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ON THE ASYMPTOTICS OF WHITTAKER FUNCTIONS

EREZ LAPID AND ZHENGYU MAO

Abstract. We study the asymptotics of Whittaker functions on split groups and relate them to the cuspidal exponents of the representation.

1. Introduction

Whittaker models are ubiquitous in the representation theory of quasi-split reductive groups over local fields. They comprise the bedrock for whole families of local zeta integrals of Rankin-Selberg type. In the analysis of the latter, it is imperative to know the asymptotics of Whittaker functions, in order to control the domain of convergence of the integrals. The basic example of $GL(2)$ was studied in detail in [God70] (cf. also Example 3.7 below). More generally, this question was studied extensively in the literature, especially in the context of GL_n (e.g. [JS90b], [JS90a], [JPSS79], [CPS]). In the Archimedean case a fairly complete answer is given in [Wal92, Chapter 15]. On the other hand, to the best of the authors' knowledge, the exact connection between the asymptotics of the Whittaker functions and the exponents of the representation in the p-adic case is not made explicit in the literature. The purpose of this short note is to partially fill this gap. The precise statement is given in Theorem 3.1 below, and is motivated by the results above (cf. [JS90b, Proposition 2.2], [JS90a, §2], [JPSS79]). As a consequence, we realize the inner product of generic square-integrable and more generally, tempered representations, on the Whittaker model. For simplicity we work with split groups.

After completing an early version of this note we learned that Yiannis Sakellaridis and Akshay Venkatesh have launched an ambitious program to study the decomposition of the L^2-space of spherical varieties. In particular, they obtained asymptotic results of the kind which appear in this paper, in a very general setup. Although strictly speaking their current setup does not include the Whittaker case, there is no doubt that this can eventually be incorporated into the general scheme. In particular, Conjectures 3.5 and 3.8 below are probably within reach. Nevertheless, we believe that the Whittaker case is both sufficiently important and elementary to merit its own exposition.

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2. Preliminaries

2.1. **Notation.** Throughout, let F be a non-archimedean local field of characteristic zero with valuation v and normalized absolute value $|\cdot| = q^{-v(\cdot)}$. If X is a variety over F, we will often denote its F-points by X as well.[1] It forms an l-space in the sense of [BZ76]. The space of compactly supported locally constant functions on X will be denoted by $\mathcal{S}(X)$.

For any algebraic group H over F we denote its center by Z_H, its derived group by H^{der} and the connected component of the identity element in H by H^0. Let $X^*(H)$ be the lattice of rational characters of H and set

$$H^1 = \bigcap_{\chi \in X^*(H)} \mathrm{Ker}|\chi|.$$

If H is reductive then $Z_H H^1$ is of finite index in H. If H is a subgroup of G, we denote its centralizer by $C_G(H)$.

If T is a torus, then T^1 is the maximal compact subgroup of T. Suppose that T is split. Then the lattice $X_*(T)$ of co-characters of T can be identified with $\mathrm{Hom}(X^*(T), \mathbb{Z})$ and the map

$$H : T \to X_*(T)$$

defined by

$$H(\mu(a)) = v(a)\mu, \ \mu \in X_*(T), \ a \in F^*$$

induces an isomorphism

$$T/T^1 \to X_*(T).$$

In other words,

$$\langle \omega, H(t) \rangle = v(\omega(t))$$

for all $t \in T$ and $\omega \in X^*(T)$. Any $\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ defines a character

$$|\lambda| : T \to \mathbb{R}_+$$

satisfying

$$|\lambda|(\mu(a)) = |a|^{\langle \lambda, \mu \rangle}$$

for any $\mu \in X_*(T)$, $a \in F^*$. Moreover, any continuous character $T \to \mathbb{R}_+$ arises this way for a unique $\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. In particular, to any continuous character

$$\chi : T \to \mathbb{C}^*$$

we can attach

$$\mathrm{Re}\,\chi \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$$

such that

$$|\chi(t)| = |\mathrm{Re}\,\chi|(t)$$

for all $t \in T$. Similarly, any $\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ defines a character

$$T/T^1 \to \mathbb{C}^*,$$

by the formula

$$t \mapsto q^{-\langle \lambda, H(t) \rangle}.$$

All characters of T/T^1 are of this form and λ is uniquely determined modulo $\frac{2\pi \mathrm{i}}{\log q} X^*(T)$.

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[1] An unfortunate outcome of this convention is the possible ambiguity in the notation $H\backslash G$ when G and H are groups over F. However, in this paper such quotients show up only when the first Galois cohomology group of H is trivial, in which case there is no ambiguity.

We say that a function $f : H \to \mathbb{C}$ is *smooth* if it is invariant under right translation by an open subgroup of H. We denote the space of smooth functions on H by $C^\infty(H)$ and by R the right regular representation of H on $C^\infty(H)$.

Throughout, G will be a connected reductive group which is split over F. Fix a Borel subgroup B of G and a maximal torus T_0 contained in B, both defined over F. Let U_0 be the unipotent radical of B, so that $B = T_0U_0$. We choose a maximal compact K of G in good position with respect to T_0. In particular, $G = BK$.

2.2. **Roots and weights.** We set

$$\mathfrak{a}_0^* = X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_0 = X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$$

with the canonical pairing. Let

$$\Delta_0 \subseteq X^*(T_0) \subseteq \mathfrak{a}_0^*$$

denote the set of simple roots of T_0 on $\mathrm{Lie}(U_0)$. For each $\alpha \in \Delta_0$ let U_α be the one-parameter unipotent subgroup corresponding to α. We extend $H : T_0 \to X_*(T_0)$ to G by requiring that H is left-U_0 and right-K invariant.

Henceforth, $P = MU$ will always denote a parabolic subgroup containing B, U its unipotent radical and M its Levi part containing T_0. Similarly, for $P' = M'U'$, etc. The simple root Δ_0^P of T_0 on $\mathrm{Lie}(U_0 \cap M)$ is a subset of Δ_0. The set Δ_0^P determines P uniquely and any subset of Δ_0 arises in this way. The torus

$$T_0^M = T_0 \cap M^{\mathrm{der}}$$

is a maximal (split) torus in M^{der} and we have

$$Z_M = \bigcap_{\chi \in \Delta_0^P} \operatorname{Ker} \chi. \tag{2.1}$$

Setting

$$T_M = Z_M^0$$

we have $M = C_G(T_M)$. Moreover, M/T_MM^1 is a finite abelian group. We write

$$\mathfrak{a}_M^* = X^*(T_M) \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$(\mathfrak{a}_0^M)^* = X^*(T_0^M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\mathfrak{a}_M$ and $\mathfrak{a}_0^M$ be the respective dual spaces. The set Δ_0^P is a basis for $(\mathfrak{a}_0^M)^*$ and the restriction maps

$$X^*(T_0) \to X^*(T_M), \quad X^*(T_0) \to X^*(T_0^M)$$

induce the isomorphisms

$$\mathfrak{a}_0^* = \mathfrak{a}_M^* \oplus (\mathfrak{a}_0^M)^*, \;\; \mathfrak{a}_0 = \mathfrak{a}_M \oplus \mathfrak{a}_0^M.$$

Similarly, the restriction map

$$X^*(M) \to X^*(T_M)$$

induces an isomorphism

$$\mathfrak{a}_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Thus, any character $\chi : T_M \to \mathbb{R}_+$ uniquely extends to M.

We let

$$\hat{\Delta}_0^\vee = \{\varpi_\alpha^\vee : \alpha \in \Delta_0\}$$

be the dual basis of Δ_0 in $\mathfrak{a}_0^G$. Let

$$\Delta_P \subseteq X^*(T_M) \subseteq \mathfrak{a}_M^*$$

be the set of "simple roots" of T_M on U, i.e., the restrictions of the roots $\Delta_0 \setminus \Delta_0^P$ to T_M. Then

$$\hat{\Delta}_P^\vee = \hat{\Delta}_0^\vee \cap \mathfrak{a}_M$$

is a dual basis of Δ_P in $\mathfrak{a}_M^G = \mathfrak{a}_M \cap \mathfrak{a}_0^G$.

Remark 2.1. Suppose that $\alpha \in \Delta_P$ is the restriction of $\beta \in \Delta_0 \setminus \Delta_0^P$. Then in $\mathfrak{a}_0^*$ we have

$$\alpha = \beta + \sum_{\gamma \in \Delta_0^P} m_\gamma \gamma$$

where $m_\gamma \geq 0$ for all $\gamma \in \Delta_0^P$.

Indeed, let $\lambda = \alpha - \beta \in (\mathfrak{a}_0^M)^*$. Then for any $\gamma \in \Delta_0^P$, if $\gamma^\vee \in X_*(T_0)$ denotes the corresponding co-root, then

$$0 = \langle \alpha, \gamma^\vee \rangle = \langle \beta, \gamma^\vee \rangle + \langle \lambda, \gamma^\vee \rangle \leq \langle \lambda, \gamma^\vee \rangle .$$

Therefore, λ is in the closure of the positive Weyl chamber of $(\mathfrak{a}_0^M)^*$, and in particular, it is a non-negative linear combination of Δ_0^P.

If $P_i \supseteq B$, $i = 1, 2$, we will denote by $P_1 \bullet P_2$ the parabolic subgroup generated by P_1 and P_2. Its Levi subgroup $M_1 \bullet M_2$ is the group generated by M_1 and M_2. We have

$$\begin{aligned} Z_{M_1 \bullet M_2} &= Z_{M_1} \cap Z_{M_2}, \\ \Delta_0^{P_1 \bullet P_2} &= \Delta_0^{P_1} \cup \Delta_0^{P_2}, \\ \hat{\Delta}_{P_1 \bullet P_2}^\vee &= \hat{\Delta}_{P_1}^\vee \cap \hat{\Delta}_{P_2}^\vee, \\ \mathfrak{a}_0^{P_1 \bullet P_2} &= \mathfrak{a}_0^{P_1} + \mathfrak{a}_0^{P_2}, \\ \mathfrak{a}_{P_1 \bullet P_2} &= \mathfrak{a}_{P_1} \cap \mathfrak{a}_{P_2}, \end{aligned}$$

and similarly for the dual spaces.

We denote by δ_P the modulus function of P, viewed as a character of M. Observe that if $P' \subseteq P$, then

$$\delta_{P'} \equiv \delta_P \text{ on } Z_M. \tag{2.2}$$

2.3. **Eventually polynomial exponential sequences.**

Definition 2.2. A sequence a_n, $n \in \mathbb{Z}$ of complex numbers is called *eventually polynomial exponential* (e.p.e.) if there exists a finite subset Λ of $\mathbb{C}/\frac{2\pi \mathrm{i}}{\log q}\mathbb{Z}$ and polynomials P_λ, $\lambda \in \Lambda$ such that $a_n = 0$ for $n \ll 0$ and

$$a_n = \sum_{\lambda \in \Lambda} P_\lambda(n) q^{\lambda n} \tag{2.3}$$

for $n \gg 0$.

Clearly, for an e.p.e. sequence as above, the set Λ is uniquely determined provided that P_λ is non-zero for all $\lambda \in \Lambda$. We call Λ the set of exponents of the sequence a_n and denote it by $\mathcal{E}((a_n))$.

The *q-transform* $Q((a_n), s)$ of a sequence a_n is by definition the formal power series

$$\sum_{n\in\mathbb{Z}} a_n q^{-ns}.$$

If T is the right shift operation on sequences, then

$$Q(T((a_n)), s) = q^s Q((a_n), s). \tag{2.4}$$

The following lemma is elementary.

Lemma 2.3. *Let a_n be a sequence of complex numbers such that $a_n = 0$ for $n \ll 0$ and $\Lambda \subseteq \mathbb{C}/\frac{2\pi \mathrm{i}}{\log q}\mathbb{Z}$ be a finite set. The following conditions are equivalent:*

1. *a_n is e.p.e. with $\mathcal{E}((a_n)) \subseteq \Lambda$.*
2. *There exists $m \geq 0$ such that the sequence*
   $$\prod_{\lambda\in\Lambda} (T - q^\lambda)^m (a_n)$$
   has only finitely many non-zero terms.
3. *$Q((a_n), s)$ converges absolutely for $\operatorname{Re} s > \max_\Lambda \operatorname{Re}\lambda$ and extends to a rational function in q^s whose poles are contained in*
   $$\{s \in \mathbb{C} : q^s \in q^\Lambda\}.$$

Moreover, if a_n has the form (2.3) *and $q^{s_0} = q^\lambda$, $\lambda \in \Lambda$, then $Q((a_n), s)$ admits a pole of order $d+1$ at s_0 where $d = \deg P_\lambda$ and the leading coefficient in the Laurent expansion of $Q((a_n), s)$ around s_0 is $\frac{d!}{(\log q)^{d+1}} b$ where b is the leading coefficient of P_λ.*

Proof. It is easy to see that the first condition implies the second. The converse direction reduces by induction to the following statement. If $a_n = 0$ for $n \ll 0$ and

$$b_n = (T - q^\lambda) a_n$$

is e.p.e., then a_n is e.p.e. and

$$\exp((a_n)) \subseteq \exp((b_n)) \cup \{\lambda\}.$$

To show this, note that

$$a_n = \sum_{m=0}^{\infty} q^{\lambda m} b_{n-m-1}$$

where only finitely many terms are non-zero in the sum. It is enough to consider the case where

$$b_n = \begin{cases} Q(n) q^{\mu n} & \text{if } n \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for some polynomial Q and $\mu \in \mathbb{C}$. By writing $Q(x)$ as a linear combination of Newton polynomials $\binom{x}{k}$, the assertion now follows from the identity

$$\sum_{n=0}^{m} \binom{n}{k} x^n = \frac{x^k}{k!} \left[\frac{1 - x^{m+1}}{1 - x}\right]^{(k)}, \quad x \neq 1. \tag{2.5}$$

To show that the first condition implies the third we express each P_λ as a linear combination of Newton polynomials and use the identity

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = x^k (1 - x)^{-(k+1)}, \ |x| < 1.$$

This also shows the very last part of the lemma.

Finally, suppose that $Q((a_n),s)$ satisfies condition (3). Then for some m,

$$\left[\prod_{\lambda\in\Lambda}(q^s-q^\lambda)^m\right]Q((a_n),s)$$

is a rational function in q^s without poles, and hence a Laurent polynomial in q^s. This implies the second condition by (2.4). $\square$

Corollary 2.4. *Suppose that $a_n=0$ for $n\ll 0$ and there exists a positive integer h and $\lambda\in\mathbb{C}/\frac{2\pi\mathrm{i}}{\log q}\mathbb{Z}$ such that $b_n=a_{n+h}-q^\lambda a_n$ is e.p.e. Then a_n is e.p.e. and*

$$\mathcal{E}((a_n))\subseteq\mathcal{E}((b_n))\cup\left\{\frac{\lambda+\frac{2\pi\mathrm{i}}{\log q}j}{h},0\le j<h\right\}.$$

In particular,

$$\{|q^\mu|:\mu\in\mathcal{E}((a_n))\}\subseteq\{|q^\mu|:\mu\in\mathcal{E}((b_n))\}\cup\{q^{\operatorname{Re}\lambda/h}\}.$$

In the case of a sequence with non-negative real numbers we can say a little more.

Lemma 2.5. *Suppose that the sequence a_n, $n\in\mathbb{Z}$ is e.p.e. and $a_n\ge 0$ for all n. Let Λ and P_λ be as in (2.3) with $P_\lambda\not\equiv 0$ for all $\lambda\in\Lambda$. Suppose further that $\operatorname{Re}\lambda\le 0$ for all $\lambda\in\Lambda$. Then either $0\notin\Lambda$, in which case $\operatorname{Re}\lambda<0$ for all $\lambda\in\Lambda$, a_n is rapidly decreasing, and*

$$\sum_{n\in\mathbb{Z}}a_n=Q((a_n),0),$$

or else we have:

1. *The highest coefficient of P_0 is positive.*
2. *$\deg P_\lambda\le d:=\deg P_0$ for all $\lambda\in\Lambda^*:=\Lambda\cap\mathrm{i}\mathbb{R}/\frac{2\pi\mathrm{i}}{\log q}\mathbb{Z}$.*
3. *$a_n=O(n^d)$.*
4. *$Q((a_n),s)$ converges for $\operatorname{Re}s>0$ and has a pole of order $r=d+1$ at $s=0$. Let $\kappa=\lim_{s\to 0}s^rQ((a_n),s)$ be the leading coefficient. Then*

$$\sum_{n\le m}a_n\sim\frac{(\log q)^r\kappa}{r!}m^r\quad\text{as } m\to\infty.$$

Proof. The lemma is straightforward if $\operatorname{Re}\lambda<0$ for all $\lambda\in\Lambda$, i.e., if $\Lambda^*=\emptyset$. Assume that $\Lambda^*\ne\emptyset$ and let $l=\max_{\lambda\in\Lambda^*}\deg P_\lambda\ge 0$. Let h_λ be the coefficient of x^l in P_λ. We have

$$b_n:=\sum_{\lambda\in\Lambda^*}h_\lambda q^{\lambda n}=c_n+d_n \tag{2.6}$$

where $c_n=\frac{a_n}{n^l}\ge 0$ and $d_n=O(\frac{1}{n})$ as $n\to\infty$. For $m\ge 0$ consider

$$x_m=\frac{1}{m}\sum_{n=1}^m b_n,\quad y_m=\frac{1}{m}\sum_{n=1}^m|b_n|,\quad z_m=\frac{1}{m}\sum_{n=1}^m|b_n|^2.$$

By summing the geometric series we get

$$|x_m-h_0|=o(1).$$

By (2.6),

$$|x_m - y_m| \le \frac{1}{m}\sum_{n=1}^{m}|d_n| = o(1).$$

Therefore,

$$|y_m - h_0| = o(1). \tag{2.7}$$

On the other hand,

$$|b_n|^2 = \sum_{\lambda,\lambda'\in\Lambda^*} h_\lambda \overline{h_{\lambda'}} q^{(\lambda-\lambda')n}$$

so that

$$|z_m - H| = o(1) \tag{2.8}$$

where

$$H = \sum_{\lambda\in\Lambda^*}|h_\lambda|^2 > 0.$$

Since b_n is bounded, z_m is bounded by a constant multiple of y_m. From (2.7) and (2.8) we conclude that $h_0 > 0$ so that $\deg P_0 = l$. The first three parts of the lemma follow immediately.

Observe that for any polynomial P and $\lambda \in \mathbb{C} \setminus \frac{2\pi\mathrm{i}}{\log q}\mathbb{Z}$ with $\operatorname{Re}\lambda \le 0$ we have

$$|\sum_{n=0}^{m} P(n) q^{\lambda n}| = \begin{cases} O(m^{\deg P}) & \text{if } \operatorname{Re}\lambda = 0, \\ O(1) & \text{if } \operatorname{Re}\lambda < 0, \end{cases}$$

as $m \to \infty$. Indeed, this follows from (2.5) by decomposing P into Newton polynomials. On the other hand, by the last part of Lemma 2.3 we have

$$\frac{(\log q)^r}{r!}\kappa = \frac{h_0}{r}.$$

Therefore, the last part of the lemma reduces to the relation

$$\sum_{n=0}^{m} P_0(n) \sim \frac{h_0}{r} m^r \quad \text{as } m \to \infty.$$

By ignoring lower order terms in P_0 we can assume that $P_0(n) = \binom{n}{d}$. It remains to apply the identity

$$\sum_{n=0}^{m}\binom{n}{d} = \binom{m+1}{d+1}. \qquad \square$$

2.4. **A space of functions on T_0.** Let V be a representation space of a locally compact abelian group A. For any character χ of A and $n \in \mathbb{N}$ we write

$$V_{\chi,n} = \{u \in V : (a_1 - \chi(a_1)) \dots (a_n - \chi(a_n))u = 0 \quad \text{for all } a_1, \dots, a_n \in A\}.$$

We also denote by

$$V_\chi = \bigcup_{n=0}^{\infty} V_{\chi,n}$$

the generalized χ-eigenspace and by

$$V_{A\text{-fin}} = \bigoplus_{\chi\in\hat{A}} V_\chi$$

the space of A-finite vectors in V. In particular, for $V =$ functions on A, we write

$$\mathcal{F}(A) = V_{A\text{-fin}}$$

for the space of A-finite functions on A. Note that for any $\chi_i \in \hat{A}$, $i = 1, 2$,

$$\mathcal{F}(A)_{\chi_1}\mathcal{F}(A)_{\chi_2} \subseteq \mathcal{F}(A)_{\chi_1\chi_2}. \tag{2.9}$$

For example, if $A = \mathbb{Z}^r$, then $\mathcal{F}(A)$ is the space of polynomial exponential functions on A. Similarly, if T is a split torus and $\chi \in \hat{T}$, then it is easy to see by induction on n that

$$\mathcal{F}(T)_{\chi,n} = \{\chi(t)Q(H(t))\} \tag{2.10}$$

where Q ranges over the polynomials on $X_*(T)$ of degree $< n$.

We consider the affine line $\mathbb{A}^1$ as a monoid with respect to multiplication. The multiplicative group $\mathbb{G}_m$ is an open subgroup of $\mathbb{A}^1$. For any parabolic subgroup P we consider the monoids

$$\mathbb{M}_P = \prod_{\alpha\in\Delta_0} \begin{cases} \mathbb{G}_m & \text{if } \alpha \in \Delta_0^P, \\ \mathbb{A}^1 & \text{otherwise.} \end{cases}$$

In particular,

$$\mathbb{M}_G = \mathbb{G}_m^{\Delta_0}.$$

For any $P' \subseteq P$, $\mathbb{M}_P$ is an open submonoid of $\mathbb{M}_{P'}$. We view $\mathcal{S}(\mathbb{M}_P)$ as the ideal of $\mathcal{S}(\mathbb{M}_{P'})$ (under pointwise multiplication) consisting of those functions which are supported in $\mathbb{M}_P$, that is, vanish on $\mathbb{M}_{P'} \setminus \mathbb{M}_P$. More generally, we have

$$\mathcal{S}(\mathbb{M}_{P_1})\mathcal{S}(\mathbb{M}_{P_2}) \subseteq \mathcal{S}(\mathbb{M}_{P_1\bullet P_2}) \tag{2.11}$$

for any P_1, P_2. Each space $\mathcal{S}(\mathbb{M}_P)$ is invariant under right translation by $\mathbb{M}_G$. Let

$$\mathfrak{r} : T_0 \to \mathbb{M}_G$$

be the homomorphism defined by

$$\mathfrak{r}(t)_\alpha = \alpha(t).$$

Note that by (2.1) we have

$$\operatorname{Ker}\mathfrak{r} = Z_G. \tag{2.12}$$

For any P and $\chi \in \widehat{Z_M}$, we denote by

$$\mathfrak{F}_{P,\chi} = \mathfrak{F}_{P,\chi}^G$$

(resp. $\mathfrak{F}_{P,\chi,n}$) the subspace of $C^\infty(T_0)$ spanned by functions of the form

$$\xi(t)\varphi(\mathfrak{r}(t))$$

where $\xi \in \mathcal{F}(T_0)_\chi$ (resp. $\xi \in \mathcal{F}(T_0)_{\chi,n}$) and $\varphi \in \mathcal{S}(\mathbb{M}_P)$. Recall that by our convention $\mathcal{F}(T_0)_\chi$ is the space of T_0-finite functions on T_0 which are generalized (Z_M, χ)-eigenfunctions. Alternatively, we have

$$\mathcal{F}(T_0)_\chi = \sum_{\chi'\in\widehat{T_0}:\chi'|_{Z_M}=\chi} \mathcal{F}(T_0)_{\chi'}. \tag{2.13}$$

For any $\alpha \in \Delta_0$ let $Z_\alpha = Z_{L_\alpha}$ where L_α is the Levi part of the maximal parabolic subgroup defined by α. Explicitly,

$$Z_\alpha = \bigcap_{\alpha\neq\beta\in\Delta_0} \operatorname{Ker}\beta.$$

Also, if P is a parabolic subgroup and $\alpha \in \Delta_0 \setminus \Delta_0^P$, then we write P^α for the parabolic subgroup defined by

$$\Delta_0^{P^\alpha} = \Delta_0^P \cup \{\alpha\}$$

and M^α its Levi subgroup. Thus, M is a co-rank one Levi subgroup of M^α. Note that

$$\mathbb{M}_{P^\alpha} = \{x \in \mathbb{M}_P : x_\alpha \neq 0\}$$

and

$$\mathcal{S}(\mathbb{M}_{P^\alpha}) = \{f \in \mathcal{S}(\mathbb{M}_P) : f(x) = 0 \text{ if } x_\alpha = 0\}. \tag{2.14}$$

We have the following elementary properties.

Lemma 2.6. (1) *For any $\chi \in \widehat{Z_G}$, $\mathfrak{F}_{G,\chi}$ is the space of smooth functions on T_0 which are compactly supported modulo Z_G and have generalized eigenvalue χ under Z_G.*

(2) *For any $\phi \in \mathfrak{F}_{P,\chi}$ which is T_G-invariant there exist constants C, n, k such that*

$$|\phi(t)| \le C q^{-\langle \operatorname{Re}\chi, H(t)\rangle} \prod_{\alpha\in\Delta_0} \theta_{P,\alpha}^{n,k}(t), \quad t \in T_0 \tag{2.15}$$

where for any $\alpha \in \Delta_0$ and $t \in T_0$ we set

$$\theta_{P,\alpha}^{n,k}(t) = \begin{cases} 1 & \text{if } \alpha \in \Delta_0^P \text{ and } |v(\alpha(t))| \le n, \\ (v(\alpha(t)) + n + 1)^k & \text{if } \alpha \notin \Delta_0^P \text{ and } v(\alpha(t)) \ge -n, \\ 0 & \text{otherwise.} \end{cases}$$

(3) *Suppose that $P' \subseteq P$ and $\chi' \in \widehat{Z_{M'}}$ with $\chi'|_{Z_M} = \chi$. Then*

$$\mathfrak{F}_{P,\chi} \subseteq \mathfrak{F}_{P',\chi'}.$$

(4) *For $t \in T_0$ set*

$$D_{t_0}\phi(t) = \phi(tt_0) - \chi(t_0)\phi(t).$$

Then for any $\alpha \in \Delta_0 \setminus \Delta_0^P$, $t_0 \in Z_\alpha$, $\chi \in \widehat{Z_M}$ and $n > 0$ we have

$$D_{t_0}\mathfrak{F}_{P,\chi,n} \subseteq \mathfrak{F}_{P,\chi,n-1} + \mathfrak{F}_{P^\alpha,\chi|_{Z_{M^\alpha}},n}. \tag{2.16}$$

(5) *Suppose that $\phi_i \in \mathfrak{F}_{P_i,\chi_i}$, $i = 1, 2$. Then*

$$\phi_1\phi_2 \in \mathfrak{F}_{P_1\bullet P_2,\chi_1\chi_2|_{Z_{M_1\bullet M_2}}}.$$

In particular, for any $\phi \in \mathfrak{F}_{P,\chi}$ and $\mu \in \widehat{T_0}$ we have

$$\phi\mu \in \mathfrak{F}_{P,\chi\mu|_{Z_M}}. \tag{2.17}$$

Proof. Clearly, any $\phi \in \mathfrak{F}_{G,\chi}$ is compactly supported modulo Z_G and has generalized eigenvalue χ under Z_G. Conversely, suppose that ϕ is smooth, compactly supported modulo Z_G and has generalized eigenvalue χ under Z_G. Replacing ϕ by $\chi^{-1}\phi$ we can assume that χ is trivial. By decomposing the support of ϕ to finitely many cosets of Z_G it is enough to consider the case where ϕ is given by

$$\phi(t) = \begin{cases} f(z) & \text{if } t = zvt_0,\ z \in Z_G, v \in V, \\ 0 & \text{otherwise,} \end{cases}$$

where $f \in \mathcal{F}(Z_G)_{\mathbf{1}}$, V is a compact open subgroup of T_0^1 and $t_0 \in T_0$. The function f is a polynomial function on the lattice Z_G/Z_G^1. It can be extended to a polynomial on the lattice T_0/T_0^1, i.e., to a function $\xi \in \mathcal{F}(T_0)_{\mathbf{1}}$. By (2.12) we may write

$$\phi(t) = \xi(tt_0^{-1})\mathbf{1}_{\mathfrak{r}(Vt_0)}(\mathfrak{r}(t)).$$

Thus, $\phi \in \mathfrak{F}_{G,\chi}$ as required.

For the second part we note that any $\phi \in \mathfrak{F}_{P,\chi}$ is compactly supported modulo T_M. Therefore, it is enough to check (2.15) on any fixed coset of $T_M \cap T_0^G$. This follows from the description (2.10).

For the third part, let $\phi \in \mathfrak{F}_{P,\chi}$. To show that $\phi \in \mathfrak{F}_{P',\chi'}$ we can assume by (2.13) that

$$\phi(t) = \xi(t)\varphi(\mathfrak{r}(t))$$

with $\varphi \in \mathcal{S}(\mathbb{M}_P)$ and $\xi \in \mathcal{F}(T_0)_{\tilde{\chi}}$ where $\tilde{\chi} \in \widehat{T_0}$ and $\tilde{\chi}|_{Z_M} = \chi$. Let $\tilde{\chi}' \in \widehat{T_0}$ be an arbitrary extension of χ'. Then

$$\eta := \tilde{\chi}'\tilde{\chi}^{-1} \in \widehat{T_0}$$

is trivial on Z_M. Using (2.1) we can write

$$\eta(t)^{-1} = \tilde{\varphi}(\mathfrak{r}(t))$$

where $\tilde{\varphi}$ is a smooth function on $\mathbb{M}_G$ depending only on the coordinates in Δ_0^P. Thus, we have

$$\phi(t) = \xi'(t)\varphi'(\mathfrak{r}(t))$$

where

$$\xi' = \xi\eta \in \mathcal{F}(T_0)_{\tilde{\chi}'}$$

and

$$\varphi' = \varphi\tilde{\varphi} \in \mathcal{S}(\mathbb{M}_P) \subseteq \mathcal{S}(\mathbb{M}_{P'}).$$

To show the fourth part, suppose that $\phi(t) = \xi(t)\varphi(\mathfrak{r}(t))$ where $\xi \in \mathcal{F}(T_0)_\chi$ and $\varphi \in \mathcal{S}(\mathbb{M}_P)$. Then

$$D_{t_0}\phi(t) = [\xi(tt_0) - \chi(t_0)\xi(t)]\varphi(\mathfrak{r}(t)\mathfrak{r}(t_0)) + \chi(t_0)\xi(t)[\varphi(\mathfrak{r}(t)\mathfrak{r}(t_0)) - \varphi(\mathfrak{r}(t))].$$

Since $\mathfrak{r}(t_0)$ is 1 in all coordinates except α, we have

$$\varphi(\cdot\mathfrak{r}(t_0)) - \varphi(\cdot) \in \mathcal{S}(\mathbb{M}_{P^\alpha})$$

by (2.14).

Finally, the last part follows from (2.9), (2.11) and (2.13). □

Remark 2.7. The definition of $\mathfrak{F}_{P,\chi}$ is tailor-made for split groups. In order to deal with general quasi-split groups we have to modify the definition of $\mathfrak{F}_{P,\chi}$ to a space of functions on $C_G(T_0)$ (where T_0 is a maximal split torus). One has to take into account that the simple roots of T_0 do not necessarily extend to $C_G(T_0)$.

For any $\alpha \in \Delta_0$ the lattice $T_G Z_\alpha^1 \backslash Z_\alpha \simeq X_*(T_G \backslash Z_\alpha)$ is one-dimensional. We fix an element $t_\alpha \in Z_\alpha$ which lies above a generator of $T_G Z_\alpha^1 \backslash Z_\alpha$ and such that $|\alpha(t_\alpha)| < 1$. We also fix a generating set $\omega_1^\vee, \dots, \omega_d^\vee$ for $X_*(T_G \backslash T_0)$.

For any $\chi \in \widehat{Z_M}$ we set

$$\mathfrak{S}(\chi) = \{\alpha \in \Delta_0 \setminus \Delta_0^P : \chi|_{Z_\alpha^1} \equiv 1\}.$$

Note that if $P' \subseteq P$ and $\chi' \in \widehat{Z_{M'}}$, then

$$\mathfrak{S}(\chi'|_{Z_M}) = \mathfrak{S}(\chi') \setminus \Delta_0^P.$$

Lemma 2.8. *Let* $\phi \in \mathfrak{F}_{P,\chi}$ *and suppose that* ϕ *is* T_G*-invariant. (In particular,* $\chi|_{T_G} \equiv 1$.*) Then:*

(1) *The integral*

$$I(\phi, \lambda) = \int_{T_G \backslash T_0} \phi(t) q^{-\langle \lambda, H(t) \rangle} \, dt$$

converges in the cone

$$\{\lambda \in (\mathfrak{a}_0^G)^*_{\mathbb{C}} : \langle \operatorname{Re} \lambda + \operatorname{Re} \chi, \varpi^\vee \rangle > 0 \textit{ for all } \varpi^\vee \in \hat{\Delta}_P^\vee\}$$

and defines a rational function in $q^{\langle \lambda, \omega_i^\vee \rangle}$, $i = 1, \dots, d$ *with poles at most along the hyperplanes*

$$q^{\langle \lambda, H(t_\alpha) \rangle} = \chi(t_\alpha), \quad \alpha \in \mathfrak{S}(\chi).$$

(2) *Let* $\omega = \sum_{\alpha \in \Delta_0} n_\alpha \alpha \in X^*(T_0)$ *with* $n_\alpha > 0$. *Define*

$$a_n = a_n(\phi) = \int_{T_G \backslash T_0} \delta_{\langle \omega, H(t) \rangle, n} \phi(t) \, dt$$

where $\delta_{m,n}$ *is the Kronecker delta. Then* a_n *is e.p.e. and*

$$\operatorname{Re} \mathcal{E}((a_n)) \subseteq \bigcup_{\alpha \in \mathfrak{S}(\chi)} \{\lambda : |\omega(t_\alpha)|^{-\lambda} = |\chi(t_\alpha)|\}.$$

Proof. For the convergence part we use (2.15) to reduce it to the convergence of

$$\int_{T_G \backslash T_0} q^{\langle -(\operatorname{Re} \lambda + \operatorname{Re} \chi), H(t) \rangle} \prod_{\alpha \in \Delta_0} \theta_{P,\alpha}^{m,k}(t) \, dt.$$

The integrand is clearly invariant under T_0^1. By writing

$$H(t) = \sum_{\alpha \in \Delta_0} v(\alpha(t)) \varpi_\alpha^\vee \bmod \mathfrak{a}_G$$

the convergence further reduces to that of

$$\sum_{n \geq -m} q^{-n(\langle \operatorname{Re} \lambda + \operatorname{Re} \chi, \varpi^\vee \rangle)} (1 + |n|)^k$$

for all $\varpi^\vee \in \hat{\Delta}_P^\vee$.

Similarly, the integrand in the definition of a_n is compactly supported in the domain of integration. Therefore, a_n is well defined and it is easy to see that $a_n = 0$ for $n \ll 0$.

Suppose that $\phi \in \mathfrak{F}_{P,\chi,m}$. We will prove the remaining two statements by induction on m and r where r is the co-rank of P. If $m = 0$, then $\phi = 0$ and there is nothing to prove. If $r = 0$, then $P = G$ and $\phi \in \mathcal{S}(T_G \backslash T_0)$. In this case $a_n = 0$ for almost all n and $I(\phi, \lambda)$ reduces to a polynomial in $q^{\pm \langle \lambda, \omega_i^\vee \rangle}$.

For the induction step, suppose that $P \neq G$ and $m > 0$. Let $\alpha \in \Delta_0 \setminus \Delta_0^P$. For any $t_0 \in Z_\alpha$ we have

$$I(D_{t_0} \phi, \lambda) = (q^{\langle \lambda, H(t_0) \rangle} - \chi(t_0)) I(\phi, \lambda),$$
$$a_n(D_{t_0} \phi) = a_{n + v(\omega(t_0))}(\phi) - \chi(t_0) a_n(\phi).$$

If $\alpha \notin \mathfrak{S}(\chi)$, we take $t_0 \in Z_\alpha^1$ such that $\chi(t_0) \neq 1$. Otherwise, take $t_0 = t_\alpha$. In the first case $H(t_0) \in \mathfrak{a}_G$ and thus $q^{\langle \lambda, H(t_0) \rangle} = 1$. In both cases, the lemma follows from (2.16) and Corollary 2.4 by applying the induction hypothesis to $D_{t_0} \phi$. ☐

2.5. **Germs.** For any parabolic subgroup $P = MU$ and $\epsilon > 0$ let

$$M_{<\epsilon} = \{m \in M : |\alpha|(m) < \epsilon, \quad \forall \alpha \in \Delta_P\}.$$

We set an equivalence relation on $C^\infty(M)$ by saying that f_1 and f_2 have the same germ (at 0) if they agree on $M_{<\epsilon}$ for some $\epsilon > 0$. This equivalence relation clearly respects the left and right action of M and therefore the space $\mathcal{G}(M)$ of equivalence classes is also a representation space of M. Similarly, we can define the set $(Z_M)_{<\epsilon}$ and the space $\mathcal{G}(Z_M)$ of germs of functions on Z_M.

Lemma 2.9. *The map $f \mapsto [f]$ sending f to its germ (i.e., its equivalence class) induces an isomorphism of Z_M-modules*

$$\Gamma : \mathcal{F}(Z_M) \to \mathcal{G}(Z_M)_{Z_M\text{-fin}}.$$

Moreover, there exists $\delta > 0$ with the following property. Let V be an open subgroup of Z_M^1, $\chi \in \widehat{Z_M/V}$, $n \in \mathbb{N}$ and $\mathcal{B} \subseteq Z_M$ a finite set such that $\mathcal{B}V$ generates Z_M. Assume that $\mathcal{B}$ contains t_α, $\alpha \in \Delta_0 \setminus \Delta_0^P$ as well as a set of elements $z_1, \dots, z_s \in T_G$ such that $H(z_i)$, $i = 1, \dots, s$ form a basis for $\mathfrak{a}_G$. Suppose that $f \in C^\infty(Z_M)$ is V-invariant and

$$\prod_{i=1}^n (R(b_i) - \chi(b_i)) f(t) = 0$$

for all $t \in (Z_M)_{<\epsilon}$ and $b_1, \dots, b_n \in \mathcal{B}$. Then

$$f(t) = \Gamma^{-1}([f])(t)$$

for all $t \in (Z_M)_{<\epsilon\delta}$.

For the proof we will use the following well-known fact whose elementary proof will be omitted.

Lemma 2.10. *Let $r, s \in \mathbb{N}$ and let X be the monoid $\mathbb{N}^r \times \mathbb{Z}^s$ with the standard generators $e_1, \dots, e_r, \pm e_{r+1}, \dots, \pm e_{r+s}$. Let φ be a function on X. Suppose that for any $i = 1, \dots, r + s$ the function $\varphi(e_i + x) - \varphi(x)$ is a polynomial on X. Then φ is a polynomial on X.*

Proof of Lemma 2.9. The injectivity of Γ reduces to the corresponding statement for T_M, which in turn follows from the description (2.10) of $\mathcal{F}(T_M)$.

To show surjectivity and the last part, let $f \in C^\infty(Z_M)$ be such that $[f] \in \mathcal{G}(Z_M)_{\chi,n}$. Thus, there exists $\epsilon > 0$ such that

$$\prod_{i=1}^n (R(b_i) - \chi(b_i)) f(t) = 0$$

for all $b_i \in \mathcal{B}$ and $t \in (Z_M)_{<\epsilon}$. We will show that for an appropriate $\delta > 0$ (independent of f and ϵ) there exists $g \in \mathcal{F}(Z_M)$ such that

$$g \equiv f \text{ on } (Z_M)_{<\epsilon\delta}.$$

Replacing f by $f\chi^{-1}$, we can assume that χ is trivial. Observe that f is Z_M^1-invariant on $(Z_M)_{<\epsilon}$, since Z_M^1 is compact. Thus, the restriction of f to $(Z_M)_{<\epsilon}$ depends only on $H(t)$. Let X (resp. $\tilde{X}$) be the monoid (resp. group) generated by t_α, $\alpha \in \Delta_0 \setminus \Delta_0^P$ and $z_i^{\pm 1}$, $i = 1, \dots, s$. Note that $\tilde{X}$ is free. Fix a finite set of representatives $\mathcal{Y}$ for $Z_M / Z_M^1 \tilde{X}$. We can choose $\mathcal{Y}$ so that

$$\epsilon\delta < |\alpha|(y) < \epsilon, \quad y \in \mathcal{Y}, \alpha \in \Delta_0 \setminus \Delta_0^P$$

where δ is a uniform constant. Thus,

$$(Z_M)_{<\epsilon\delta} \subseteq Z_M^1 X \mathcal{Y} \subseteq (Z_M)_{<\epsilon}.$$

We will show that for any $y \in \mathcal{Y}$ there exists a polynomial Q_y on $\mathfrak{a}_M$ such that

$$f(ty) = Q_y(H(t)) \text{ for all } t \in X.$$

We use induction on n. The statement is clear for $n = 0$. For the induction step, we know that for each $y \in \mathcal{Y}$ there exist polynomials $Q_{y,\alpha}$, $\alpha \in \Delta_0 \setminus \Delta_0^P$ and $Q'_{y,i}$, $i = 1, \dots, s$ such that for all $t \in X$ we have

$$\begin{aligned} f(t_\alpha ty) - f(ty) &= Q_{y,\alpha}(H(t)) \quad \alpha \in \Delta_0 \setminus \Delta_0^P, \\ f(z_i ty) - f(ty) &= Q'_{y,i}(H(t)) \quad i = 1, \dots, s. \end{aligned}$$

Our claim now follows from Lemma 2.10.

Thus, the function g on Z_M defined by

$$g(aty) = Q_y(H(t)) \quad y \in \mathcal{Y}, t \in \tilde{X}, a \in Z_M^1$$

is Z_M-finite and agrees with f on $(Z_M)_{<\epsilon\delta}$. □

Corollary 2.11. *The map $f \mapsto [f]$ induces a bi-M-equivariant isomorphism*

$$\iota_M : C^\infty(M)_{Z_M\text{-fin}} \to \mathcal{G}(M)_{Z_M\text{-fin}}.$$

Proof. Let $f \in C^\infty(M)$ and for each $m \in M$ let $f_m(t) = f(tm)$, $t \in Z_M$. Suppose that f is Z_M-finite and $\iota_M(f) = 0$. Then for any $m \in M$, $f_m \in \mathcal{F}(Z_M)$ and $\Gamma(f_m) = 0$. By the lemma we conclude that $f_m \equiv 0$ and therefore $f \equiv 0$. Thus, ι_M is injective.

To show surjectivity, suppose that $f \in C^\infty(M)$ and $[f] \in \mathcal{G}(M)_{\chi,n}$ for some $\chi \in \widehat{Z_M}$ and n. Then $[f_m] \in \mathcal{G}(Z_M)_{\chi,n}$ for all $m \in M$. Define

$$\tilde{f}(m) = \Gamma^{-1}([f_m])(1).$$

Then $\tilde{f} \in C^\infty(M)$, since for any m_0 in a small neighborhood of 1 we have $f_m = f_{mm_0}$ as $f \in C^\infty(M)$. Since Γ is Z_M-equivariant, we have

$$\tilde{f}(tm) = \Gamma^{-1}([f_m])(t)$$

for all $t \in Z_M$. Therefore,

$$\tilde{f}_m \in \mathcal{F}(Z_M)_{\chi,n}$$

for all $m \in M$. Thus,

$$\tilde{f} \in C^\infty(M)_{\chi,n}.$$

Finally, by the last part of the Lemma there exists $\epsilon > 0$ such that

$$\tilde{f}(m) = f_m(1) = f(m)$$

for all $m \in M_{<\epsilon}$. It follows that $\iota_M(\tilde{f}) = [f]$ as required. □

3. The main result

Let π be a smooth representation of G. For any parabolic subgroup $P \supseteq B$ let $J_P(\pi)$ denote the Jacquet module of π with respect to P, viewed as a smooth representation of M. Let $\mathcal{E}_P(\pi)$ denote the set of cuspidal exponents of π along P, i.e., those $\chi \in \widehat{Z_M}$ such that $J_P(\pi)_\chi$, the χ-generalized eigenspace of $J_P(\pi)$, contains a supercuspidal constituent. Set

$$\mathcal{E}(\pi) = \bigcup_P \mathcal{E}_P(\pi) \neq \emptyset.$$

If π is of finite length, then $\mathcal{E}(\pi)$ is finite. If π is irreducible, then there exists P such that $\mathcal{E}_Q(\pi) = \emptyset$ unless Q and P are associated.

Fix a non-degenerated character $\psi : U_0 \to \mathbb{C}^*$ of U_0, that is, $\psi|_{U_\alpha} \not\equiv 1$ for all $\alpha \in \Delta_0$. Let $\Omega(G)$ be the G-space of smooth function $W : G \to \mathbb{C}$ such that $W(ug) = \psi(u)W(g)$ for all $u \in U_0$, $g \in G$, with G acting by right translation.

Suppose that π is an irreducible generic representation of G. That is, π can be realized as a subspace $\mathcal{W}(\pi)$ of $\Omega(G)$. The space $\mathcal{W}(\pi)$ is uniquely determined by the equivalence class of π and is called the Whittaker model of π. (Cf. [Sha74], [GK75], [BZ76], [Rod73].)

Theorem 3.1. *Let $(\pi, \mathcal{W}(\pi))$ be a subrepresentation of $\Omega(G)$ of finite length. Then there exists n such that any $W \in \mathcal{W}(\pi)$ can be written as*

$$W(utk) = \psi(u) \sum_{P \supseteq B} \delta_P^{\frac{1}{2}}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}(t,k) \quad t \in T_0, u \in U_0, k \in K$$

where $\phi_{P,\chi}(\cdot, k) \in \mathfrak{F}_{P,\chi,n}$ for all $k \in K$ and $\phi_{P,\chi}$ is invariant under an open subgroup of K.

Proof. We will prove the theorem by induction on the semi-simple rank of G, the case where G is a torus being trivial. Of course, we are primarily interested in irreducible representations. However, we need the finite length assumption to make the induction work.

First note that by considering finitely many translates of W it is enough to prove the statement for k being the identity element. Consider the map

$$W \in \mathcal{W}(\pi) \mapsto [\delta_P^{-\frac{1}{2}} W|_M].$$

We claim that it factors through the Jacquet module $J_P(\pi)$ and gives rise to an intertwining map

$$\kappa_M : J_P(\pi) \to \mathcal{G}(M).$$

Indeed, let $u \in U$. Then for $m \in M_{<\epsilon}$ for ϵ sufficiently small we have

$$W(mu) = W(mum^{-1}m) = \psi(mum^{-1})W(m) = W(m)$$

so that the germs of W and $\pi(u)W$ coincide. The equivariance property is clear (because of the $\delta_P^{\frac{1}{2}}$-shift in the definition of the Jacquet functor).

Since $J_P(\pi)$ is of finite length, all vectors in $J_P(\pi)$ and the image of κ_M are Z_M-finite. Therefore, we get an M-equivariant map

$$\Xi_M := \iota_M^{-1} \circ \kappa_M : J_P(\pi) \to C^\infty(M)_{Z_M\text{-fin}}$$

where ι_M is as in Corollary 2.11. Clearly, the image lies in the space $\Omega(M)$ of Whittaker functions on M.

Next, observe that $W(t) = 0$ if $|\alpha(t)| \gg 1$ for some $\alpha \in \Delta_0$. (Cf. [JPSS81, p. 204] for the GL_n case.) Indeed, fix $u \in U_\alpha$ so that $\psi(u) \neq 1$. By the property of t, $u' = t^{-1}ut$ is very close to 1, and therefore W is right invariant under u'. Hence,

$$W(t) = W(tu') = W(ut) = \psi(u)W(t)$$

so that $W(t) = 0$ as required.

We are now ready for the induction step. By passing to a direct summand we may assume that $\mathcal{W}(\pi) = \mathcal{W}(\pi)_\mu$ for some $\mu \in \widehat{Z_G}$. For any $\emptyset \neq I \subseteq \Delta_0$ let $P_I = M_I U_I$ be the proper parabolic subgroup of G such that $\Delta_0 \setminus \Delta_0^{P_I} = I$. Let $j_I : \pi \to J_{P_I}(\pi)$ be the canonical projection. By the transitivity of the Jacquet functor and the induction hypothesis applied to $J_{P_I}(\pi)$ we can write

$$[\Xi_{M_I} j_I(W)](t) = \sum_{P \subseteq P_I} \delta_{P\cap M_I}^{\frac{1}{2}}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi^I_{P,\chi}(t)$$

where $\phi^I_{P,\chi} \in \mathfrak{F}^{M_I}_{P\cap M_I,\chi}$. In other words, there exists $\epsilon > 0$ such that

$$W(t) = \sum_{P \subseteq P_I} \delta_P^{\frac{1}{2}}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi^I_{P,\chi}(t) \tag{3.1}$$

provided that $|\alpha|(t) < \epsilon$ for all $\alpha \in \Delta_{P_I}$. Note that both sides of (3.1) vanish if $|\alpha(t)|$ is large for some $\alpha \in \Delta_0^{P_I}$. Therefore, it follows from Remark 2.1 that for an appropriate $\epsilon > 0$, (3.1) holds whenever $|\alpha(t)| < \epsilon$ for all $\alpha \in I$. Fix such ϵ which works for all $I \neq \emptyset$. Set

$$\phi_{P,\chi}(t) = \sum_{\emptyset \neq I \subseteq \Delta_0 \setminus \Delta_0^P} (-1)^{|I|-1} \phi^I_{P,\chi}(t) \prod_{\alpha \in I} \mathbf{1}_{<\epsilon}(|\alpha(t)|)$$

where $\mathbf{1}_{<\epsilon}$ denotes the characteristic function of $(0, \epsilon)$. Then $\phi_{P,\chi} \in \mathfrak{F}_{P,\chi}$ and

$$\sum_{P \subsetneq G} \sum_{\chi \in \mathcal{E}_P(\pi)} \delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t) = W(t) \sum_{\emptyset \neq I \subseteq \Delta_0} (-1)^{|I|-1} \prod_{\alpha \in I} \mathbf{1}_{<\epsilon}(|\alpha(t)|).$$

By the inclusion-exclusion principle we get

$$\sum_{P \subsetneq G} \sum_{\chi \in \mathcal{E}_P(\pi)} \delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t) = \begin{cases} W(t) & \text{if } |\alpha(t)| < \epsilon \text{ for some } \alpha \in \Delta_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let Q be such that $\mathcal{E}_Q(\pi) \neq \emptyset$ and take any $\omega \in \mathcal{E}_Q(\pi)$. Then $\omega|_{Z_G} = \mu$ and therefore

$$\phi'(t) := \delta_Q^{-\frac{1}{2}}(t) W(t) \prod_{\alpha \in \Delta_0} \mathbf{1}_{\geq\epsilon}(|\alpha(t)|) \in \mathfrak{F}_{G,\mu} \subseteq \mathfrak{F}_{Q,\omega}$$

by Lemma 2.6. The conclusion of the theorem holds upon replacing $\phi_{Q,\omega}$ by $\phi_{Q,\omega} + \phi'$. □

The theorem just proved shows that the asymptotic behavior of the Whittaker functions are completely governed by the exponents of the representation. It is natural to ask to what extent the converse is true. Namely,

Question 3.2. What is the kernel of the maps κ_M?

Suppose that π is irreducible, generic and has a unitary central character. For $W_1, W_2 \in \mathcal{W}(\pi)$ and $\lambda \in (\mathfrak{a}_0^G)^*_{\mathbb{C}}$, define

$$I(W_1, W_2, \lambda) = \int_{T_G U_0 \backslash G} q^{-\langle \lambda, H(g) \rangle} W_1(g) \overline{W_2(g)}\, dg.$$

Corollary 3.3. *The integral* $I(W_1, W_2, \lambda)$ *is absolutely convergent whenever*

$$\langle \operatorname{Re} \lambda + 2 \operatorname{Re} \chi, \varpi^\vee \rangle > 0$$

for all P, $\varpi^\vee \in \hat{\Delta}_P^\vee$ *and* $\chi \in \mathcal{E}_P(\pi)$. *It extends to a rational function in* $q^{\langle \lambda, \omega_i^\vee \rangle}$, $i = 1, \dots, d$ *with poles at most along*

$$q^{\langle \lambda, H(t_\alpha) \rangle} = \chi_1 \overline{\chi_2}(t_\alpha)$$

where $\chi_i \in \mathcal{E}_{P_i}(\pi)$, $i = 1, 2$ *and* $\alpha \in \Delta_0 \setminus (\Delta_0^{P_1} \cup \Delta_0^{P_2})$ *satisfy*

$$\chi_1 \equiv \chi_2 \text{ on } Z_\alpha^1.$$

Here $\omega_1^\vee, \dots, \omega_d^\vee$ *and* t_α *are as in Lemma* 2.8.

Proof. Denote the integrand of $I(W_1, W_2, \lambda)$ by $F(g)$. It is evidently left $T_G U_0$-invariant. We use the integration formula

$$\int_{T_G U_0 \backslash G} f(g)\, dg = \int_K \int_{T_G \backslash T_0} f(tk) \delta_B(t)^{-1}\, dt\, dk$$

for any left $T_G U_0$-invariant continuous function on G. By Theorem 3.1, $F(tk)$ is a linear combination of functions of the form

$$q^{-\langle \lambda, H(t) \rangle} \delta_{P_1}^{\frac{1}{2}}(t) \delta_{P_2}^{\frac{1}{2}}(t) \phi^1_{P_1, \chi_1}(t, k) \overline{\phi^2_{P_2, \chi_2}(t, k)}$$

where $\phi^i_{P_i, \chi_i}(\cdot, k) \in \mathfrak{F}_{P_i, \chi_i}$ for $i = 1, 2$. Thus the integral $I(W_1, W_2, \lambda)$ is a finite sum of integrals of the form

$$\int_{T_G \backslash T_0} q^{-\langle \lambda, H(t) \rangle} \delta_{P_1}^{\frac{1}{2}}(t) \delta_{P_2}^{\frac{1}{2}}(t) \phi_1(t) \overline{\phi_2(t)} \delta_B(t)^{-1}\, dt$$

where $\phi_i \in \mathfrak{F}_{P_i, \chi_i}$ for $i = 1, 2$. From Lemma 2.6,

$$\phi_1 \overline{\phi_2} \in \mathfrak{F}_{P_1 \bullet P_2, \chi_1 \overline{\chi_2}|_{T_{M_1 \bullet M_2}}}.$$

On the other hand, it follows from (2.2) that

$$\delta_{P_1}^{\frac{1}{2}} \delta_{P_2}^{\frac{1}{2}} \equiv \delta_B \text{ on } T_{M_1 \bullet M_2}.$$

By (2.17) we infer that

$$\delta_{P_1}^{\frac{1}{2}} \delta_{P_2}^{\frac{1}{2}} \delta_B^{-1} \phi_1 \overline{\phi_2} \in \mathfrak{F}_{P_1 \bullet P_2, \chi_1 \overline{\chi_2}|_{T_{M_1 \bullet M_2}}}$$

as well. We note that if

$$\langle \operatorname{Re} \lambda + 2 \operatorname{Re} \chi, \varpi^\vee \rangle > 0$$

for all P, $\varpi^\vee \in \hat{\Delta}_P^\vee$ and $\chi \in \mathcal{E}_P(\pi)$, then

$$\langle \operatorname{Re} \lambda + \operatorname{Re} \chi_1 + \operatorname{Re} \chi_2, \varpi^\vee \rangle > 0$$

for all P_1, P_2, $\varpi^\vee \in \hat{\Delta}^\vee_{P_1 \bullet P_2}$, $\chi_1 \in \mathcal{E}_{P_1}(\pi)$, $\chi_2 \in \mathcal{E}_{P_2}(\pi)$. The corollary now follows from Lemma 2.8, part 1. $\square$

Corollary 3.4. *Suppose that π is square-integrable and generic. Then the form*

$$(W_1, W_2) = \int_{T_G U_0 \backslash G} W_1(g)\overline{W_2(g)}\, dg$$

defines a non-zero G-invariant inner product on $\mathcal{W}(\pi)$.

Proof. The above integral, if convergent, is clearly G-invariant and positive-definite. We only need to check convergence. This follows from the previous corollary and the characterization of square-integrable representations by the positivity of their exponents, namely $\langle \operatorname{Re}\chi, \varpi^\vee \rangle > 0$ for all $\varpi^\vee \in \hat{\Delta}_P^\vee$ and $\chi \in \mathcal{E}_P(\pi)$. □

We conjecture the following converse to Corollary 3.4, which is related to Question 3.2 above.

Conjecture 3.5. Let π be a generic representation, and suppose that

$$\int_{T_G U_0 \backslash G} |W(g)|^2\, dg < \infty$$

for any $W \in \mathcal{W}(\pi)$. (It is enough to require this for a single $0 \neq W \in \mathcal{W}(\pi)$.) Then π is square-integrable.

Finally, we extend Corollary 3.4 to the tempered case.

Corollary 3.6. *Suppose that π is tempered. Fix $\omega = \sum_{\alpha \in \Delta_0} n_\alpha \alpha \in X^*(T_0)$ with $n_\alpha > 0$ for all $\alpha \in \Delta_0$. Then there exists $r = r(\pi, \omega) \in \mathbb{N}$ such that for any $W_1, W_2 \in \mathcal{W}(\pi)$,*

$$\int_{g \in T_G U_0 \backslash G : \langle \omega, H(g) \rangle \leq n} W_1(g)\overline{W_2(g)}\, dg \sim [W_1, W_2] n^r \quad \text{as } n \to \infty$$

where $[\cdot,\cdot]$ is a non-zero invariant (positive-definite) inner product on $\mathcal{W}(\pi)$.

Proof. As before, it follows from Corollary 3.3 that $I(W_1, W_2, s\omega)$ is absolutely convergent for $\operatorname{Re}(s) > 0$ and extends to a rational function in q^s. Let $r = r(W)$ be the order of the pole of $I(W, W, s\omega)$ at $s = 0$. Fix $W \in \mathcal{W}(\pi)$ and write

$$I(W, W, s\omega) = \sum a_n q^{-ns}$$

for $\operatorname{Re} s > 0$ where we set

$$a_n = a_n(W) := \int_{T_G U_0 \backslash G} \delta_{\langle \omega, H(g) \rangle, n} |W(g)|^2\, dg \geq 0 \quad n \in \mathbb{Z}.$$

By Theorem 3.1 and Lemma 2.8, part 2 the sequence a_n is e.p.e.. By Lemma 2.5 we have

$$\int_{g \in T_G U_0 \backslash G : \langle \omega, H(g) \rangle \leq n} |W(g)|^2\, dg \sim [W, W] n^{r(W)} \tag{3.2}$$

as $n \to \infty$ where

$$[W, W] := \lim_{s \to 0} \frac{(s \log q)^{r(W)} I(W, W, s\omega)}{r(W)!}.$$

Since

$$a_n(W_1 + W_2) \leq 2(a_n(W_1) + a_n(W_2)) \quad n \in \mathbb{Z}$$

for all $W_1, W_2 \in \mathcal{W}(\pi)$, it also follows from Lemma 2.5 that

$$r(W_1 + W_2) \leq \max(r(W_1), r(W_2)).$$

For $x \in G$, let $W_x(g) = W(gx)$. We show that

$$[W_x, W_x] = [W, W].$$

It is a well-known fact that

$$\{H(g) - H(gx) : g \in G\}$$

is a finite set (depending on x). Thus, there exists C such that

$$\begin{aligned} &\left| \int_{T_G U_0 \backslash G} \mathbf{1}_{\leq n}(\langle \omega, H(g) \rangle) |W_x(g)|^2 \, dg - \int_{T_G U_0 \backslash G} \mathbf{1}_{\leq n}(\langle \omega, H(g) \rangle) |W(g)|^2 \, dg \right| \\ &= \left| \int_{T_G U_0 \backslash G} \mathbf{1}_{\leq n}(\langle \omega, H(gx^{-1}) \rangle) |W(g)|^2 \, dg - \int_{T_G U_0 \backslash G} \mathbf{1}_{\leq n}(\langle \omega, H(g) \rangle) |W(g)|^2 \, dg \right| \\ &\leq \sum_{|m-n| \leq C} a_m. \end{aligned}$$

By Lemma 2.5 the right-hand side is $O(n^{r(W)-1})$ as $n \to \infty$. It follows once again from Lemma 2.5 that $r(W_x) = r(W)$ and

$$\lim_{s \to 0} s^r I(W_x, W_x, s\omega) = \lim_{s \to 0} s^r I(W, W, s\omega).$$

Thus $[W_x, W_x] = [W, W]$. Moreover by irreducibility, $r = r(W)$ is independent of $W \neq 0$. Thus by polarization, $[\cdot, \cdot]$ defines a G-invariant form on $\mathcal{W}(\pi)$. The fact that $[\cdot, \cdot]$ is positive-definite follows from (3.2). $\square$

Example 3.7. Consider $G = PGL_2$. Let π be the unramified tempered representation

$$\mathrm{Ind}_B^G \left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \mapsto |a|^\lambda \right), \ \lambda \in \mathrm{i}\mathbb{R}.$$

View ψ as a character of F and let $\mathfrak{c}(\psi)$ be the maximal fractional ideal on which ψ is trivial. It is well known (cf. [God70]) that the unramified Whittaker function normalized by $W(e) = 1$ is given by

$$|a|^{-\frac{1}{2}} W\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right)\right) = \begin{cases} \frac{|a|^\lambda}{1-q^{2\lambda}} + \frac{|a|^{-\lambda}}{1-q^{-2\lambda}} & \text{if } a \in \mathfrak{c}(\psi) \text{ and } \lambda \notin \frac{\pi \mathrm{i}}{\log q}\mathbb{Z}, \\ |a|^\lambda (1 + v(a)) & \text{if } a \in \mathfrak{c}(\psi) \text{ and } \lambda \in \frac{\pi \mathrm{i}}{\log q}\mathbb{Z}, \\ 0 & \text{if } a \notin \mathfrak{c}(\psi). \end{cases}$$

It follows that for any positive ω, $r = 1$ if $\lambda \notin \frac{\pi \mathrm{i}}{\log q}\mathbb{Z}$ and $r = 3$ otherwise.

In general, let G be any split group and χ a regular unramified character of T_0, that is, $\chi^w \neq \chi$ for all $w \neq 1$ in the Weyl group of G. The representation $\mathrm{Ind}_B^G \chi$ is irreducible and it follows readily from the Casselman-Shalika formula for the unramified Whittaker function ([CS80]), that $r(\mathrm{Ind}_B^G \chi, \omega)$ is the semi-simple rank of G (regardless of ω).

One can contemplate the following conjecture related to Conjecture 3.5.

Conjecture 3.8. *Suppose that π is the generic constituent of $\mathrm{Ind}_P^G \tau$ where τ is a square-integrable generic representation of the Levi part of P. Suppose that the Plancherel measure on the component*

$$\mathrm{Ind}_P^G \tau q^{\langle \lambda, H(\cdot) \rangle}, \quad \lambda \in \mathrm{i}\mathfrak{a}_M^* / \frac{2\pi \mathrm{i}}{\log q} X^*(M)$$

is given by $\mu(\tau, \lambda) \, d\lambda$ where $\mu(\tau, \lambda) = |c_\tau(\lambda)|^{-2}$. Then $r(\pi, \omega)$ is equal to the sum of the co-rank of P and the order of zero of $\mu(\tau, s\omega)$ at $s = 0$.

Remark 3.9. Let $\tilde{G}$ be the metaplectic cover of Sp_{2n}. One can define parabolic subgroups as the inverse images of parabolic subgroups of Sp_{2n}. The Jacquet functors are defined in an analogous way and satisfy the usual properties. In particular, they control the asymptotics of the matrix coefficients of the representation and the criterion for square-integrability is the same as for linear groups. Details will appear in a forthcoming paper of Szpruch. The notions of non-degenerate characters and generic representations are also defined and the uniqueness of Whittaker model is proved in [Szp07]. The results of this paper as well as the proofs immediately carry over to $\tilde{G}$.

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INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM ISRAEL, 91904
E-mail address: `erezla@math.huji.ac.il`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NEW JERSEY 07102
E-mail address: `zmao@rutgers.edu`