi}(k_o, k)$$

of $K \times K$ into $\mathbb{C}$ is C^{∞}; for $h \in L^2(K)$ we set

$$B_{f,\varphi}(h)(k_o) = \int_K \text{trace } A(k_o, k)h(k)\, dk, \quad k_o \in K.$$

Then it follows, ([23, I, p. 470]), that the operator $B_{f,\varphi}$ is of trace class and

$$\text{trace } B_{f,\varphi} = \int_K \text{trace } A_{f,\varphi}(k,k)\, dk$$

$$= \int_{M_\varphi(\mathbb{R}) \times N_\varphi(\mathbb{R}) \times K} f(k^{-1}mnk)\,\delta_{P_\varphi}^{\frac{1}{2}}(m)\,\chi_\varphi^{(M_\varphi)}(m)\, dm\,dn\,dk$$

Choosing a suitable basis for $\mathcal{H}$ we see that

$$\text{trace } B_{f,\varphi} = \text{trace } \pi_\varphi^G(f)\ ;$$

this computation is indicated in [13]. We conclude that

$$\text{trace } \pi_\varphi^G(f) = \int_{M_\varphi(\mathbb{R}) \times N_\varphi(\mathbb{R}) \times K} f(k^{-1}mnk)\,\delta_{P_\varphi}^{\frac{1}{2}}(m)\,\chi_\varphi^{(M_\varphi)}(m)\, dm\,dn\,dk$$

$$= \int_{M_\varphi(\mathbb{R}) \times N_\varphi(\mathbb{R}) \times K} f(k^{-1}n^{-1}mnk)\,\delta_{P_\varphi}^{\frac{1}{2}}(m)\,\mu_{P_\varphi}(m)\,\chi_\varphi^{(M_\varphi)}(m)\, dm\,dn\,dk$$

where

$$\mu_{P_\varphi}(m) = \left|\det(\mathrm{Ad}(m^{-1})-1)_{n_\varphi(\mathbb{R})}\right|, \quad m \in M_\varphi(\mathbb{R}),$$

(using the argument of [9, p. 93]).

Let $T(\mathbb{R})$ be a Cartan subgroup of $M_\varphi(\mathbb{R})$. We choose a Haar measure on $T(\mathbb{R})$ and normalize the right-invariant measure on $T(\mathbb{R})\backslash M_\varphi(\mathbb{R})$ so that

$$\int_{M_\varphi(\mathbb{R})} f(m)dm = \int_{T(\mathbb{R})\backslash M_\varphi(\mathbb{R})} \int_{T(\mathbb{R})} f(tm)\, dt d\overline{m}, \quad f \in C_c(M_\varphi(\mathbb{R})).$$

Suppose that $\{T_1(\mathbb{R}), \ldots T_n(\mathbb{R})\}$ is a maximal set of Cartan subgroups of $M_\varphi(\mathbb{R})$ which are not conjugate under $M_\varphi(\mathbb{R})$. Then we have (cf. [10, Lemma 91]) that trace $\pi_\varphi^G(f)$

$$= \sum_{i=1}^{n} C_i^{M_\varphi} \int_{T_i(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash M_\varphi(\mathbb{R})} F_\varphi(\overline{m}, t)\nu_i^{M_\varphi}(t)\, d\overline{m}dt$$

where

$$F_\varphi(\overline{m}, t) = \int_{N_\varphi(\mathbb{R}) \times K} f(k^{-1}n^{-1}m^{-1}tmnk)\, dndk\, \chi_\varphi^{(M_\varphi)}(t)\mu_{P_\varphi}(t)\delta_{P_\varphi}^{\frac{1}{2}}(t),$$

$C_i^{M_\varphi}$ is the order of the Weyl group of $T_i(\mathbb{R})$ in $M_\varphi(\mathbb{R})$ and

$$\nu_i^{M_\varphi}(t) = \left|\det(\mathrm{Ad}\, t^{-1}-1)_{\mathfrak{m}_\varphi(\mathbb{R})/\mathfrak{t}_i(\mathbb{R})}\right|, \quad t \in T_i(\mathbb{R}),$$

$\mathfrak{m}_\varphi(\mathbb{R})$, $\mathfrak{t}_i(\mathbb{R})$ being the Lie algebras of $M_\varphi(\mathbb{R})$ and $T_i(\mathbb{R})$ respectively.

On the other hand, $T_i(\mathbb{R})$ is also a Cartan subgroup of $G(\mathbb{R})$ (cf. Lemma 5.1.). We normalize right-invariant measure on $T_i(\mathbb{R})\backslash G(\mathbb{R})$ so that

$$\int_{G(\mathbb{R})} f(g)dg = \int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} \int_{T_i(\mathbb{R})} f(tg)dtd\overline{g}, \quad f \in C_c(G(\mathbb{R})).$$

But

$$\int_{G(\mathbb{R})} f(g)dg = \int_{K \times N_\varphi(\mathbb{R}) \times M_\varphi(\mathbb{R})} f(mnk)\, dmdndk, \quad f \in C_c(G(\mathbb{R}))$$

$$= \int_{K \times N_\varphi(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash M_\varphi(\mathbb{R})} \int_{T_i(\mathbb{R})} f(tmnk)\, dtd\overline{m}dndk.$$

It follows then that

$$\int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)\, d\overline{g} = \int_{K \times N_\varphi(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash M_\varphi(\mathbb{R})} f(k^{-1}n^{-1}m^{-1}tmnk)\, d\overline{m}dndk$$

for each regular element t of $T_i(\mathbb{R})$. We may change the order of integration on the right hand side of this equation and so, for $f \in C_c^\infty(G(\mathbb{R}))$,

$$\text{trace}\ \pi_\varphi^G(f) =$$

$$\sum_{i=1}^{n} C_i^{M_\varphi} \int_{T_i(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\overline{g}\,\mu_{P_\varphi}(t)\,\delta_{P_\varphi}^{\frac{1}{2}}(t)\,\chi_\varphi^{(M_\varphi)}(t)\,\nu_i^{M_\varphi}(t)\, dt\,.$$

For a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ we define

$$\nu_T^G(t) = \left|\det(\mathrm{Ad}(t^{-1}) - 1)_{\mathfrak{g}(\mathbb{R})/\mathfrak{t}(\mathbb{R})}\right|, \quad t \in T(\mathbb{R})\,.$$

Then $\nu_T^G(t) \neq 0$ if and only if t is regular; we denote the set of regular elements in $T(\mathbb{R})$ by $T(\mathbb{R})_{reg}$. Suppose also that $T(\mathbb{R})$ is contained in $M(\mathbb{R})$. Then we define

$$\zeta^{(M_\varphi)}(t) = \frac{\delta_{P_\varphi}^{\frac{1}{2}}(t)\mu_{P_\varphi}(t)\nu_T^{M_\varphi}(t)}{\nu_T^G(t)}, \quad t \in T(\mathbb{R})_{reg}$$

$$= \frac{|\det(\mathrm{Ad}t)_{n_\varphi(\mathbb{R})}|^{\frac{1}{2}}}{|\det(\mathrm{Ad}(t^{-1})-1)_{n_\varphi(\mathbb{R})^-}|}$$

where $n_\varphi(\mathbb{R})^-$ has its usual meaning (eg. [23, 1.2.4.]). Thus trace $\pi_\varphi^G(f) =$

$$\sum_{i=1}^{n} C_i^{M_\varphi} \int_{T_i(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\bar{g}\ \zeta^{(M_\varphi)}(t)\chi_\varphi^{(M_\varphi)}(t)\nu_{T_i}^G(t)\, dt$$

But $\int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\bar{g}$ is invariant under the action on t by the Weyl group $\Omega_i^G(\mathbb{R})$ of $T_i(\mathbb{R})$ in $G(\mathbb{R})$. Thus we have

$$\text{trace } \pi_\varphi^G(f) =$$

$$\sum_{i=1}^{n} \frac{1}{C_i^{M_\varphi} C_i^G} \int_{T_i(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\bar{g} \sum_{\omega \in \Omega_i^G(\mathbb{R})} \zeta^{(M_\varphi)}(\omega t)\chi_\varphi^{(M_\varphi)}(\omega t)\nu_{T_i}^G(t)dt$$

where $C_i^G = |\Omega_i^G(\mathbb{R})|$.

Suppose now that $x \in G(\mathbb{R})$ is such that $xT_i(\mathbb{R})x^{-1} = T_j(\mathbb{R})$. Then there is a unique bijection $\omega \longrightarrow \omega'$ between $\Omega_i^G(\mathbb{R})$ and $\Omega_j^G(\mathbb{R})$ satisfying $\omega'(xtx^{-1}) = x(\omega t)x^{-1}$, $t \in T_i(\mathbb{R})$. A computation shows that

$$\int_{T_j(\mathbb{R})} \int_{T_j(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\overline{g} \sum_{\omega' \in \Omega_j^G(\mathbb{R})} \zeta^{(M_\varphi)}(\omega' t)\chi_\varphi^{(M_\varphi)}(\omega' t)\ \nu_j^G(t)\ dt$$

$$= \int_{T_i(\mathbb{R})} \int_{T_i(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}tg)d\overline{g} \sum_{\omega \in \Omega_i^G(\mathbb{R})} \zeta^{(M_\varphi)}(x(\omega t)x^{-1})\chi_\varphi^{(M_\varphi)}(x(\omega t)x^{-1})\nu_i^G(t)dt$$

For each Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ we define

$$\Omega_T^{G/M_\varphi}(\mathbb{R}) = \{x \in G(\mathbb{R}) : xT(\mathbb{R})x^{-1} \subseteq M_\varphi(\mathbb{R})\}/M_\varphi(\mathbb{R}) .$$

If $\Omega_T^{G/M_\varphi}(\mathbb{R})$ is nonempty we set

$$\chi_{\varphi, T}^{(G)}(t) = \sum_{\omega \in \Omega_T^{G/M_\varphi}(\mathbb{R})} \zeta^{(M_\varphi)}(\omega t)\chi_\varphi^{(M_\varphi)}(\omega t), \quad t \in T(\mathbb{R})_{reg} .$$

If $\Omega^{G/M_\varphi}(\mathbb{R})$ is empty we define $\chi_{\varphi, T}^{(G)} \equiv 0$ on $T(\mathbb{R})_{reg}$. We note that if $T(\mathbb{R}) \not\subseteq M_\varphi(\mathbb{R})^{G(\mathbb{R})}$ then $T(\mathbb{R})_{reg} \cap M_\varphi(\mathbb{R})^{G(\mathbb{R})}$ is empty so that $\chi_{\varphi, T}^{(G)} \equiv 0$. Finally, if g is a regular element in $G(\mathbb{R})$ we define

$$\Psi_\varphi^{(G)}(g) = \chi_{\varphi, T_g}^{(G)}(g)$$

where $T_g(\mathbb{R})$ is the (unique) Cartan subgroup of $G(\mathbb{R})$ containing g; if g is singular we define $\Psi_\varphi^{(G)}(g) = 0$.

Returning to our formula for trace $\pi_\varphi^G(f)$ we conclude (cf. [10, Lemma 91]) that

$$\text{trace } \pi_\varphi^G(f) = \int_{G(\mathbb{R})} f(g)\Psi_\varphi^{(G)}(g)dg$$

We thus identify $\chi_{\varphi}^{(G)}$, the character of π_{φ}^{G}, with $\Psi_{\varphi}^{(G)}$; $\chi_{\varphi}^{(G)}$ is then a locally L^1 class function on $G(\mathbb{R})$ vanishing outside the set of regular elements. Furthermore

Lemma 4.1.

If $\varphi \in \Phi_o(G)$ *then*

$$\chi_{\varphi}^{(G)}(g) = \sum_{\omega \in \Omega_{T_g}^{G/M_{\varphi}(\mathbb{R})}} \zeta^{(M_{\varphi})}(\omega g)\chi_{\varphi}^{(M_{\varphi})}(\omega g)$$

for each regular element g *of* $G(\mathbb{R})$.

§5. Some Cartan Subgroups.

We now introduce a family $\mathcal{J}_{\mu}(G)$ of Cartan subgroups of $G(\mathbb{R})$. First, we recall that a Cartan subgroup of $G(\mathbb{R})$ is the centralizer in $G(\mathbb{R})$ of a Cartan subalgebra of the Lie algebra $g(\mathbb{R})$ of $G(\mathbb{R})$. The following result is well-known.

Lemma 5.1.

The group of $\mathbb{R}$-*rational points on a maximal torus in* G *defined over* $\mathbb{R}$, *is a Cartan subgroup of* $G(\mathbb{R})$. *Moreover all Cartan subgroups of* $G(\mathbb{R})$ *are of this form.*

Proof: Let $h(\mathbb{R})$ be a Cartan subalgebra of $g(\mathbb{R})$. Then by [4, Lemma 4] the centralizer of $h(\mathbb{R})$ in $G(\mathbb{C})$ is both connected and abelian. The proof now follows.

We recall that $\mu : G \longrightarrow H$ is an isomorphism defined over $\mathbb{C}$ such that $\sigma(\mu)\mu^{-1}$ is inner for $\sigma \in \mathcal{G}(\mathbb{C}/\mathbb{R})$. We define the family $\mathcal{J}_\mu(G)$ as follows. Let T_G be a maximal torus in G, defined over $\mathbb{R}$. Then $T_G(\mathbb{R})$ belongs to $\mathcal{J}_\mu(G)$ if and only if there exists an element x of $H(\mathbb{C})$ such that the restriction of $\mathrm{ad}\, x \circ \mu$ to T_G is defined over $\mathbb{R}$. We remark that $T_\mu = x\mu(T_G)x^{-1}$ is then defined over $\mathbb{R}$ and moreover the restriction of $\mathrm{ad}\, x \circ \mu$ to $T_G(\mathbb{R})$ is an isomorphism of $T_G(\mathbb{R})$ with the Cartan subgroup $T_H(\mathbb{R})$ of $H(\mathbb{R})$.

Lemma 2.1. of [18] shows that if H is quasisplit then $\mathcal{J}_\mu(G)$ contains all Cartan subgroups of $G(\mathbb{R})$. In general, $\mathcal{J}_\mu(G)$ is non-empty; we indicate its structure in an appendix (§8).

Suppose now that $T_G(\mathbb{R}) \in \mathcal{J}_\mu(G)$. Suppose that $x_1, x_2 \in H(\mathbb{C})$ and that both $\mathrm{ad}\, x_1 \circ \mu$ and $\mathrm{ad}\, x_2 \circ \mu$, when restricted to T_G, are defined over $\mathbb{R}$. Then the restriction of

$$\mathrm{ad}(x_2 \;\; x_1^{-1}) = (\mathrm{ad}\, x_2 \circ \mu)(\mathrm{ad}\, x_1 \circ \mu)^{-1}$$

to $x_1\mu(T_G)x_1^{-1}$ is also defined over $\mathbb{R}$. This suggests the following definition.

Let T be a maximal torus in G defined over $\mathbb{R}$. Then we define $\mathfrak{A}(T)$ to be the set of all elements x of $G(\mathbb{C})$ for which the restriction of $\mathrm{ad}\, x$ to T is defined over $\mathbb{R}$. We will need to know the structure of $\mathfrak{A}(T)$. Thus;

<u>Lemma</u> 5.2.

<u>Let</u> S <u>be the maximal</u> $\mathbb{R}$-<u>split torus in</u> T. <u>Let</u> M <u>be the</u>

centralizer of S in G and $N_{M(\mathbb{C})}(T(\mathbb{C}))$ the normalizer of $T(\mathbb{C})$ in $M(\mathbb{C})$. Then

$$\mathfrak{A}(T) = G(\mathbb{R})\, N_{M(\mathbb{C})}(T(\mathbb{C})) .$$

Proof: Suppose that $x \in \mathfrak{A}(T)$. Then it follows that $\sigma(x\sigma(t)x^{-1}) = xtx^{-1}$, $t \in T(\mathbb{C})$, where σ, the nontrivial element of $\mathcal{A}(\mathbb{C}/\mathbb{R})$, acts on $G(\mathbb{C})$ in the usual manner. Therefore $x^{-1}\sigma(x)$ centralizes $T(\mathbb{C})$ and so belongs to $T(\mathbb{C})$. Let P be a parabolic subgroup of G, defined over $\mathbb{R}$ and containing S as a maximal $\mathbb{R}$-split torus in its radical (cf. [1, 4.15.]). Then

$$\sigma(xPx^{-1}) = x(x^{-1}\sigma(x)P\sigma(x)^{-1}x)x^{-1} = xPx^{-1}$$

since T is contained in P so that xPx^{-1} is defined over $\mathbb{R}$. From [1, 4.15.] it follows that P and xPx^{-1} are conjugate under $G(\mathbb{R})$. Thus let $y \in G(\mathbb{R})x$ be such that y normalizes P and so lies in P. We recall that M, the centralizer of S in G, is a Levi subgroup of P defined over $\mathbb{R}$, [1, 4.16]. But $\sigma(yMy^{-1}) = yMy^{-1}$ since T is contained in M. Therefore the Levi subgroup yMy^{-1} is also defined over $\mathbb{R}$ and so is conjugate to M under the group $N(\mathbb{R})$ of $\mathbb{R}$-rational points on the unipotent radical N of P (cf. [1, 0.8.]). We may then choose $z \in N(\mathbb{R})y \subseteq G(\mathbb{R})x \cap P$ such that z normalizes M; z must lie in M and $\operatorname{ad} z : T \longrightarrow zTz^{-1}$ is defined over $\mathbb{R}$. In particular, zTz^{-1} is defined over $\mathbb{R}$. Let M_1 be the derived group of M and Z_M be the connected component of the identity in the center of M. Then

$M = Z_M M_1$, $T_1 = Z_M (T \cap M_1)$ and $zTz^{-1} = Z_M z(T \cap M_1)z^{-1}$; $T \cap M_1$ and $z(T \cap M_1)z^{-1}$ are maximal tori in M_1 defined over $\mathbb{R}$. Moreover both $T \cap M_1$ and $z(T \cap M_1)z^{-1}$ are anisotropic over $\mathbb{R}$. This implies that T and zTz^{-1} are conjugate under $M_1(\mathbb{R})$ so that $z \in M_1(\mathbb{R})\, N_{M(\mathbb{C})}(T(\mathbb{C}))$. We conclude that

$$\mathfrak{A}(T) \subseteq G(\mathbb{R})\, N_{M(\mathbb{C})}(T(\mathbb{C})) .$$

Let $T_1 = T \cap M_1$. Then to complete the proof it is sufficient to show that if $x \in N_{M_1(\mathbb{C})}(T_1(\mathbb{C}))$ then the restriction of $\operatorname{ad} x$ to T_1 is defined over $\mathbb{R}$. This is a consequence of the following:

<u>Proposition</u> 5.3.

<u>Suppose that</u> T <u>is a torus defined over</u> $\mathbb{R}$. <u>Suppose moreover that</u> T <u>is anisotropic over</u> $\mathbb{R}$. <u>Then every (rational) automorphism of</u> T <u>is defined over</u> $\mathbb{R}$.

<u>Proof</u>: Suppose that ϕ is a rational automorphism of T. Then there is a unique automorphism $\phi^\wedge$ of the group L of rational characters on T which satisfies

$$\langle \phi^\wedge \lambda, t \rangle = \langle \lambda, \phi t \rangle, \quad \lambda \in L, \quad t \in T .$$

On the other hand the action of σ, the nontrivial element of $\mathfrak{G}(\mathbb{C}/\mathbb{R})$, on L is given by $\sigma\lambda = -\lambda$, $\lambda \in L$. A computation then shows that $\sigma(\phi)^\wedge = \phi^\wedge$. This implies that ϕ is defined over $\mathbb{R}$, as required. The proof of Lemma 5.2. is thus complete.

§6. A property of $\chi_{\varphi}^{(G)}$

Associated to the sets $\mathfrak{A}(T)$ of the last section is the notion of a stable distribution on $G(\mathbb{R})$. For our purposes the following definition will be adequate. Thus, a (locally L^1) function f on $G(\mathbb{R})$ is said to be stable if for each Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$

$$f(xtx^{-1}) = f(t), \quad t \in T(\mathbb{R}), \quad x \in \mathfrak{A}(T),$$

(whenever $f(xtx^{-1})$ and $f(t)$ are defined).

To formulate our character relations we will need;

Lemma 6.1.

If φ is an essentially tempered class in $\Phi(G)$ then $\chi_{\varphi}^{(G)}$ is stable.

Proof: We first prove the assertion for discrete classes. Thus, suppose that $\varphi \in \Phi_d(G)$ and that T is a maximal torus in G, defined over $\mathbb{R}$. We recall that $G^{\sim}$ is the simply-connected covering group of G_{der} and $p : G^{\sim} \longrightarrow G_{der}$ is the natural projection. Let $T^{\sim}$ be the inverse image under p of $T_{der} = T \cap G_{der}$. Then $T^{\sim}$ is a maximal torus in $G^{\sim}$, defined over $\mathbb{R}$. Moreover, if $x \in \mathfrak{A}(T)$ then $p^{-1}(x) \in \mathfrak{A}(T^{\sim})$. This implies, in particular, that

$$xT_{der}^{(o)}(\mathbb{R})x^{-1} \subseteq G_{der}^{o}(\mathbb{R})$$

for each $x \in \mathfrak{A}(T)$. Thus, if we write

$$\Theta^{*}_{(\Lambda_{\varphi})} = \sum_{\omega \in \Omega_o(\mathbb{R}) \backslash \Omega(\mathbb{C})} \det \omega \, \Theta_{(\omega\Lambda_{\varphi})}$$

and

$$\Theta^*_{\Lambda^\sim_\varphi} = \sum_{\omega^\sim \in \Omega^\sim(\mathbb{R})\backslash\Omega^\sim(\mathbb{C})} \det\omega^\sim \, \Theta_{\omega^\sim\Lambda^\sim_\varphi}$$

(using the notation of §3) then it is sufficient to prove that

$$\Theta^*_{(\Lambda_\varphi)}(xtx^{-1}) = \Theta^*_{(\Lambda_\varphi)}(t)$$

for $t \in T^o_{der}(\mathbb{R})$ and $x \in \mathfrak{A}(T)$, or, equivalently, that

$$\Theta^*_{\Lambda^\sim_\varphi}(x^\sim t^\sim x^{\sim -1}) = \Theta^*_{\Lambda^\sim_\varphi}(t^\sim)$$

for $t^\sim \in T^\sim(\mathbb{R})$ and $x^\sim \in \mathfrak{A}(T^\sim)$. Thus we may assume that

$$G = G_{der} = G^\sim \quad (*)$$

As before let S be the maximal $\mathbb{R}$-split torus in T and M be the centralizer of S in G. Under our assumptions on G, $G(\mathbb{R})$ is connected, semisimple and acceptable in the sense of [7, §18]. It is then a consequence of [8, Lemma 62] that the restriction of $\Theta^*_{\Lambda_\varphi}$ to $T(\mathbb{R})$ is invariant under the normalizer of $T(\mathbb{C})$ in $M(\mathbb{C})$; (a direct proof of this (nontrivial) fact is indicated in [22]). Since $\Theta^*_{\Lambda_\varphi}$ is a class function on $G(\mathbb{R})$ Lemma 5.2. then implies that

$$\Theta^*_{\Lambda_\varphi}(xtx^{-1}) = \Theta^*_{\Lambda_\varphi}(t)$$

---

* By Θ_{Λ_φ} we then mean $\Theta_{\Lambda^\sim_\varphi}$; in the notation of [8] this would be $\Theta_{\Lambda_\varphi+\delta}$, $\Lambda_\varphi+\delta$ now being well-defined.

for $t \in T(\mathbb{R})$ and $x \in \mathfrak{A}(T)$. We thus conclude that $\chi_\varphi^{(G)}$ is stable when φ is discrete.

Suppose now that φ is any class in $\Phi_o(G)$ and that T, S and M are as above. Our formula for $\chi_\varphi^{(G)}$ shows that we need consider only those T which are contained in the group M_φ of §2. Again, by Lemma 5.2., we need consider only those elements of $\mathfrak{A}(T)$ which lie in the normalizer $N_{M(\mathbb{C})}(T(\mathbb{C}))$ of $T(\mathbb{C})$ in $M(\mathbb{C})$.

Thus suppose that T is contained in M_φ and that $x \in N_{M(\mathbb{C})}(T(\mathbb{C}))$. We recall that

$$\Omega_T^{G/M_\varphi}(\mathbb{R}) = \{x \in G(\mathbb{R}) : xT(\mathbb{R})x^{-1} \subseteq M_\varphi(\mathbb{R})\}/M_\varphi(\mathbb{R}) .$$

Suppose that $\omega \in \Omega_T^{G/M_\varphi}(\mathbb{R})$; we choose a representative w for ω in $G(\mathbb{R})$ and define $T_1 = wTw^{-1}$, $S_1 = wSw^{-1}$, $M_1 = wMw^{-1}$ and $x_1 = wxw^{-1}$. Then $x_1 \in N_{M_1(\mathbb{C})}(T_1(\mathbb{C}))$ and M_1 is just the centralizer of S_1 in M_φ. From the first part of the proof it follows that

$$\chi_\varphi^{(M_\varphi)}(x_1 t_1 x_1^{-1}) = \chi_\varphi^{(M_\varphi)}(t_1), \quad t_1 \in T_1(\mathbb{R})$$

(in the notation of §4). Therefore if $t \in T(\mathbb{R})$ then $\chi_\varphi^{(M_\varphi)}(\omega(xtx^{-1})) = \chi_\varphi^{(M_\varphi)}(wxtx^{-1}w^{-1}) = \chi_\varphi^{(M_\varphi)}(x_1 wtw^{-1}x_1^{-1}) = \chi_\varphi^{(M_\varphi)}(wtw^{-1}) = \chi_\varphi^{(M_\varphi)}(\omega t)$. Moreover $\zeta^{(M_\varphi)}(\omega(xtx^{-1})) = \zeta^{(M_\varphi)}(\omega t)$. Our formula for $\chi_\varphi^{(G)}$ (Lemma 4.1) then implies that

$$\chi_\varphi^{(G)}(xtx^{-1}) = \chi_\varphi^{(G)}(t), \quad t \in T(\mathbb{R}) .$$

Therefore $\chi_\varphi^{(G)}$ is stable and so the proof of the lemma is complete.

§7. Character Relations

We may now prove the desired character relations. Thus, suppose that T is a maximal torus in G, defined over $\mathbb{R}$. Suppose that $x, x' \in H(\mathbb{C})$ and that the restrictions to T of both $\operatorname{ad} x \circ \mu$ and $\operatorname{ad} x' \circ \mu$ are defined over $\mathbb{R}$. We write μ_T, μ'_T for the restriction to $T(\mathbb{R})$ of $\operatorname{ad} x \circ \mu$, $\operatorname{ad} x' \circ \mu$ respectively. Then, summarizing the results of §§5 and 6, we have:

Theorem 7.1.

Suppose that φ is an essentially tempered class in $\Phi(H)$. Then

$$\chi_{\varphi}^{(H)} \circ \mu_T = \chi_{\varphi}^{(H)} \circ \mu'_T .$$

Proof: We have $\mu'_T = \operatorname{ad} y \circ \mu_T$ where $y = x'x^{-1} \in \mathfrak{A}(x\mu(T)x^{-1})$. Therefore if $t \in T(\mathbb{R})$ then

$$\chi_{\varphi}^{(H)}(\mu'_T(t)) = \chi_{\varphi}^{(H)}(y\mu_T(t)y^{-1}) = \chi_{\varphi}^{(H)}(\mu_T(t))$$

by Lemma 6.1., so that the theorem is proved.

We recall that $\mathcal{J}_{\mu}(G)$ is the set of Cartan subgroups $T(\mathbb{R})$ of $G(\mathbb{R})$ for which μ_T is well-defined. Our main result is then:

Theorem 7.2.

Suppose that φ is an essentially tempered class in $\Phi_{\mu}(G)$. Then

$$\chi_{\varphi}^{(G)} = (-1)^{q_G - q_H} \chi_{\mu^{\wedge}(\varphi)}^{(H)} \circ \mu_T$$

whenever $T(\mathbb{R}) \in \mathcal{J}_{\mu}(G)$.

Here we define q_G, q_H as in §3.

Proof: We first observe that it is sufficient to prove the theorem for the case $H = G_1$ (G_1 was defined in §2). For then we have, in general, that

$$\chi_{\varphi}^{(G)} = (-1)^{q_G - q_{G_1}} \chi_{\lambda^\wedge(\varphi)}^{(G_1)} \circ \lambda_{T_G}$$

on each Cartan subgroup $T_G(\mathbb{R})$ of $G(\mathbb{R})$, and

$$\chi_{\mu^\wedge(\varphi)}^{(H)} = (-1)^{q_H - q_{G_1}} \chi_{\nu^\wedge\mu^\wedge(\varphi)}^{(G_1)} \circ \nu_{T_H}$$

on each Cartan subgroup $T_H(\mathbb{R})$ of $H(\mathbb{R})$, where $\nu = \lambda \circ \mu^{-1}$. Thus, if $T_G(\mathbb{R}) \in \mathcal{J}_\mu(G)$ and $T_H = \mu_{T_G}(T_G)$ for some choice of μ_{T_G}, we have

$$\chi_{\mu^\wedge(\varphi)}^{(H)} \circ \mu_{T_G} = (-1)^{q_H - q_{G_1}} \chi_{\nu^\wedge(\varphi)}^{(G_1)} \circ \nu_{T_H} \circ \mu_{T_G}$$

on $T_G(\mathbb{R})$. But $\nu_{T_H} \circ \mu_{T_G}$ is of the form $\operatorname{ad} x \circ \lambda$, $x \in G_1(\mathbb{C})$, and is defined over $\mathbb{R}$. Therefore, by Theorem 7.1.,

$$\chi_{\mu^\wedge(\varphi)}^{(H)} \circ \mu_{T_G} = (-1)^{q_H - q_{G_1}} \chi_{\mu^\wedge\nu^\wedge(\varphi)}^{(G_1)} \circ \lambda_{T_G}$$

on $T_G(\mathbb{R})$. But $\mu^\wedge \nu^\wedge = \lambda^\wedge$ (by construction) and so it follows that the theorem is true in general.

Suppose now that $\varphi \in \Phi_d(G)$. Then $\mu^\wedge(\varphi) \in \Phi_d(G_1)$ and we write S for T_φ^G and S_1 for $T_{\mu^\wedge(\varphi)}^{G_1}$. We have associated to φ an orbit $X_\varphi(G, S)$ in the group of quasicharacters on $S(\mathbb{R})$ under the action of $\Omega(\mathbb{C}) = N_{G(\mathbb{C})}(S(\mathbb{C}))/S(\mathbb{C})$ and also an orbit $X_{\mu^\wedge(\varphi)}(G_1, S_1)$

in the group of quasicharacters on $S_1(\mathbb{R})$ under the action of $\Omega_1(\mathbb{C}) = N_{G_1(\mathbb{C})}(S_1(\mathbb{C}))/S_1(\mathbb{C})$. Moreover we have chosen the spaces $\Phi(G)$ and $\Phi(G_1)$ (cf. §2) so that

$$X_{\mu^\wedge(\varphi)}(G_1, S_1) = \{\Lambda \circ \mu_\varphi^{-1} : \Lambda \in X_\varphi(G, S)\} .$$

We let $s(\mathbb{C})$ and $s_1(\mathbb{C})$ be the Lie algebras of $S(\mathbb{C})$ and $S_1(\mathbb{C})$ respectively and fix a system P_S of positive roots for $s(\mathbb{C})$ in $g(\mathbb{C})$; the isomorphism μ_φ then determines a system P_{S_1} of positive roots for $s_1(\mathbb{C})$ in $g_1(\mathbb{C})$. We choose an element Λ of $X_\varphi(G, S)$ which lies in the closure of the dominant Weyl chamber for P_S (in the sense of §3) and set $\Lambda_1 = \Lambda \circ \mu_\varphi^{-1}$; Λ_1 then lies in the closure of the dominant Weyl chamber for P_{S_1}.

Suppose now that T is a maximal torus in G, defined over $\mathbb{R}$. As usual, we choose $x \in G_1(\mathbb{C})$ such that the restriction of $\mathrm{ad}\, x \circ \mu$ to T is defined over $\mathbb{R}$ and write μ_T for the restriction of $\mathrm{ad}\, x \circ \mu$ to $T(\mathbb{R})$. We recall that $G^\sim$ is the simply-connected covering group of G and that $p : G^\sim \longrightarrow G_{der}$ is the natural projection; $G_1^\sim$ and p_1 are the corresponding objects for G_1. Let $\mu^\sim$ be the (unique) isomorphism of $G^\sim$ with $G_1^\sim$ which satisfies $\mu \circ p = p_1 \circ \mu^\sim$. Then $\sigma(\mu^\sim)\mu^{\sim -1}$ is inner for $\sigma \in \mathcal{G}(\mathbb{C}/\mathbb{R})$ and moreover, if $x^\sim \in p^{-1}(x)$ then the restriction of $\mathrm{ad}\, x^\sim \circ \mu^\sim$ to $T^\sim = p^{-1}(T \cap G_{der})$ is defined over $\mathbb{R}$. This implies then that $\mu_T(T \cap G^o_{der}(\mathbb{R})) = T_1 \cap G^o_{1_{der}}(\mathbb{R})$. We note also that

$$\mu_T(Z_G(\mathbb{R})) = Z_{G_1}(\mathbb{R})$$

since the restriction of μ_T to Z_G is defined over $\mathbb{R}$. It then follows that to prove the theorem (for discrete classes φ) it is enough to show that

$$\Theta^*_{(\Lambda)} = \Theta^*_{(\Lambda_1)} \circ \mu_T$$

on $T^{(o)}_{der}(\mathbb{R}) = T \cap G^o_{der}(\mathbb{R})$ or, equivalently, that

$$\Theta^*_{\Lambda^\sim} = \Theta^*_{\Lambda_1^\sim} \circ \mu_{T^\sim}^\sim$$

on $T^\sim(\mathbb{R})$ (in the notation of p. 32) Thus we may assume, without loss of generality, that $G = G_{der} = G^\sim$ (cf. footnote p. 32); then also $G_1 = G_{1_{der}} = G_1^\sim$.

Under these assumptions we define a function Ψ_Λ on $G(\mathbb{R})$ by

$$\Psi_\Lambda(g) = \Theta^*_{\Lambda_1} \circ \mu_{T_g}(g)$$

if g is regular and $T_g(\mathbb{R})$ is the (unique) Cartan subgroup of $G(\mathbb{R})$ containing g, and by

$$\Psi_\Lambda(g) = 0$$

if g is singular. We have then to show that $\Psi_\Lambda = \Theta^*_\Lambda$.

We remark first that Ψ_Λ is stable and so, in particular, Ψ_Λ is invariant (under $G(\mathbb{R})$); this is an immediate consequence of Theorem 7.1..

We will need some more notation. Thus, suppose that $T(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$ and that P_T is a set of positive roots for $t(\mathbb{C})$ in $g(\mathbb{C})$. We set $\delta = \frac{1}{2} \sum_{\alpha \in P_T} \alpha$ and $\Delta_T(\exp H) = e^{\delta(H)} \prod_{\alpha \in P_T} (1-e^{-\alpha(H)})$, $H \in t(\mathbb{C})$ (recalling that $G(\mathbb{C})$ is simply-connected). Suppose again that $x \in G_1(\mathbb{C})$ and that the restriction of $\mu_x = \text{ad}\, x \circ \mu$ to T is defined over $\mathbb{R}$; we set $T_1 = \mu_x(T)$. Then μ_x and P_T determine a system P_{T_1} of positive roots for $t_1(\mathbb{C})$ in $g_1(\mathbb{C})$ and if we set $\delta_1 = \frac{1}{2} \sum_{\alpha_1 \in P_{T_1}} \alpha_1$ and

$$\Delta_{T_1}(\exp H) = e^{\delta_1(H)} \prod_{\alpha_1 \in P_{T_1}} (1-e^{-\alpha_1(H)}), \quad H \in t_1(\mathbb{C})$$

then

$$\Delta_T = \Delta_{T_1} \circ \mu_x .$$

For $g \in G(\mathbb{R})$ (or $G_1(\mathbb{R})$) we define $D(g)$ to be the coefficient of t^ℓ in $\det(t+1-\text{Ad}g)$ where ℓ is the rank of $G(\mathbb{R})$. The set $G(\mathbb{R})_{reg}$ of regular elements in $G(\mathbb{R})$ then consists of those elements g for which $D(g) \neq 0$. It is clear that

$$D(t) = D(\mu_x(t)), \quad t \in T(\mathbb{R}) .$$

Next, if $f \in C^\infty(T(\mathbb{R}))$ (or $C^\infty(T(\mathbb{R})_{reg})$) we define $\mu_x f \in C^\infty(T_1(\mathbb{R}))$ (or $C^\infty(T_1(\mathbb{R})_{reg})$) by $\mu_x f(t) = f(\mu_x^{-1} t)$, $t \in T_1(\mathbb{R})$. Let $\mathcal{U}_T$ be the universal enveloping algebra of $t(\mathbb{C})$ and $\mathfrak{Z}_T$ be the subalgebra of elements invariant under the Weyl group of $t(\mathbb{C})$ in $g(\mathbb{C})$; we regard the elements of $\mathcal{U}_T$ as differential operators on $T(\mathbb{R})$ (or $T(\mathbb{R})_{reg}$). The isomorphism μ_x then determines a unique isomorphism of $\mathcal{U}_T$ with $\mathcal{U}_{T_1}$, again denoted μ_x, which satisfies

$$\mu_x(Df) = (\mu_x D)(\mu_x f)$$

for $D \in \mathcal{U}_T$ and $f \in C^{\infty}(T(\mathbb{R}))$.

We identify the algebra $\mathfrak{Z}_G$ of left and right invariant differential operators on $G(\mathbb{R})$ with the center of the universal enveloping algebra of $g(\mathbb{C})$; the isomorphism μ_x then determines an isomorphism of $\mathfrak{Z}_G$ with $\mathfrak{Z}_{G_1}$ again denoted μ_x. We may then show that the isomorphisms $\gamma_T : \mathfrak{Z}_G \longrightarrow \mathfrak{I}_T$ and $\gamma_{T_1} : \mathfrak{Z}_{G_1} \longrightarrow \mathfrak{I}_{T_1}$ of [4, Lemma 19] satisfy

$$\mu_x \circ \gamma_T = \gamma_{T_1} \circ \mu_x$$

We recall that associated to the character Λ on $S(\mathbb{R})$ is a character χ_Λ on $\mathfrak{Z}_G$ which satisfies

$$\chi_\Lambda(z) = \Lambda(\gamma_S(z)), \quad z \in \mathfrak{Z}_G$$

where Λ is extended to $\mathfrak{I}_S$ in the natural manner ([8, §19]). It is clear that

$$\chi_\Lambda = \chi_{\Lambda_1} \circ \mu_\varphi$$

We define $\varpi_T \in \mathcal{U}_T$ by $\varpi_T = \prod_{\alpha \in P_T} H_\alpha$ where H_α is the coroot associated to the root α. If $\varpi_{T_1} = \prod_{\alpha_1 \in P_{T_1}} H_{\alpha_1}$ then $\varpi_{T_1} = \mu_x \varpi_T$. We remark that while ϖ_T and Δ_T depend on the choice of P_T the function $\varpi_T(\Delta_T \Psi_\Lambda)$, in which we will be interested, is independent of P_T.

Finally, let $P_T^{\mathbb{R}}$ be the set of real roots ([6, §4]) in P_T and set

$$\Delta_T^{\mathbb{R}}(\exp H) = \prod_{\alpha \in P_T^{\mathbb{R}}} (e^{\alpha(H)} - 1), \quad H \in t(\mathbb{C}).$$

Then we define

$$T(\mathbb{R})^{\mathbb{R}} = \{t \in T(\mathbb{R}) : \Delta_T^{\mathbb{R}}(t) \neq 0\} .$$

If we define $T_1(\mathbb{R})^{\mathbb{R}}$ similarly, then

$$\mu_x(T(\mathbb{R})^{\mathbb{R}}) = T_1(\mathbb{R})^{\mathbb{R}} .$$

We now continue the proof of Theorem 7.2.. Thus

Proposition 7.3.

The restriction of Ψ_Λ to $G(\mathbb{R})_{reg}$ is analytic and an eigenfunction for $\mathfrak{Z}_G$.

Proof: Let $g \in G(\mathbb{R})_{reg}$ and suppose that $T(\mathbb{R})$ is the Cartan subgroup of $G(\mathbb{R})$ containing g. Then we may choose a neighborhood N_g of g in $G(\mathbb{R})_{reg}$ which is diffeomorphic to a neighborhood in $T(\mathbb{R})_{reg} \times T(\mathbb{R})\backslash G(\mathbb{R})$ under the map $(g, \bar{x}) \longrightarrow x^{-1}gx$. It is clear that Ψ_Λ is analytic on N_g.

Suppose now that $z \in \mathfrak{Z}_G$. Then by [4, Theorem 2]

$$z\Psi_\Lambda = (\Delta_T^{-\frac{1}{2}} \circ \gamma(z) \circ \Delta_T^{\frac{1}{2}}) \Psi_\Lambda$$

on $T(\mathbb{R})_{reg}$. We recall that

$$z_1 \Theta_{\Lambda_1} = \chi_{\Lambda_1}(z_1)\Theta_{\Lambda_1}, \quad z_1 \in \mathfrak{Z}_{G_1} .$$

A computation then shows that

$$z\Psi_\Lambda = \chi_\Lambda(z)\Psi_\Lambda$$

on $T(\mathbb{R})_{reg}$ and so also on $G(\mathbb{R})_{reg}$. This proves the proposition.

From [4, Theorem 4] and [7, Lemma 53] we conclude that Ψ_Λ is locally L^1 on $G(\mathbb{R})$.

Regarding Ψ_Λ as a distribution on $G(\mathbb{R})$ we next prove:

Lemma 7.4.

Ψ_Λ is an eigendistribution for $\mathfrak{Z}_G$.

Proof: We have to show that

$$\int_{G(\mathbb{R})} (\Psi_\Lambda(g) z^* f(g) - f(g) z \Psi_\Lambda(g))\, dg = 0$$

for $f \in C_c^\infty(G(\mathbb{R}))$ and $z \in \mathfrak{Z}_G$ where z^* denotes the adjoint of z. Because the character $\Lambda^\sim$ $(= \Lambda+\delta)$ is regular we may apply [22; 2, Theorem 2], (cf. [14]). Concerning the proof of this theorem we refer to the comments in [22] and omit the details. Lemma 7.4. is then a consequence of:

Lemma 7.5.

Let $T(\mathbb{R})$ be a Cartan subgroup of $G(\mathbb{R})$. Then

(i) $\Delta_T \Psi_\Lambda$ extends to an analytic function on $T(\mathbb{R})^{\mathbb{R}}$,

(ii) $\varpi_T(\Delta_T \Psi_\Lambda)$ extends to a continuous function on $T(\mathbb{R})$ and

if $T'(\mathbb{R})$ is also a Cartan subgroup of $G(\mathbb{R})$ then

(iii) $\varpi_{T'}(\Delta_{T'} \Psi_\Lambda) = \varpi_T(\Delta_T \Psi_\Lambda)$ on $T(\mathbb{R}) \cap T'(\mathbb{R})$.

Proof: We note that

$$\Delta_T \Psi_\Lambda = (\Delta_{T_1} \Theta^*_{\Lambda_1}) \circ \mu_x$$

and

$$\varpi_T(\Delta_T \Psi_\Lambda) = (\varpi_{T_1}(\Delta_{T_1} \Theta^*_{\Lambda_1})) \circ \mu_x$$

on $T(\mathbb{R})_{reg}$; (i) and (ii) then follow immediately because $\Theta^*_{\Lambda_1}$ satisfies the corresponding conditions on $T_1(\mathbb{R})$, [7, Lemmas 31, 34]. We will use the same symbols to denote the extensions of the functions in (i) and (ii).

For (iii) we remark first that if $y \in \mathfrak{A}(T_1)$ and $T_1^y = yT_1y^{-1}$ then

$$\varpi_{T_1^y}(\Delta_{T_1^y}\Theta^*_{\Lambda_1})(yty^{-1}) = \varpi_{T_1}(\Delta_{T_1}\Theta^*_{\Lambda_1})(t)$$

for $t \in T_1(\mathbb{R})$.

If $T = T'$ then we interpret (iii) as stating that $\varpi_T(\Delta_T\Psi_\Lambda)$ is independent of the choice of P_T, which is so because $\varpi_{T_1}(\Delta_{T_1}\Theta^*_{\Lambda_1})$ is independent of the choice of P_{T_1}.

Suppose now that both $T(\mathbb{R})$ and $T'(\mathbb{R})$ are compact. Let $t \in T(\mathbb{R}) \cap T'(\mathbb{R})$. Then because t is semisimple C_t, the connected component of identity of the centralizer of t in G, is reductive. Moreover both T and T' are contained in C_t. Therefore since both $T(\mathbb{R})$ and $T'(\mathbb{R})$ are compact there exists $g \in C_t(\mathbb{R})$ such that $T' = gTg^{-1}$. But

$$\mu_{x'} \circ \operatorname{ad} g = \operatorname{ad} y \circ \mu_x$$

for some $y \in \mathfrak{A}(T_1)$ and so, in particular,

$$\mu_{x'}(t) = y\mu_x(t)y^{-1}.$$

Therefore

$$\varpi_{T'}(\Delta_{T'}\Psi_{\Lambda})(t) = \varpi_{T_1'}(\Delta_{T_1'}\Theta^*_{\Lambda_1})(\mu_{x'}(t))$$

$$= \varpi_{T_1}(\Delta_{T_1}\Theta^*_{\Lambda_1})(\mu_x(t)) = \varpi_T(\Delta_T\Psi_{\Lambda})(t) .$$

Thus we have verified (iii) when both $T(\mathbb{R})$ and $T'(\mathbb{R})$ are compact.

To complete the proof of (iii) we now need only consider the case where $T(\mathbb{R}) \cap T'(\mathbb{R})$ contains a semiregular element of noncompact type [6, §3]. For then (iii) follows in general by a well-known inductive argument (cf. [22, § 3.6.]).

Thus suppose that $T(\mathbb{R}) \cap T'(\mathbb{R})$ contains a semiregular element t of noncompact type. We remark that the set of such elements in $T(\mathbb{R}) \cap T'(\mathbb{R})$ is dense in $T(\mathbb{R}) \cap T'(\mathbb{R})$ (the proof of this proceeds as for [4, Lemma 8]) and so it is sufficient to prove that

$$\varpi_T(\Delta_T\Psi_{\Lambda})(t) = \varpi_{T'}(\Delta_{T'}\Psi_{\Lambda})(t).$$

But

$$\varpi_T(\Delta_T\Psi_{\Lambda})(t) = \varpi_{T_1}(\Delta_{T_1}\Theta^*_{\Lambda_1})(\mu_x(t)) = \varpi_{T_1^y}(\Delta_{T_1^y}\Theta^*_{\Lambda_1})(y\mu_x(t)y^{-1})$$

for $y \in \mathfrak{A}(T_1)$ and, on the other hand,

$$\varpi_{T'}(\Delta_{T'}\Psi_{\Lambda})(t) = \varpi_{T_1'}(\Delta_{T_1'}\Theta^*_{\Lambda_1})(\mu_{x'}(t))$$

Therefore, since

$$\varpi_{T_1^y}(\Delta_{T_1^y}\Theta^*_{\Lambda_1}) = \varpi_{T_1'}(\Delta_{T_1'}\Theta^*_{\Lambda_1})$$

on $T_1^y(\mathbb{R}) \cap T_1'(\mathbb{R})$ we have only to show that

$$\mu_{x'}(t) = y\mu_x(t)y^{-1}$$

for some $y \in \mathfrak{G}(T_1)$.

Thus let L be the connected component of the identity in the centralizer of t in G and $L^{(1)}$ be the connected component of the identity in the centralizer of $\mu_{x'}(t)$ in G_1. Then $\mu_{x'} : L \longrightarrow L^{(1)}$ and $\sigma(\mu_{x'})\mu_{x'}^{-1}$ is an inner automorphism of $L^{(1)}$. Moreover the element $\mu_{x'}(t)$ is semiregular and of noncompact type and so the derived group $L^{(1)}_{der}$ of $L^{(1)}$ splits over $\mathbb{R}$. Since T is contained in L it then follows from [18, Lemma 2.1.] that there exists $y' \in L_{der}(\mathbb{C})$ such that the restriction of $\mathrm{ad}\, y' \circ \mu_{x'}$ to $T \cap L_{der}$ is defined over $\mathbb{R}$. It is clear that the restriction of $\mathrm{ad}\, y' \circ \mu_{x'}$ to T is then also defined over $\mathbb{R}$. Therefore

$$\mathrm{ad}\, y' \circ \mu_{x'} = \mathrm{ad}\, y \circ \mu_x$$

for some $y \in \mathfrak{G}(T_1)$. Hence

$$\mu_{x'}(t) = \mathrm{ad}\, y' \circ \mu_{x'}(t) = \mathrm{ad}\, y \circ \mu_x(t)$$

as required. Thus (iii) is proved. The proof of Lemmas 7.4 and 7.5 is then complete.

From the characterization of $\Theta^*_{\Lambda_1}$ ([10, Theorem 3]) it now follows that

$$z\Psi_\Lambda = \chi_\Lambda(z)\Psi_\Lambda, \quad z \in \mathfrak{Z}_G .$$

Moreover

$$\sup_{x \in G(\mathbb{R})_{reg}} |D(x)|^{\frac{1}{2}} |\Psi_\Lambda(x)| < \infty$$

and

$$\Psi_\Lambda = \Delta_S^{-1} \sum_{\omega \in \Omega(\mathbb{C})} \det\omega \; \omega(\Lambda+\delta)$$

on $S(\mathbb{R})$ (here we are using δ to denote the character $\exp H \longrightarrow e^{\delta(H)}$ on $S(\mathbb{R})$).

From the characterization of Θ_Λ^* (again [10, Theorem 3]) it then follows that $\Psi_\Lambda = \Theta_\Lambda^*$. The proof of Theorem 7.2., in the case that φ is discrete, is then complete.

Suppose now that φ is any essentially tempered class in $\Phi_\mu(G)$. We recall that we have chosen subgroups P_φ^G, M_φ^G and T_φ^G of G, subgroups $P_{\mu^\wedge(\varphi)}^{G_1}$, $M_{\mu^\wedge(\varphi)}^{G_1}$ and $T_{\mu^\wedge(\varphi)}^{G_1}$ of G_1 and an isomorphism μ_φ of the form $\operatorname{ad} x_\varphi \circ \mu$, such that $\mu_\varphi : T_\varphi^G \longrightarrow T_{\mu^\wedge(\varphi)}^{G_1}$ and the restriction of μ_φ to T_φ^G is defined over $\mathbb{R}$; by construction

$$\mu_\varphi : P_\varphi^G \longrightarrow P_{\mu^\wedge(\varphi)}^{G_1} \quad \text{and} \quad \mu_\varphi : M_\varphi^G \longrightarrow M_{\mu^\wedge(\varphi)}^{G_1} .$$

Suppose now that T is a maximal torus in G, defined over $\mathbb{R}$. As usual, suppose that the restriction of $\mu_x = \operatorname{ad} x \circ \mu = \operatorname{ad} y \circ \mu_\varphi$ to T is defined over $\mathbb{R}$ and set $T_1 = \mu_x(T)$. We recall that

$$\Omega_T^{G/M_\varphi^G}(\mathbb{R}) = \{g \in G(\mathbb{R}) : gT(\mathbb{R})g^{-1} \subseteq M_\varphi^G(\mathbb{R})\}/M_\varphi^G(\mathbb{R});$$

$\Omega_{T_1}^{G_1/M_{\mu^\wedge(\varphi)}^{G_1}}(\mathbb{R})$ is then defined similarly. We will need:

<u>Lemma</u> 7.6.

<u>There is a unique bijection</u> $\omega \longrightarrow \omega_1$ <u>of</u> $\Omega_T^{G/M_\varphi^G}(\mathbb{R})$ <u>with</u>

$\Omega_{T_1}^{G_1/M^{G_1}_{\mu^\wedge(\varphi)}}(\mathbb{R})$ such that if s is a representative for ω in $G(\mathbb{R})$ then there exists a representative s_1 for ω_1 in $G_1(\mathbb{R})$ and an element x_o of $M^{G_1}_{\mu^\wedge(\varphi)}(\mathbb{C})$ such that

$$\mathrm{ad}\, s_1 \circ \mu_x = \mu_{x_0} \circ \mathrm{ad}\, s$$

where μ_{x_o} is the restriction of $\mathrm{ad}\, x_o \circ \mu_\varphi$ to sTs^{-1}.

Proof: Let $s \in G(\mathbb{R})$ and suppose that $sT(\mathbb{R})s^{-1} \subseteq M^G_\varphi(\mathbb{R})$. Then we shall define $x_o \in M^{G_1}_{\mu^\wedge(\varphi)}(\mathbb{C})$ and $s_1 \in G_1(\mathbb{R})$ such that

$$s_1 T_1(\mathbb{R}) s_1^{-1} \subseteq M^{G_1}_{\mu^\wedge(\varphi)}(\mathbb{R})$$

and

$$\mathrm{ad}\, s_1 \circ \mu_x = \mu_{x_o} \circ \mathrm{ad}\, s.$$

We have that $\mu_x \circ \mathrm{ad}\, s^{-1} : sTs^{-1} \longrightarrow T_1$ is defined over $\mathbb{R}$. Since

$$\mu_x \circ \mathrm{ad}\, s^{-1} = \mathrm{ad}\, z \circ \mu_\varphi \quad \text{where} \quad z = y\mu_\varphi(s^{-1}),$$

it follows, as in the proof of Lemma 5.2., that $P_1 = zP^{G_1}_{\mu^\wedge(\varphi)}z^{-1}$ is a parabolic subgroup of G_1, defined over $\mathbb{R}$ and that $M_1 = zM^{G_1}_{\mu^\wedge(\varphi)}z^{-1}$ is a Levi subgroup of P_1, also defined over $\mathbb{R}$. By [1, 4.15.] there exists $h \in G_1(\mathbb{R})$ such that $P_1 = hP^{G_1}_{\mu^\wedge(\varphi)}h^{-1}$ and $M_1 = hM^{G_1}_{\mu^\wedge(\varphi)}h^{-1}$. It then follows then $h^{-1}z \in M^{G_1}_{\mu^\wedge(\varphi)}$. Also

$$h^{-1}T_1 h = \mathrm{ad}\, h^{-1} \circ \mu_x(T) = \mathrm{ad}\,(h^{-1}z) \circ \mu_\varphi \circ \mathrm{ad}\, s(T)$$

and so is contained in $M^{G_1}_{\mu^\wedge(\varphi)}$. Thus we set $s_1 = h^{-1}$ and $x_o = h^{-1}z$.

Similarly for given s_1 we may define s and x_o so that the same conditions are satisfied. We observe that if $s, s' \in G(\mathbb{R})$ and $s_1, s'_1 \in G_1(\mathbb{R})$ are such that s_1 corresponds to s and s'_1 corresponds to s' then $s' \in s\, M^G_\varphi(\mathbb{R})$ if and only if $s'_1 \in s_1\, M^{G_1}_{\mu^\wedge(\varphi)}(\mathbb{R})$.

We may now define a bijection from $\Omega_T^{G/M_\varphi}(\mathbb{R})$ to $\Omega_T^{G_1/M^{G_1}_{\mu^\wedge(\varphi)}}(\mathbb{R})$ by mapping $sM^G_\varphi(\mathbb{R})$ to $s_1M^{G_1}_{\mu^\wedge(\varphi)}(\mathbb{R})$; this is the only bijection between these sets which satisfies the requirements of Lemma 7.6..

Lemma 7.6. is thus proved.

Continuing the proof of Theorem 7.2., it follows from what we have already shown that

$$\chi_\varphi^{(M^G_\varphi)} = (-1)^{q_{M^G_\varphi} - q_{M^{G_1}_{\mu^\wedge(\varphi)}}} \chi_{\mu^\wedge_\varphi(\varphi)}^{(M^{G_1}_{\mu^\wedge(\varphi)})} \circ \mu_{x_o}$$

on $T(\mathbb{R})_{reg}$. We observe that $\mu^\wedge_\varphi(\varphi) = \mu^\wedge(\varphi)$ and that:

Proposition 7.7.

$$(-1)^{q_{M^G_\varphi} - q_{M^{G_1}_\varphi}} = (-1)^{q_G - q_{G_1}}$$

Proof: We shall compute $q_G - q_{M^G_\varphi}$. By definition (p. 15)

$$q_G = \frac{1}{2}(\dim G(\mathbb{R}) - \dim K)$$

where K is a maximal compact subgroup of $G(\mathbb{R})$. We choose K so that $M^G_\varphi(\mathbb{R})$ is invariant under the Cartan involution on $G(\mathbb{R})$ determined by K. Then $K \cap M^G_\varphi(\mathbb{R})$ is a maximal compact subgroup of $M^G_\varphi(\mathbb{R})$ and

$K \cap (M_\varphi^G)^o_{der}(\mathbb{R})$ is a maximal compact subgroup of $(M_\varphi^G)^o_{der}(\mathbb{R})$, the connected component of the identity in $(M_\varphi^G)_{der}(\mathbb{R})$, ([20, §2]). Moreover $\dim(K \cap M_\varphi^G(\mathbb{R})) = \dim(K \cap (M_\varphi^G)^o_{der}(\mathbb{R})) + \dim A_\varphi^G(\mathbb{R})$, where A_φ^G is the maximal $\mathbb{R}$-anisotropic torus in the center $Z_{M_\varphi^G}$ of M_φ^G. We note also that

$$\dim M_\varphi^G(\mathbb{R}) = \dim (M_\varphi^G)^o_{der}(\mathbb{R}) + \dim Z_{M_\varphi^G}(\mathbb{R})$$

and that

$$\dim Z_{M_\varphi}^G(\mathbb{R}) = \dim A_\varphi^G(\mathbb{R}) + \dim S_\varphi^G(\mathbb{R})$$

where S_φ^G is the maximal $\mathbb{R}$-split torus in $Z_{M_\varphi^G}$. Therefore

$$q_{M_\varphi^G} = \frac{1}{2}(\dim (M_\varphi^G)^o_{der}(\mathbb{R}) - \dim (K \cap (M_\varphi^G)^o_{der}(\mathbb{R}))$$

$$= \frac{1}{2}(\dim M_\varphi^G(\mathbb{R}) - \dim (K \cap M_\varphi^G(\mathbb{R})) + \dim S_\varphi^G(\mathbb{R}))$$

On the other hand

$$G = K M_\varphi^G(\mathbb{R}) N_\varphi^G(\mathbb{R})$$

$$K \cap M_\varphi^G(\mathbb{R}) N_\varphi^G(\mathbb{R}) = K \cap M_\varphi^G(\mathbb{R})$$

and $M_\varphi^G(\mathbb{R}) \cap N_\varphi^G(\mathbb{R}) = 1$, N_φ^G being the unipotent radical of P_φ^G. Therefore

$$\dim G(\mathbb{R}) - \dim K = \dim M_\varphi^G(\mathbb{R}) - \dim K \cap M_\varphi^G(\mathbb{R})$$
$$+ \dim N_\varphi^G(\mathbb{R})$$

and so we conclude that

$$q_G - q_{M_\varphi^G} = \frac{1}{2}(\dim N_\varphi^G(\mathbb{R}) + \dim S_\varphi^G(\mathbb{R})).$$

To complete the proof of the proposition we observe that $\mu_\varphi : N_\varphi^G \longrightarrow N_{\mu^\wedge(\varphi)}^{G_1}$ so that

$$\dim N_\varphi^G(\mathbb{R}) = \dim_{\mathbb{C}} N_\varphi^G(\mathbb{C}) = \dim_{\mathbb{C}} N_{\mu^\wedge(\varphi)}^{G_1}(\mathbb{C}) = \dim N_{\mu^\wedge(\varphi)}^{G_1}(\mathbb{R}),$$

and that the restriction of μ_φ to $Z_{M_\varphi^G}$ is defined over $\mathbb{R}$ so that

$$\dim S_\varphi^G(\mathbb{R}) = \dim S_{\mu^\wedge(\varphi)}^{G_1}(\mathbb{R}) .$$

Hence $q_G - q_{M_\varphi^G} = q_{G_1} - q_{M_{\mu^\wedge(\varphi)}^{G_1}}$ and the proposition is proved.

We may now complete the proof of Theorem 7.2.. Thus suppose that $t \in T(\mathbb{R})_{reg}$. Then by Lemma 4.1. we have

$$\chi_{\mu^\wedge(\varphi)}^{(G_1)}(\mu_x(t)) = \sum_{\omega \in \Omega_{T_1}^{G_1/M_{\mu^\wedge(\varphi)}^{G_1}(\mathbb{R})}} \zeta^{(M_{\mu^\wedge(\varphi)}^{G_1})}(\omega\mu_x t)\chi_{\mu^\wedge(\varphi)}^{(M_{\mu^\wedge(\varphi)}^{G_1})}(\omega\mu_x t)$$

$$= \sum_{\omega \in \Omega_T^{G/M_\varphi^G(\mathbb{R})}} \zeta^{(M_{\mu^\wedge(\varphi)}^{G_1})}(\mu_{x_o}\omega t)\chi_{\mu^\wedge(\varphi)}^{M_{\mu^\wedge(\varphi)}^{G_1}}(\mu_{x_o}\omega t)$$

(by Lemma 7.6.)

$$= (-1)^{q_G - q_{G_1}} \sum_{\omega \in \Omega_T^{G/M_\varphi^G(\mathbb{R})}} \zeta^{(M_\varphi^G)}(\omega t) \chi_\varphi^{(M_\varphi^G)}(\omega t)$$

$$= (-1)^{q_G - q_{G_1}} \chi_\varphi^{(G)}(t)$$

Here we have used the identity

$$\zeta^{(M_{\mu^\wedge(\varphi)}^{G_1})} \circ \mu_{x_o} = \zeta^{(M_\varphi^G)}$$

which follows immediately from the definition of $\zeta^{(\;)}$. Hence

$$\chi_\varphi^{(G)}(t) = (-1)^{q_G - q_{G_1}} \chi_{\mu^\wedge(\varphi)}^{(G_1)} (\mu_x(t)) .$$

Since $\chi_\varphi^{(G)}$ and $\chi_{\mu^\wedge(\varphi)}^{(G_1)} \circ \mu_x$ both vanish on the set of singular elements in $T(\mathbb{R})$ we conclude that

$$\chi_\varphi^{(G)} = (-1)^{q_G - q_{G_1}} \chi_{\mu^\wedge(\varphi)}^{(G_1)} \circ \mu_x$$

on $T(\mathbb{R})$, which completes the proof of Theorem 7.2..

Finally we remark that the Cartan subgroup $T_\varphi^G(\mathbb{R})$ belongs to $\mathcal{J}_\mu(G)$ and moreover $\chi_\varphi^{(G)}$ is not identically zero on $T_\varphi^G(\mathbb{R})$. Hence the formula of Theorem 7.2. is (valid and) nontrivial for at least one conjugacy class of Cartan subgroups in $G(\mathbb{R})$.

§8. Appendix

The purpose of this appendix is to describe a property of the family $\mathcal{J}_\mu(G)$ (Lemma 8.1.) and to indicate the relation between $\mathcal{J}_\mu(G)$

and $\Phi_\mu(G)$ (Lemma 8.2.). Since $\mathcal{J}_\mu(G)$ is closed under conjugation with respect to $G(\mathbb{R})$ we consider instead the set $t_\mu(G)$ of conjugacy classes in $\mathcal{J}_\mu(G)$.

As in [14, §3] we introduce a partial ordering on the set $t(G)$ of all conjugacy classes of Cartan subgroups in $G(\mathbb{R})$. Thus, if T is a maximal torus in G defined over $\mathbb{R}$, S_T will denote the maximal $\mathbb{R}$-split torus in T. We recall that if $T(\mathbb{R})$ is a Cartan subgroup for which S_T is not maximal among the $\mathbb{R}$-split tori in G then there exists a Cartan subgroup $T'(\mathbb{R})$ such that

(i) $\dim S_{T'} = \dim S_T + 1$

and

(ii) $T(\mathbb{R}) \cap T'(\mathbb{R})$ contains a semiregular element (of noncompact type)

(cf. [6, §7]). Moreover $T'(\mathbb{R})$ is determined up to conjugacy under $G(\mathbb{R})$ ([6, Lemma 12]) and a Cartan subgroup conjugate to $T(\mathbb{R})$ yields a Cartan subgroup conjugate to $T'(\mathbb{R})$. Thus for $\tau, \tau' \in t(G)$ we may define τ' to be a successor of τ if for some $T(\mathbb{R}) \in \tau$ there exists $T'(\mathbb{R}) \in \tau'$ such that (i) and (ii) are satisfied. This relation extends uniquely to a partial ordering on $t(G)$ which we denote by $\leq$. There is a unique minimal element for this ordering, namely the class of fundamental Cartan subgroups ([5, §8]).

We remark that μ determines a natural correspondence between $\mathcal{J}_\mu(G)$ and $\mathcal{J}_{\mu^{-1}}(H)$. For if $T_G(\mathbb{R}) \in \mathcal{J}_\mu(G)$ and the restriction of $\mathrm{ad}\, x \circ \mu$ to T_G is defined over $\mathbb{R}$ then $x\mu(T_G(\mathbb{R}))x^{-1} \in \mathcal{J}_{\mu^{-1}}(H)$ and conversely, if $T_H(\mathbb{R}) \in \mathcal{J}_{\mu^{-1}}(H)$ and the restriction of $\operatorname{ad} y \circ \mu^{-1}$ to T_H is defined over $\mathbb{R}$ then $y\mu^{-1}(T_H(\mathbb{R}))y^{-1} \in \mathcal{J}_\mu(G)$. Lemma 5.2. shows that on passing to conjugacy classes we obtain a bijection, denoted $\mu^{(t)}$, between $t_\mu(G)$ and $t_{\mu^{-1}}(H)$. It is clear that $\mu^{(t)}$ is order-preserving (cf. the argument on p. 44). In particular, if H is quasisplit then $\mu^{(t)}$ is an order-preserving embedding of $t(G)$ in $t(H)$.

We now have:

Lemma 8.1.

(i) $t_\mu(G)$ contains the class of fundamental Cartan subgroups in $G(\mathbb{R})$ and

(ii) if $\tau \in t_\mu(G)$ and $\tau' \leq \tau$ then $\tau' \in t_\mu(G)$.

Proof: Suppose that H is quasisplit so that $t_\mu(G) = t(G)$. Then from [5, Lemma 3] it follows that the image of the class τ_0^G of fundamental Cartan subgroups in $G(\mathbb{R})$ is the class of fundamental Cartan subgroups in $H(\mathbb{R})$. If we now remove the assumption that H is quasisplit then, since

$$\mu^{(t)} = ((\lambda \circ \mu^{-1})^{(t)})^{-1} \circ \lambda^{(t)}$$

(where λ is as in §2), it follows that τ_0^G lies in the domain of $\mu^{(t)}$; that is, $\tau_0^G \in t_\mu(G)$, as asserted.

The argument on p. 44 proves (ii).

Finally we recall that to each class φ in $\Phi(G)$ we have associated a Cartan subgroup $T^G_\varphi(\mathbb{R})$. The group $T^G_\varphi(\mathbb{R})$ is determined up to conjugacy under $G(\mathbb{R})$ (Lemma 5.2.) so that we then have associated to φ a class τ^G_φ in $t(G)$.

Lemma 8.2.

Let $\varphi \in \Phi(G)$. Then $\varphi \in \Phi_\mu(G)$ if and only if $\tau^G_\varphi \in t_\mu(G)$. Moreover, if $\varphi \in \Phi_\mu(G)$ then

$$\mu^{(t)}(\tau^G_\varphi) = \tau^H_{\mu^\wedge(\varphi)} .$$

For the proof of this lemma we refer to the constructions in §2 and omit the details.

We remark that by reversing the argument on pp. 7-8 we have associated to each $\tau \in t(G)$ a (nonempty) family of essentially tempered classes in $\Phi(G)$. From Lemmas 8.1. and 8.2. we then conclude that $\Phi_\mu(G)$ always contains a family of essentially tempered classes.

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