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A relative Kuznietsov trace formula on G_2

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Abstract. In this paper, we set up the general formulation to study distinguished residual representations of a reductive group G by the relative trace formula approach. This approach simplifies the argument of [JR], which deals with this type of relative trace formula for a special symmetric pair $(GL(2n), Sp(2n))$ and also works for non-symmetric, spherical pairs. To illustrate our idea and method, we complete our relative trace formula (both the geometric side identity and the spectral side identity) for the case $(G_2, SL(3))$.

1. Introduction

One of the major problems in the modern theory of automorphic forms is to understand the discrete spectrum of the space of square integrable automorphic functions over $G(\mathbb{A})$, where G is a reductive algebraic group defined over a number field F and $\mathbb{A}$ is the ring of adeles of F. The discrete spectrum of G is known to break into two parts, cuspidal spectrum and non-cuspidal spectrum. From the theory of Eisenstein series, the non-cuspidal spectrum can be realized by the residual representations of Eisenstein series of G associated to various cuspidal data.

The classification and parameterization of the cuspidal spectrum are extremely difficult in general and far from being understood. For recent progress, we refer to the work of J. Arthur [A] and of C. Moeglin [M]. However, the non-cuspidal residual spectrum seems more accessible. Following basically the traditional method, suggested by R. Langlands in his fundamental work on the theory of Eisenstein series of general reductive groups [L], some more cases of the classification and parameterization of residual spectrum appeared in [M,MW1,K,KS], cte.

Our objective on this topic is to formulate a relative trace formula method approach to understand the distinguished residual representations. The significant applications of the distinguished residual representations of various types of groups have been found in the recent publications, [GRS], [Jng] and [Jng1].

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The idea to formulate this type of relative trace formula is due to Jacquet and Rallis [JR1], where the case of the $Sp(2n)$-distinguished representations of $GL(2n)$ was considered. In this paper, a very simple argument will be provided to show the matching of the global geometric sides of the trace formulas and the matching of the local orbital integrals, and to prove the fundamental lemma in this situation. It is easy to see that the refined argument works for general cases, symmetric space cases and non-symmetric, spherical variety cases.

The more precise description of the set-up is as follows. A subgroup H of G is called spherical if over the algebraic closure of F, the Borel subgroup B of G has a Zariski open dense orbit of the quotient space G/H. Typical examples of spherical subgroups are (1) parabolic subgroups and (2) the subgroup of all fixed points of G under the action of an involutive automorphism. The classification of reductive spherical pairs (G, H) was given by M. Brion [B1]. When H is *not* reductive, the classification is not known at present as yet [B].

Assume that G is F-split reductive algebraic group. Let $P = MN$ be a maximal parabolic subgroup of G and σ an irreducible cuspidal automorphic representation of $M(\mathbb{A})$. Then one may form an induced representation

$$I(s, \sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes exp < s, H_P(\cdot) >)$$

and for a section $\Phi_{s,\sigma} \in I(s, \sigma)$, one define

$$E(g, s; \Phi_{s,\sigma}) = \sum_{\gamma \in P \backslash G} \Phi_{s,\sigma}(\gamma g),$$

the Eisenstein series associated to the section $\Phi_{s,\sigma}$. From the theory of Eisenstein series [L] and [MW], $E(g, s; \Phi_{s,\sigma})$ is meromorphic in s and has possibly finite number of simple poles when $Re(s) > 0$. The residual representation of $E(g, s; \Phi_{s,\sigma})$ at $s = s_0$ is denoted by $E_{s_0}(g, \sigma)$. The residual representation $E_{s_0}(g, \sigma)$ is called H-distinguished if the following integral

$$\mathcal{P}_H(E_{s_0}(\cdot, \sigma))(g) := \int_{[Z_G(\mathbb{A}) \cap H(\mathbb{A})] \cdot H(F) \backslash H(\mathbb{A})} E_{s_0}(hg, \sigma)\omega_\pi(h)^{-1}\, dh$$

exists and does not vanish. If H is not reductive, certain modification is needed. The integral is called period integral of the residue $E_{s_0}(\cdot, \sigma)$ over H.

To formulate the relative Kuznietsov trace formula, we simply assume that the spherical subgroup H is reductive and has a finite center. Let $f \in C_c^\infty(G(\mathbb{A}))$. One may form an automorphic kernel associated to f as follows:

$$K_f(x, y) := \sum_{\xi \in G(F)} f(x^{-1}\xi y).$$

The relative trace formula is defined by

$$I_{H,G}(f, \psi) := \int_{H(F) \backslash H(\mathbb{A})} \int_{U(F) \backslash U(\mathbb{A})} K_f(h, u)\psi(u) dh\, du$$

where U is the standard maximal unipotent subgroup of G and $\psi(u)$ is a character of $U(F)\backslash U(\mathbb{A})$. The choice of this additive character depends on the functorial lifting data. In this paper, we deal with the distinguished residual representations induced from the generic cuspidal data. The character ψ is naturally chosen as follows: for any $u \in U$, one can write $u = nu_M$, where $u_M \in M \cap U$, and

$$\psi(u) := \psi_M(u_M)$$

where ψ_M is the generic character of U_M, the maximal unipotent radical in M. On the other hand, we can define a relative Kuznietsov trace formula on M. Let $f' \in C_c^\infty(M(\mathbb{A}))$. An automorphic kernel associated to f' is defined by

$$K_{f'}(x, y) := \sum_{\xi \in M(F)} f'(x^{-1}\xi y).$$

The relative trace formula is defined by

$$I_{H_M,M}(f', \psi_M) := \int_{H_M(F)\backslash H_M(\mathbb{A})} \int_{U_M(F)\backslash U_M(\mathbb{A})} K_{f'}(h, u)\psi_M(u)dh\, du,$$

where $H_M := H \cap M$. The idea is to prove the following identity

$$I_{H,G}(f, \psi) = I_{H_M,M}(f', \psi_M) \tag{1.1}$$

for any matching pair (f, f'). The spectral side of this identity reflects the Langlands lifting from M to G.

One of the key points of this approach is to parameterize the distinguished residual representations by means of the cuspidal data, based on the idea developed in [JR] and [Jng] that the "outer" periods of the residual representations of $G(\mathbb{A})$ is comparable with the "inner" periods of the cuspidal representations on the Levi subgroup $M(\mathbb{A})$. The non-vanishing of certain special values of L-functions will eventually play the role in the parameterization by studying the relations between the special values or residues of L-functions and periods of automorphic forms, which is another important focus in the recent study of automorphic representations and L-functions, see [Jng1] for instance.

It must be mentioned that in general, one may not have an identity like (1.1), as observed in [Jng2]. However, we believe that if H is a subgroup consisting of the fixed points of an involution, the identity (1.1) should be true. In fact, one can reformulate the calculation for the case $\mathrm{GL}(2n), \mathrm{Sp}(2n))$ done in [JR1] as (1.1).

To illustrate our idea and method to establish identity (1.1) for general spherical varieties, we show in this paper that under the cuspidality assumption at one finite prime, one can have identity (1.1) for the case where $G = G_2$ and $H = SL(3)$. Based on the result concerning the period of residual representations in [Jng], we give a complete description of the spectral identity of the relative Kuznietsov trace formula. For the other cases like $(\mathrm{SO}(7), G_2)$ and (D_4, G_2) treated in [Jng] and $(\mathrm{Sp}(4n), \mathrm{Sp}(2n) \times \mathrm{Sp}(2n))$ in [GRS], the same approach works. The example discussed in this paper will serve as a model for this relative trace formula approach to study the distinguished residual representations.

2. Geometric side of the trace formula

We recall the some basic structure of the exceptional group G_2 from [Jng].

Let $G = G_2$. It has two simple roots: the long simple root α and the short simple root β. Then the positive roots are

$$\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta.$$

For each root γ, we denote by w_γ the Weyl group element associated to the simple reflection along γ and by $\chi_\gamma(t)$ the additive subgroup attached to γ

The following double coset decomposition is from [Jng], which will be needed to establish the trace formula.

Lemma 2.1. *Let $P = MN$ be the maximal parabolic subgroup of $G = G_2$, whose Levi part is generated by the short root α, and H be the subgroup generated by all long roots, which is isomorphic to $SL(3)$. Then one has*

(1) *$G = [HP] \cup [H\mu P]$, where $\mu = w_\alpha \chi_{-\alpha-\beta}(1)$;*
(2) *for each μ_i ($\mu_0 = e$ the identity element of G and $\mu_1 = \mu$), $H\mu_i P = H\mu_i N[M_{\mu_i} \backslash M]$, where $M_{\mu_i} = p_M(NM \cap \mu_i^{-1} H\mu_i)$, the projection to M of the intersection. More precisely, we have*

$$M_e = H \cap M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } M_{\mu_1} = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}.$$

2.1. Relative Kuznietsov trace formula

Let $f \in C_c^\infty(G(\mathbb{A}))$. One may form an automorphic kernel associated to f as follows:

$$K_f(x, y) := \sum_{\xi \in G(F)} f(x^{-1}\xi y).$$

We consider the following type of distribution, which is called the relative Kuznietsov trace formula ([JR1]),

$$I_{H,G}(f, \psi) = \int_{H(F)\backslash H(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} K_f(h, u)\psi(u) dh\, du$$

where U is the standard maximal unipotent subgroup of G and $\psi(u)$ is a character of $U(F)\backslash U(\mathbb{A})$, defined by

$$\psi(u) = \psi_M(u_M)$$

where we write $U = U_M \cdot N$ and ψ_M is a fixed generic character of U_M. The point here is to relate this distribution on G to a similar type of distribution of the Levi subgroup M, so that the spectral side of both distributions provides the information about the Langlands lifting from M to G. By Lemma 2.1, we unfold the integral as follows.

$$\begin{aligned} I_{H,G}(f,\psi) &= \int_{H(F)\backslash H(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})}\sum_{\xi\in G(F)} f(h^{-1}\xi u)\psi(u)dh\,du \\ &= \sum_{\xi\in H\backslash G}\int_{H(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} f(h\xi u)\psi(u)dh\,du \\ &= \sum_{\xi\in H\backslash HNM}\int_{H(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} f(h\xi u)\psi(u)dh\,du \\ &\quad + \sum_{\xi\in H\backslash H\mu NM}\int_{H(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} f(h\xi u)\psi(u)dh\,du \\ &= I^e + I^\mu. \end{aligned}$$

We remark that from representation theoretic point of view, the second term I^μ is mysterious. In the following, we shall show that there are abundant functions f in $C_c^\infty(G(\mathbb{A}))$, such that the second term I^μ vanishes. Let v_0 be a fixed finite place of the number field F. Let π_{v_0} be an irreducible admissible supercuspidal representation of $M(F_{v_0}) = GL_2(F_{v_0})$ with trivial central character, (therefore is self-dual). Choose a nonzero vector $u_0 \in V_{\pi_{v_0}}$. By admissibility of the representation π_{v_0}, the stabilizer of u_0 is open and compact in $M(F_{v_0})$ modulo the center $Z(F_{v_0})$ in $M(F_{v_0})$. Let $S_M(u_0)$ be a compact open group with $ZS_M(u_0)$ being the stabilizer of u_0. Take K to be a compact open subgroup of $G(F_{v_0})$, the intersection of which with $M(F_{v_0})$ equals $S_M(u_0)$. Define a function ϕ on $G(F_{v_0})$ by

$$\phi(g) = \begin{cases} 0, & \text{if } g \notin MK, \\ \pi_{v_0}(m)\cdot u_0, & \text{if } g = mk \in MK. \end{cases} \tag{2.1}$$

Then the function ϕ belongs to $C_c^\infty(G(F_{v_0}), V_{\pi_{v_0}})$ with the following properties:

(1) The function ϕ has a compact support modulo M to the left.
(2) For $m \in M(F_{v_0})$,

$$\phi(mg) = \pi_{v_0}(m)\cdot\phi(g).$$

Define a function $\Phi(g)$ by

$$\Phi(g) := <\phi(g), u_0'> \tag{2.2}$$

the matrix coefficient attached to the vectors $\phi(g)$ and some nonzero vector u_0' in $V_{\pi_{v_0}}$. Since π_{v_0} is supercuspidal, it is easy to check that the function $\Phi(g)$ belongs to $C_c^\infty(G(F_{v_0}))$. Take f in $C_c^\infty(G(\mathbb{A}))$ to be

$$f = \otimes_v f_v, \tag{2.3}$$

where for $v = v_0$, $f_{v_0} = \Phi$, as defined above. We call such a function a test function with local supercuspidal data (M, π_{v_0}).

With any such chosen function f in $C_c^\infty(G(\mathbb{A}))$, we shall show that the second term I^μ vanishes. In fact, we have

$$I^\mu = \sum_{\xi \in H\backslash H\mu NM} \int_{H(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} f(h\xi u)\psi(u) dh\, du$$

$$= \sum_{\xi \in H\backslash H\mu NM} \int_{U_M(F)\backslash U_M(\mathbb{A})} \int_{H(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} f(h\xi nu)\psi_M(u) dh\, dn\, du.$$

By changing variables, the summation over $H\backslash H\mu NM$ can be expressed as a summation over $\mu(M_\mu\backslash M) \times (N_\mu\backslash N)$, where $N_\mu = \chi_{\alpha+3\beta} = \mu^{-1}H\mu \cap N$. Then we have

$$I^\mu = \sum_{\xi \in M_\mu\backslash M} \int_{U_M(F)\backslash U_M(\mathbb{A})} \int_{H(\mathbb{A})} \int_{N_\mu(F)\backslash N(\mathbb{A})} f(h\mu n\xi u)\psi_M(u) dh\, dn\, du$$

$$= \sum_{\xi \in M_\mu\backslash M} \int_{U_M(F)\backslash U_M(\mathbb{A})} \int_{H(\mathbb{A})} \int_{[N_\mu\backslash N](\mathbb{A})} f(h\mu n\xi u)\psi_M(u) dh\, dn\, du.$$

Now we can show that for any $m \in M(\mathbb{A})$, the following integral

$$\int_{H(\mathbb{A})} \int_{[N_\mu\backslash N](\mathbb{A})} f(h\mu nm) dh\, dn$$

vanishes. This implies the vanishing of the second term I^μ. Since f is factorizable, the integral is an Euler product of the local integrals

$$\int_{H(F_v)} \int_{[N_\mu\backslash N](F_v)} f_v(h\mu nm) dh\, dn.$$

For $v = v_0$, we have

$$\int_{H(F_v)} \int_{[N_\mu\backslash N](F_v)} f_{v_0}(h\mu nm) dh\, dn$$

$$= \int_{H(F_v)} \int_{[N_\mu\backslash N](F_v)} \Phi(h\mu nm) dh\, dn$$

$$= \int_{H(F_v)} \int_{[N_\mu\backslash N](F_v)} < \phi(h\mu nm), u_0' > dh\, dn.$$

We observe the following relation

$$\chi_\alpha(x)\chi_{\alpha+3\beta}(-x)\mu = \mu\chi_\beta(x)\chi_{\alpha+2\beta}(x)\chi_{\alpha+3\beta}(x^2)\chi_{2\alpha+3\beta}(x)$$

Since $U_M = \chi_\beta$, after changing the variables in the integration over N, we notice that the integral defines an U_M-invariant functional

$$L(u_0') = \int_{H(F_v)} \int_{[N_\mu\backslash N](F_v)} < \phi(h\mu nm), u_0' > dh\, dn$$

over $V_{\pi_{v_0}}$. By the supercuspidality of the representation π_{v_0}, the functional $L(u_0')$ must be zero, thus the integral I^μ must be zero. We have shown:

Proposition 2.1. *Let $f \in C_c^\infty(G(\mathbb{A}))$ be as in (2.3). Then $I^\mu(f, \psi) = 0$ and*

$$I_{H,G}(f, \psi) = I^e(f, \psi).$$

2.2. Unwind the distributions

We will compare the distribution $I^e(f, \psi)$ with a distribution $I_{M_e,M}(f', \psi_M)$ on M, defined by

$$I_{M_e,M}(f', \psi_M) = \int_{Z(\mathbb{A})M_e(F)\backslash M_e(\mathbb{A})} \int_{U_M(F)\backslash U_M(\mathbb{A})} K_{f'}(t, u)\psi_M(u)dt\, du$$

where for $f' \in C_c^\infty(M(\mathbb{A}))$, the kernel function $K_{f'}(x, y)$ is defined to be:

$$K_{f'}(x, y) = \int_{Z(F)\backslash Z(\mathbb{A})} \sum_{\xi \in M(F)} f'(zx^{-1}\xi y)dz.$$

We will unwind the distributions $I^e(f, \psi)$ and $I_{M_e,M}(f', \psi_M)$ into the sums of orbital integrals.

Consider the Iwasawa decomposition of $H(\mathbb{A})$, i.e.

$$H(\mathbb{A}) = K_H T_H(\mathbb{A})U_H(\mathbb{A}).$$

For $f_v \in C_c^\infty(G(F_v))$, define:

$$F_{f,v}(m) := |\det m|_v^2 \cdot \int_{K_H(F_v)} \int_{N(F_v)} f(kmn)dk\, dn \tag{2.4}$$

and

$$F_f := \otimes_v F_{f,v}. \tag{2.5}$$

We have:

Proposition 2.2. *Let $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}))$, then*

$$I^e(f, \psi) = \sum_{\xi_m \in H\cap M\backslash M/U_M} c(\xi_m) \prod_v \mathcal{O}(\xi_m, F_{f,v}, \psi_M)$$

Here

$$\mathcal{O}(\xi_m, F_{f,v}, \psi_M) = \int_{M_e(F_v)} \int_{U_{\xi_m}(F_v)\backslash U_M(F_v)} F_{f,v}(t\xi_m u)\psi_M(u)dt\, du$$

where U_{ξ_m} is the fixator of the coset $M_e\xi_m$ in U_M, and

$$c(\xi_m) = \int_{U_{\xi_m}(F)\backslash U_{\xi_m}(\mathbb{A})} \psi_M(u)du$$

Remark 2.1. If $c(\xi_m) \neq 0$, then we say the orbit of ξ_m is *relevant*. For the purpose of comparing $I^e(f, \psi)$ and $I_{M_e,M}(f', \psi_M)$, one does not need to classify which orbit is relevant, unlike in some other cases of the relative trace formula.

Proof. From Lemma 2.1, one has

$$H\backslash HNM = [H\cap N\backslash N][H\cap M\backslash M],$$

any ξ can be written as $\xi = \xi_n\xi_m$ with $\xi_n \in H\cap N\backslash N$ and $\xi_m \in H\cap M\backslash M$. It follows that

$$I^e(f,\psi) = \sum_{\xi\in H\backslash HNM}\int_{H(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})}\int_{U_M(F)\backslash U_M(\mathbb{A})} f(h\xi nu)\psi(u)dh\,dn\,du$$
$$= \sum_{\xi_m\in H\cap M\backslash M}\int_{H(\mathbb{A})}\int_{[H\cap N\backslash N](\mathbb{A})}\int_{U_M(F)\backslash U_M(\mathbb{A})} f(hn\xi_m u)\psi(u)dh\,dn\,du.$$

One notice from Lemma 2.1 that $H\cap M$ is the maximal torus T_H in H (and also a maximal torus in M) and $U_H = H\cap N$. It follows that

$$I^e(f,\psi) =$$
$$\sum_{\xi_m\in H\cap M\backslash M}\int_{K_H}\int_{T_H(\mathbb{A})}\int_{N(\mathbb{A})}\int_{U_M(F)\backslash U_M(\mathbb{A})} f(ktn\xi_m u)\psi(u)\delta_{B_H}(t)dk\,dt\,dn\,du$$
$$= \sum_{\xi_m\in H\cap M\backslash M}\int_{T_H(\mathbb{A})}\int_{U_M(F)\backslash U_M(\mathbb{A})}\int_{K_H}\int_{N(\mathbb{A})} f(kt\xi_m un)\psi(u)\delta_{B_H}(t)dk\,dn\,dt\,du.$$

With the definition of $F_f = \otimes_v F_{f,v}$ (as in (2.5)), and note that for $m = h(a,b)$, one has

$$\delta_{B_H}(m) = |\det m|^2.$$

we see the integral I^e can be written as

$$I^e(f,\psi) = \sum_{\xi_m\in H\cap M\backslash M}\int_{T_H(\mathbb{A})}\int_{U_M(F)\backslash U_M(\mathbb{A})} F_f(t\xi u)\psi_M(u)dt\,du$$

The proposition follows from the fact that $M_e = T_H$. $\square$

The unwinding of $I_{M_e,M}(f',\psi_M)$ is standard, indeed it is given in [J], (also, in [J] the relevant orbits are classified). We have

Proposition 2.3. *If $f' = \otimes_v f'_v \in C_c^\infty(M(\mathbb{A}))$, then*

$$I_{M_e,M}(f',\psi_M) = \sum_{\xi_m\in H\cap M\backslash M/U_M} c(\xi_m)\prod_v \mathcal{O}(\xi_m, f', \psi_M)$$

Remark 2.2. If we take f'_v to be $F_{f,v}$, then from the above two propositions, we have $I^e(f,\psi) = I_{M_e,M}(f',\psi_M)$. To complete the comparison of $I^e(f,\psi)$ and $I_{M_e,M}(f',\psi_M)$, we need to show the above map from f_v to f'_v agrees with a Hecke algebra homomorphism.

2.3. *Matching of local orbital integrals: unramified case*

In this subsection, everything will be considered over a p-adic field F. The fundamental lemma for trace formula is to match the relevant local orbital integrals for the corresponding Hecke functions. We will see that in case under consideration, the correspondence of the Hecke functions on G and the Hecke functions on M is essentially given by the Jacquet functor. Let K_G and K_M are the maximal compact open subgroup in G and M, respectively, such that $K_G \cap M = K_M$. Let f be a function in the Hecke algebra $\mathcal{H}(G//K_G)$, we define f' by (2.5) which clearly is a function in the Hecke algebra $\mathcal{H}(M//K_M)$.

We shall prove that when restricted to the Hecke algebra elements, the map

$$f \mapsto f'$$

gives rise to a Hecke algebra homomorphism from $\mathcal{H}(G//K_G)$ to $\mathcal{H}(M//K_M)$, moreover, we will give a description of the homomorphism.

It is clear that the map takes the identity element f_0 of $\mathcal{H}(G//K_G)$ to the identity element f_0' of $\mathcal{H}(M//K_M)$. In general, we let π be an irreducible admissible unramified representation of M with trivial central character and ω_π be the corresponding spherical function. Then for any Hecke function f_M in $\mathcal{H}(M//K_M)$, one has

$$\int_M f_M(m)\omega_\pi(m)dm = \hat{f_M}(\pi),$$

and the map $f_M \mapsto \hat{f_M}$ is the corresponding character of the Hecke algebra $\mathcal{H}(M//K_M)$. For an unramified representation Π of G, the associated character $f \mapsto \hat{f}(\Pi)$ of the Hecke algebra $\mathcal{H}(G//K_G)$ is similarly defined.

It is well known that the unramified representation π has a nonzero linear functional $\mathcal{P}_M$ which is M_e-invariant. Let ϕ_π^0 be a nonzero K_M-invariant vector in π. We also assume that

$$\mathcal{P}_M(\phi_\pi^0) = 1.$$

We define the unitarily induced representation from $P = MN$ to G,

$$I(\frac{1}{2}, \pi) := \mathrm{Ind}_P^G(\pi \otimes < \frac{1}{2}; H_P(\cdot) >).$$

Let $r(\pi)$ be the irreducible K_G-spherical component of $I(\frac{1}{2}, \pi)$, containing the K_G-invariant function ϕ_G^0 which has the property that

$$\phi_G^0(k) = \phi_M^0, \quad \text{for } k \in K_G.$$

Then the representation $r(\pi)$ has a nonzero H-invariant linear functional $\mathcal{P}_G$ defined by

$$\mathcal{P}_G(\phi_G) := \int_{K_G \cap H} \mathcal{P}_M(\phi_G(k))dk.$$

Then we can show

Proposition 2.4. *For any representation of M satisfying the assumption described above, we have*

$$\hat{f}(r(\pi)) = \hat{f}'(\pi). \tag{2.6}$$

In particular, the map $f \mapsto f'(= F_{f,v})$ (as defined in (2.4)) gives rise to a Hecke algebra homomorphism from $\mathcal{H}(G//K_G)$ to $\mathcal{H}(M//K_M)$.

Proof. First of all, for any Hecke function $f \in \mathcal{H}(G//K_G)$, we have

$$\int_G \phi_G^0(x) f(x) dx = \phi_G^0(e) \hat{f}(r(\pi)) = \hat{f}(r(\pi)) \cdot \phi_M^0.$$

Applying the functional $\mathcal{P}_M$, we get

$$\hat{f}(r(\pi)) = \int_G \mathcal{P}_M(\phi_G^0(x)) f(x) dx.$$

By the Iwasawa decomposition of G,

$$G = MNK_G,$$

we can manipulate the integral as follows:

$$\begin{aligned}
\hat{f}(r(\pi)) &= \int_{MNK_G} \mathcal{P}_M(\phi_G^0(mnk)) f(mnk) dm dn dk \\
&= \int_M \mathcal{P}_M(\pi(m)\phi_M^0) |\det m|^2 \int_N f(mn) dn dm \\
&= \int_M \mathcal{P}_M(\pi(m)\phi_M^0) f'(m) dm.
\end{aligned}$$

Define a function $\omega'(m)$ by

$$\omega'(m) := \int_{K_M} \mathcal{P}_M(\pi(km)\phi_M^0) dk.$$

It is easy to see that $\omega'(m)$ is bi-K_M-invariant and $\omega'(e) = 1$. In particular, ω' is the spherical function ω_π corresponding to the unramified representation π. Hence we have

$$\begin{aligned}
\hat{f}'(\pi) &= \int_M \omega_\pi(m) f'(m) dm \\
&= \int_M \omega'(m) f'(m) dm \\
&= \int_M \int_{K_M} \mathcal{P}_M(\pi(km)\phi_M^0) dk f'(m) dm \\
&= \int_M \int_{K_M} \mathcal{P}_M(\pi(m)\phi_M^0) f'(km) dk dm \\
&= \int_M \mathcal{P}_M(\pi(m)\phi_M^0) f'(m) dm.
\end{aligned}$$

This proves the matching of the characters of Hecke algebras. □

From the Propositions 2.1, 2.2, 2.3 and 2.4, we get the following:

Theorem 2.1. *Let* $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}))$ *be as in (2.3). Let* $f'_v = F_{f,v}$ *and* $f' = \otimes_v f'_v$ *(as in (2.4) and (2.5)). Then*

$$I_{H,G}(f,\psi) = I_{M_e,M}(f',\psi_M)$$

Moreover, if f_v *is a Hecke function,* f'_v *is the Hecke function associated to* f_v *satisfying (2.6).*

3. Spectral side of the trace formula

We discuss the implication of Theorem 2.1 in terms of the correspondence of automorphic representations of G and M.

Recall that $K_f(x,y)$ has a decomposition into the sum of discrete spectrum and continuous spectrum. The local supercuspidality assumption on f limits the contribution to the spectral decomposition of $K_f(x,y)$ from that of:

(1) cuspidal part of the discrete spectrum;
(2) the residue spectrum corresponding to the cuspidal representations of M;
(3) the continuous spectrum corresponding to the Eisenstein series induced from the cuspidal representations of M.

By Theorem 4.1 in [Jng] and the proof therein, one can easily see that the contribution from the part (3) to $I_{H,G}(f,\psi)$ must be 0. Then we have the spectral decomposition of $I_{H,G}(f,\psi)$

$$I_{H,G}(f,\psi) = \sum_{\Pi} I_{H,G}(\Pi, f,\psi) \oplus \sum_{\pi} I_{H,G}(\pi, f,\psi), \tag{3.1}$$

where the sum $\sum_\Pi$ is taken over all irreducible cuspidal representations Π of G and the sum $\sum_\pi$ is taken over all cuspidal representations π of M; and

$$I_{H,G}(\Pi, f,\psi) = \sum_{\phi_i} \int_{H(F)\backslash H(\mathbb{A})} \Pi(f)\phi_i(h)dh \int_{U(F)\backslash U(\mathbb{A})} \bar{\phi}_i(u)\psi(u)du$$

here $\{\phi_i\}$ is an orthonormal basis for the irreducible cuspidal representation Π of G and

$$I_{H,G}(\pi, f,\psi) = \sum_{\varphi_i} \int_{H(F)\backslash H(\mathbb{A})} R_\pi(f)\varphi_i(h)dh \int_{U(F)\backslash U(\mathbb{A})} \bar{\varphi}_i(u)\psi(u)du$$

here $\{\varphi_i\}$ is an orthonormal basis for the residue representation R_π of G coming from the cuspidal representation π of M.

By the definition of the additive character ψ, we have

$$\begin{aligned}\int_{U(F)\backslash U(\mathbb{A})} \bar{\phi}_i(u)\psi(u)du &= \int_{U_M(F)\backslash U_M(\mathbb{A})} \psi_M(u_M) \int_{N(F)\backslash N(\mathbb{A})} \bar{\phi}_i(nu_M)dndu_M \\ &= 0\end{aligned}$$

by the cuspidality of $\bar{\phi}_i$. This implies that the contribution from the cuspidal part (Part (1)) must be also zero. We thus obtain from (3.1) the following expression:

$$I_{H,G}(f,\psi)=\sum_{\pi} I_{H,G}(\pi,f,\psi), \tag{3.2}$$

where the sum $\sum_\pi$ is taken over all cuspidal representations π of M.

The spectral decomposition of $I_{M_e,M}(f',\psi_M)$ is studied in [J]. Our situation is simpler because of the local supercuspidality assumption of f. We get:

$$I_{M_e,M}(f',\psi_M)=\sum_{\pi} I_{M_e,M}(\pi,f',\psi_M) \tag{3.3}$$

where

$$I_{M_e,M}(\pi,f',\psi_M)=\sum_{\varphi'_i}\int_{M_e(F)\backslash M_e(\mathbb{A})}\pi(f')\varphi'_i(h)dh\int_{U_M(F)\backslash U_M(\mathbb{A})}\bar{\varphi}'_i(u)\psi_M(u)du.$$

Here the sum is over the cuspidal representations of M and $\{\varphi'_i\}$ is an orthonormal basis for the representation π.

Let S be a finite set of places containing v_0 and archimedean places. For $v\notin S$, we let f_v be a Hecke function. Let f' be the function associated to f as in Theorem 2.1. Let $\Xi(S)$ be the set of cuspidal representations of M which are unramified at all places not in S. If $\pi\notin\Xi(S)$, we have

$$I_{H,G}(\pi,f,\psi)=I_{M_e,M}(\pi,f',\psi_M)=0.$$

If $\pi\in\Xi(S)$, we have

$$I_{H,G}(\pi,f,\psi)=[\prod_{v\notin S}\hat{f}_v(r(\pi_v))]\cdot I_{H,G}(\Pi,\otimes_{v\in S}f_v\otimes_{v\notin S}f_{0,v},\psi)$$

(Recall that $f_{0,v}$ is the unit element of the Hecke algebra). Similarly

$$I_{M_e,M}(\pi,f,\psi_M)=[\prod_{v\notin S}\hat{f'}_v(\pi_v)]\cdot I_{M_e,M}(\pi,\otimes_{v\in S}f'_v\otimes_{v\notin S}f'_{0,v},\psi_M).$$

From Theorem 2.1, Proposition 2.4, and the above discussion, we get:

$$\sum_{\pi\in\Xi(S)}\prod_{v\notin S}\hat{f'}_v(\pi_v)[I_{H,G}(\Pi,\otimes_{v\in S}f_v\otimes_{v\notin S}f_{0,v},\psi) - I_{M_e,M}(\pi,\otimes_{v\in S}f'_v\otimes_{v\notin S}f'_{0,v},\psi_M)]=0.$$

From the linear independence of character of Hecke algebras, and the strong multiplicity one theorem for M, we get for all $\pi\in\Xi(S)$:

$$I_{H,G}(\Pi,\otimes_{v\in S}f_v\otimes_{v\notin S}f_{0,v},\psi)=I_{M_e,M}(\pi,\otimes_{v\in S}f'_v\otimes_{v\notin S}f'_{0,v},\psi_M)$$

which is just $I_{H,G}(\Pi,f,\psi)=I_{M_e,M}(\pi,f',\psi_M)$.

Since the choice of S is arbitrary, we obtain:

Theorem 3.1. *Let f and f' be those as defined in Theorem 2.1. For all cuspidal representations π of M with at least one local supercuspidal component,*

$$I_{H,G}(\Pi, f, \psi) = I_{M_e,M}(\pi, f', \psi_M). \tag{3.4}$$

Remark 3.1. The assumption of the local supercuspidality at one place on f simplifies technically the argument to establish both the geometric identity and the spectral identity of the relative trace formula. We believe that this assumption is removable. In order to present our idea and result transparently, we would rather keep this useful assumption.

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