A RELATIVE TRACE IDENTITY BETWEEN GL_{2n} AND $\widetilde{\mathrm{Sp}}_n$

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Abstract. We prove a relative trace identity between GL_{2n} and $\widetilde{\mathrm{Sp}}_n$, using Ginzburg-Soudry-Rallis's work on automorphic descent. This should serve as a model on using automorphic descent to establish relative trace identity.

1. Introduction

This paper establishes a geometric (relative) trace identity for the correspondence between automorphic representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_n$, the double cover of Sp_n (which is a subgroup of GL_{2n}–many authors use the notation Sp_{2n} instead). The $n = 1$ case is proven in [J1], the general case is conjectured by Jacquet (see [M]), Ginzburg-Rallis-Soudry [GRS3].

One implication of relative trace identity is the correspondence of automorphic representations on two groups. In our case it concerns the correspondence between the generic genuine cuspidal representations of $\widetilde{\mathrm{Sp}}_n$ and the subset of cuspidal representations $\{\Pi\}$ of GL_{2n} such that Π is self-dual (the exterior square $L-$function of Π has a pole at $s = 1$) and $L(\Pi, \frac{1}{2}) \neq 0$. If we combine the trace identity here with the one in [MR2] between $\widetilde{\mathrm{Sp}}_n$ and SO_{2n+1}, we can get a relative trace identity between GL_{2n} and $\mathrm{SO}(2n+1)$, which implies a functorial lifting from the odd orthogonal group to GL_{2n}.

While the functorial lifting in these cases have been studied extensively using other methods, there is another implication of this trace identity: it gives an identity for the special value of $L-$function $L(\Pi, \frac{1}{2})$. This result is worked out in full detail in the $n = 1$

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case, [BaM]. The identity in this case is a generalization of the Waldspurger's theorem relating $L(\Pi, \frac{1}{2})$ to the Fourier coefficients of the half-integral weight cusp forms.

There are several different ways to study the functorial lifting from classical groups to GL_n. These methods including the Converse theorem [CKPSS], the descent method [GRS1], and the trace formula method (see for example [Ar], [Wa]). The recent development shows that the best results follow from the combination of the different methods. This paper follows the same spirit. We will use heavily the descent method in proving the geometric trace identity. Our proof can serve as the model for other cases of the geometric trace identities, at least in the case when the automorphic descent formalism works.

1.1. **Statement of the relative trace identity.** Let k be a number field, $\mathbf{A}$ its adele ring. We use v to denote a place of k. The local field is denoted k_v; when v is nonarchimedean, $\mathcal{O}_v$ is its ring of integer. Sometimes we omit v in the notation when the place is fixed.

Let G be a reductive group. In studying relative trace formula, one considers a distribution of the following type: for $f \in \mathcal{S}(G(\mathbf{A}))$ (the space of Schwartz functions on $G(\mathbf{A})$), let

$$I_G(f : H_1, \chi_1; H_2, \chi_2) = \int_{H_1(k)\backslash H_1(\mathbf{A})} \int_{H_2(k)\backslash H_2(\mathbf{A})} K_f(h_1, h_2)\chi_1(h_1)\chi_2(h_2)\, dh_2\, dh_1. \tag{1.1}$$

Here H_1, H_2 are two closed subgroups of G, χ_i $(i = 1, 2)$ is a global automorphic character of $H_i(\mathbf{A})$ trivial on $H_i(k)$, and $K_f(x, y)$ is the kernel function for the representation $\rho(f)$ acting on $L^2(G(k)\backslash G(\mathbf{A}))$; more explicitly

$$K_f(x, y) = \sum_{\gamma \in G(k)} f(x^{-1}\gamma y).$$

Over a nonarchimedean place v, let K_v be a maximal compact subgroup of G_v. Denote by $\mathcal{H}(G_v, K_v)$ the Hecke algebra of G_v with respect to K_v. It is the algebra of all compactly supported functions on G_v satisfying $f(k_1 g k_2) = f(g)$ for any $k_1, k_2 \in K_v$ and $g \in G_v$. The operation in $\mathcal{H}(G_v, K_v)$ is given by the convolution product

$$(f_1 * f_2)(g) = \int_{G_v} f_1(h) f_2(h^{-1}g)\, dh.$$

Given two groups G and G', assume there is a homomorphism from the $L-$group of G' to $L-$group of G. Then from Satake isomorphism, the homomorphism between the $L-$groups of G and G' induces a homomorphism between $\lambda_v : \mathcal{H}(G_v, K_v) \mapsto \mathcal{H}(G'_v, K'_v)$.

If $f = \otimes f_v \in \mathcal{S}(G(\mathbf{A}))$, then at almost all places $f_v \in \mathcal{H}(G_v, K_v)$. Assume f' is also of the form $\otimes f'_v$. A relative trace identity is an identity between two distributions $I_G(f : H_1, \chi_1, H_2, \chi_2)$ and $I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2)$ where at almost all places $f'_v = \lambda_v(f_v)$. More precisely, let S_0 be the set of bad places containing archimedean places, even places and places where χ_i (or χ'_i) is nontrivial on $H_i(k_v) \cap K_v$ (or $H'_i(k_v) \cap K'_v$); we say there is relative trace identity

$$I_G(f : H_1, \chi_1; H_2, \chi_2) = I_{G'}(f' : H'_1, \chi'_1; H'_2, \chi'_2) \tag{1.2}$$

if the following is true:

> There exists maps $\epsilon_v : \mathcal{S}(G_v) \to \mathcal{S}(G'_v)$ for all places v of k, such that when $f = \otimes f_v$ where f_v is a Hecke function for $v \notin S$ a finite set of places containing S_0, the equation (1.2) holds for $f' = \otimes_{v \in S} \epsilon_v(f_v) \otimes_{v \notin S} \lambda_v(f_v)$.

We say f'_v and f_v *match* if $f'_v = \epsilon_v(f_v)$. Note that we do not require ϵ_v restricts to λ_v at a non-archimedean place v.

The case at hand is when $G = \mathrm{GL}_{2n}$, $G' = \widetilde{\mathrm{Sp}}_n$ (whose $L-$group is heuristically Sp_n). The $L-$group homomorphism is just the embedding of Sp_n in GL_{2n}. The other data is as follows: $H_1 = \mathrm{GL}_n \times \mathrm{GL}_n$, $\chi_1 = 1$ is trivial, $H_2 = N$ the maximal unipotent subgroup of GL_{2n}, $\chi_2 = \theta$ is a nondegenerate character of N; $H'_1 = H'_2 = N'$ is the maximal unipotent subgroup of Sp_n, and $\chi_1'^{-1} = \chi'_2 = \theta'$ is a nondegenerate character of N'. The theorem we prove is:

Theorem 1.1. *There is a relative trace identity in the above sense: when f and $\tilde{f}$ match*

$$I_{\mathrm{GL}_{2n}}(f : \mathrm{GL}_n \times \mathrm{GL}_n, 1; N, \theta) = I_{\widetilde{\mathrm{Sp}}_n}(\tilde{f} : N', \theta'^{-1}; N', \theta'). \tag{1.3}$$

From the trace identity (1.3), one expects to get the following distribution identity:

$$\sum_{\varphi_\alpha} \mathcal{P}(\Pi(f)\varphi_\alpha)\overline{\mathcal{W}(\varphi_\alpha)} = \sum_{\widetilde{\varphi}_\alpha} \widetilde{\mathcal{W}}(\widetilde{\Pi}(\tilde{f})\widetilde{\varphi}_\alpha)\overline{\widetilde{\mathcal{W}}(\widetilde{\varphi}_\alpha)}. \tag{1.4}$$

Here Π and $\widetilde{\Pi}$ are cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_n$, such that Π is a lift of $\widetilde{\Pi}$; φ_α and $\widetilde{\varphi}_\alpha$ are certain orthonormal basis of the spaces of Π and $\widetilde{\Pi}$; f and $\tilde{f}$ are matching functions; W and $\widetilde{\mathcal{W}}$ are Whittaker functionals; $\mathcal{P}$ is the period (GL_n^0 consists of g with $|\det g| = 1$)

$$\mathcal{P}(\phi) = \int_{(\mathrm{GL}_n(k)\times \mathrm{GL}_n(k))\backslash \mathrm{GL}_n^0(\mathbf{A})\times \mathrm{GL}_n^0(\mathbf{A})} \phi(\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix})\, dg_1\, dg_2.$$

The equation (1.4) roughly says that $|\widetilde{\mathcal{W}}(\widetilde{\varphi})|^2$ equals the product of the period $\mathcal{P}(\varphi)$ and Whittaker functional $\mathcal{W}(\varphi)$. Note $\mathcal{P}(\varphi)$ is related to $L(\Pi, \frac{1}{2})$ by [FJ, Theorem 4.1]. To make the statement more precise, one needs to either develop a local analogue of (1.4) or make a suitable choice of matching functions f and $\tilde{f}$. Both approaches are worked out in detail for some other relative trace identities. See for example [BaM], [ChJ], [LO], [MaWh].

1.2. **More relative trace identities.** The proof of Theorem 1.1 is through establishing several other relative trace identities. Automorphic descent method of Ginzburg-Rallis-Soudry constructs cusp forms on $\widetilde{\mathrm{Sp}}_n$ by taking Fourier-Jacobi coefficients of residue of Eisenstein series constructed from cusp forms on GL_{2n}. Our relative trace identities reflect the steps in their construction.

We first establish an identity between the distributions on GL_{2n} and Sp_{2n}. The group GL_{2n} is the Levi subgroup of a maximal parabolic subgroup of Sp_{2n}. The relative trace identity between a reductive group and its Levi subgroup is first studied in [JR], and formulated more generally by Jiang [JiMR]. The idea is that there is a relation between the inner period on the Levi factor and the outer period on the group. In our case, if τ is an irreducible cuspidal representation of GL_{2n} whose exterior square $L-$function has a pole at $s = 1$ and $L(\tau, \frac{1}{2}) \neq 0$, then τ has a nontrivial $\mathrm{GL}_n \times \mathrm{GL}_n$ period (the inner period). Construct the residual Eisenstein series on Sp_{2n} from τ, we see that the residual Eisenstein series has a nontrivial $\mathrm{Sp}_n \times \mathrm{Sp}_n$ period (the outer period). In fact the relation between the two periods is given by [GRS1, Theorem 2]. Reflecting this relation between periods, we prove in §2 the following trace identity (Corollary 2.11)

$$I_{\mathrm{GL}_{2n}}(f : \mathrm{GL}_n \times \mathrm{GL}_n, 1; N, \theta) = I_{\mathrm{Sp}_{2n}}(f' : \mathrm{Sp}_n \times \mathrm{Sp}_n, 1; N_3, \theta_3). \tag{1.5}$$

Here N_3 is the maximal unipotent subgroup of Sp_{2n} and θ_3 is a degenerate character, see §2 for details. (N_3, θ_3) period is a degenerate Whittaker period on Sp_{2n}.

Implicit in the above identity is that the residual Eisenstein series possess a nontrivial degenerate Whittaker model. The main theorem of [GRS2] shows that the same automorphic representations possess nontrivial Fourier-Jacobi model. Our next relative trace identity reflects the fact that the representations with $\mathrm{Sp}_n \times \mathrm{Sp}_n$-invariant linear form could have both nontrivial models. In §3 we show the trace identity (Corollary 3.6)

$$I_{\mathrm{Sp}_{2n}}(f' : \mathrm{Sp}_n \times \mathrm{Sp}_n, 1; N_3, \theta_3) = I_{\mathrm{Sp}_{2n}}(f'' : \mathrm{Sp}_n \times \mathrm{Sp}_n, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}). \tag{1.6}$$

The model $\theta_4\Theta^{\Phi}_{\psi^{-1}}$ is a mixed Fourier-Jacobi-Whittaker model.

The proof of above identity involves three other relative trace identities, each reflecting transition between different nontrivial models that the residual Eisenstein series possess, (see the definition of distributions I_1 and I_2 in §3). The transition between different models play a prominent role in the automorphic descent method. In fact the proof of the main theorem of [GRS2] is through results on transitions between different models ([GRS2, §5 Theorems 1,2 and Lemmas 1,2]). We will translate these results in the setting of relative trace identity.

We note the proof of the trace identity (1.6) is completely global. We do not decompose the distribution into a sum of orbital integrals. In fact one does not have natural matching of orbits, thus the method of comparing orbital integrals fails. For another example of global proof of relative trace identity, see [MR3].

What remains is to compare $I_{\mathrm{Sp}_{2n}}(f'' : \mathrm{Sp}_n \times \mathrm{Sp}_n, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ and $I_{\widetilde{\mathrm{Sp}}_n}(\tilde{f} : N', \theta'^{-1}, N', \theta')$. Here we go back to the standard method of comparing orbital integrals. The necessary orbital integral identities are relatively easy to establish. It is also rather easy to show the fundamental lemma for the unit Hecke element in this case. An interesting point in the current situation is: we can derive the fundamental lemma for all Hecke functions from fundamental lemma for the unit Hecke element. The argument is sketched in §6.3. It makes use of the Plancherel formula for the symmetric space $\mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ and a Jacquet module computation from the study of automorphic descent method. The details of this argument is carried out in [MR1].

1.3. **Structure of the paper.** Section 2 establishes the relative trace identity between GL_{2n} and Sp_{2n}. Section 3 establishes three relative trace identities on Sp_{2n}. Section 4 decomposes the two distribution $I_{\mathrm{Sp}_{2n}}(f'' : \mathrm{Sp}_n \times \mathrm{Sp}_n, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ and $I_{\widetilde{\mathrm{Sp}}_n}(\tilde{f} : N', \theta'^{-1}, N', \theta')$ into sums of orbital integrals. Section 5 compares the orbital integrals for arbitrary Schwartz functions, while section 6 proves the fundamental lemma. In Section 7 we prove Theorem 1.1.

1.4. **Notations and Preliminaries.** The integrals given in the form (1.1) are absolutely convergent from the argument of [J2, Proposition 2.1], as one of the integral is over a compact set in all the cases we consider. Indeed the integrals we consider here are all absolutely convergent, justifying our formal manipulations.

1.4.1. *Group elements.* • $X_{i,j}$ is the $(i,j)-$th entry of a matrix X.

- $e_{i,j}$ is the matrix where only the $(i,j)-$th entry is nonzero and the (i,j)-th entry is 1.
- 1_n is the identity matrix in GL_n.
- $\sigma_n \in \mathrm{GL}_n$ is the longest Weyl element, with 1's on the antidiagonal and 0's elsewhere.
- $J_n = \begin{pmatrix} & -\sigma_n \\ \sigma_n & \end{pmatrix}$ is an element in Sp_n.
- E and E_1 are defined in (4.1). They are elements in Sp_{2n}.
- $\mathrm{diag}[a_1, \dots, a_n]$ denotes a diagonal matrix with entries $a_1, a_2, \dots, a_n$.
- Elements in $\widetilde{\mathrm{Sp}}_n$ are denoted by $(g, \pm 1)$ with $g \in \mathrm{Sp}_n$. We let $\widetilde{g} = (g, 1)$. We denote the product of two elements g_1, g_2 of $\widetilde{\mathrm{Sp}}_n$ by $g_1 \cdot g_2$.

1.4.2. *Groups and Sets.* • $\mathrm{M}_{m,n}$ is the set of $m \times n$ matrices.

- Sp_n is the subgroup of GL_{2n} consisting of elements g with $g^t J_n g = J_n$. (Note many authors call this group Sp_{2n}).

• $\widetilde{\mathrm{Sp}}_n$ is the double cover of Sp_n. We use the Rao cocycle in definition of $\widetilde{\mathrm{Sp}}_n$. We can identify $\mathrm{Sp}_n(k)$ as a subgroup of $\widetilde{\mathrm{Sp}}_n(\mathbf{A})$ as the covering splits over this group.

• $G_1 = \mathrm{GL}_{2n}$; $G_2 = \widetilde{\mathrm{Sp}}_n$; $G_3 = \mathrm{Sp}_{2n}$;

• Z_n, N_1, N_2, N_3 are maximal unipotent subgroups of GL_n, GL_{2n}, Sp_n, Sp_{2n} respectively, consisting of upper triangular matrices with unit diagonal. We can identify $N_2(\mathbf{A})$ as a subgroup of $G_2(\mathbf{A})$ with the splitting $n \mapsto \widetilde{n}$.

• T_n is the subgroup of diagonal matrices in GL_n.

• K_1, K_2, K_3 are the maximal compact subgroups of G_1, Sp_n, G_3 respectively. When $\widetilde{\mathrm{Sp}}_n$ splits over K_2, we also use K_2 to denote its image in $\widetilde{\mathrm{Sp}}_n$.

• $H_1 = \mathrm{GL}_n \times \mathrm{GL}_n$, $H_3 = \mathrm{Sp}_n \times \mathrm{Sp}_n$.

• P_3 is the Siegel parabolic subgroup of Sp_{2n} consisting of matrices of the form $\left(\begin{smallmatrix} g & v \\ & g^* \end{smallmatrix}\right)$.

• V is the unipotent radical of P_3.

• $\mathcal{S}_n$ is the set of matrices $g \in \mathrm{GL}_n$ satisfying $\sigma_n g$ is a symmetric matrix.

• Y is the symmetric space isomorphic to $H_3\backslash G_3$ defined in §4.

• $\hat{N}^n$ is a subgroup of N_3 defined by (3.8).

• U^n is the Heisenberg group defined in (3.9). U_0^n is its normal subgroup defined right after (3.9).

• $N_1^\sigma = \sigma^{-1} H_1 \sigma \cap N_1$ where $\sigma \in G_1$.

• $N_{2,g} = g^{-1} N_2 g \cap N_2$, while $N'_{2,g} = g N_2 g^{-1} \cap N_2$ for $g \in \mathrm{Sp}_n$.

• When $y \in Y$, $N_{3,y}$ is the set of $n \in N_3$ with $n^{-1} y n = y$.

• When $\gamma \in G_3$, $N'_{3,\gamma} = \gamma^{-1} H_3 \gamma \cap N_3$.

• V_{E_1} and $U_{E_1}^1$ are subgroups of N_{3,E_1} defined after equation (4.10).

• $W(G)$ is the Weyl group of the linear group G.

1.4.3. *Maps and homomorphisms.* • $f \mapsto \hat{f}$ is the Satake transform from Hecke algebra to polynomials invariant under the Weyl group action.

• $\mathrm{tr}(X)$ is the trace of a matrix X.

• g^t is the transpose of g.

• $g^* = \sigma_n (g^t)^{-1} \sigma_n$ if $g \in \mathrm{GL}_n$.

- m is map from G_1 to G_3: $m(g) = \begin{pmatrix} g & \\ & g^* \end{pmatrix}$.
- j is an injection from Sp_n to Sp_{2n} defined by (3.7).
- η is map from $k^n \times k^n \times k$ to U^n.
- P is a map from G_2 to Y defined by (4.6).
- u is a map with image in N_{3,E_1} defined in (4.10).
- i is an embedding from N_2 to $H_3 \cap N_3$ defined by (4.12).

1.4.4. *Characters.* • ψ is either a nontrivial additive character of $\mathbf{A}/k$ or of k_v.

- θ_1 is a character of N_1 with

$$\theta_1(n) = \psi(n_{1,2} + \ldots + n_{2n-1,2n}).$$

- θ_2 is a character of N_2 with

$$\theta_2(n) = \psi(n_{1,2} + \ldots + n_{n-1,n} + \frac{n_{n,n+1}}{2}).$$

- θ_3 is a character of N_3 with

$$\theta_3(n) = \psi(n_{1,2} + \ldots + n_{2n-1,2n}).$$

- θ_4 is a character of N_3 defined by (3.31).

1.4.5. *Weil representation and Theta function.* Recall for a fixed ψ, the Weil representation is defined for the metaplectic group $\widetilde{\mathrm{Sp}}_n$. We use $\gamma(*, \psi)$ to denote the Weil constant, and ω_ψ to denote the Weil representation. We describe explicitly a model of the Weil representation.

Let $\Phi \in \mathcal{S}(\mathbf{A}^n)$. Then

$$\omega_\psi(\widetilde{m(g)})\Phi(X) = |\det g|^{1/2} \frac{\gamma(1,\psi)}{\gamma(\det g, \psi)} \Phi(Xg), \; g \in \mathrm{GL}_n . \tag{1.7}$$

$$\omega_\psi(\begin{pmatrix} 1_n & V \\ & 1_n \end{pmatrix}, 1)\Phi(X) = \psi(\mathrm{tr}(X^t V \sigma_n X))\Phi(X), \; V \in \mathcal{S}_n. \tag{1.8}$$

$$\omega_\psi(\widetilde{J_n})\Phi(X) = \gamma(1,\psi)^{-n}\widehat{\Phi}(X), \tag{1.9}$$

where

$$\widehat{\Phi}(X) = \int_{\mathbf{A}^n} \psi(\mathrm{tr}(X^t \sigma_n Y))\Phi(Y)\, dY.$$

The above describes the action of the metaplectic group on $\mathcal{S}(\mathbf{A}^n)$ under the Weil representation.

We use $\Theta^{\Phi}_{\psi^{-1}}$ to denote the Theta function defined by (3.30).

1.4.6. *Functions, places, measures.* When v is a non-archimedean place, the space of Schwartz functions $\mathcal{S}(G(k_v))$ consists of smooth functions on $G(k_v)$ of compact support. At an archimedean place v, we use the definition in [Ca] (see also [AGo]) for Schwartz functions. Roughly speaking, a Schwartz function is smooth and all its derivatives are rapidly decreasing functions. Functions ϕ on $\widetilde{\mathrm{Sp}}_n$ that we consider are always genuine, namely $\phi(g,-1) = -\phi(g,1)$.

We say a place v bad if it is an archimedean place or an even places, or a place where ψ is not unramified. We call a place v good if it is not a bad place.

Measures are fixed as follows. Measure of $G(\mathbf{A})$ is the Tamagawa measure of G. Over a given local field k_v, when v is a good place, we fix measure on $G(k_v)$ so that $G(\mathcal{O}_v)$ has volume 1, where $\mathcal{O}_v$ is the ring of integers in k_v. At bad places, we only require the choices of measures on different groups to be compatible.

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2. The trace identity between GL_{2n} and Sp_{2n}

Let $G_1 = \mathrm{GL}_{2n}$, $G_3 = \mathrm{Sp}_{2n}$. Let $H_1 = \mathrm{GL}_n \times \mathrm{GL}_n$ and $H_3 = \mathrm{Sp}_n \times \mathrm{Sp}_n$ subgroups of G_1 and G_3 as in the introduction. Let N_1 and N_3 be the maximal unipotent subgroups of G_1 and G_3, and θ_1, θ_3 be their characters as defined in the introduction. Let P_3 be the Siegel parabolic of G_3 containing N_3, let V be its unipotent radical. For $g \in G_1$, let $m(g) = \left(\begin{smallmatrix} g & \\ & g^* \end{smallmatrix}\right) \in G_3$.

Fix v a place of k, for $f_v \in \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$, define for $g \in G_1$:

$$f'_v(g) = \int_{u\in V(k_v)} \int_{k\in K_3\cap H_3(k_v)} f_v(km(g)u)|\det(g)|_v^{n+1}\, dk\, du \tag{2.1}$$

where K_3 is the maximal compact subgroup of G_3. Then $f'_v \in \mathcal{S}(\mathrm{GL}_{2n}(k_v))$. We prove:

Theorem 2.1. *Let $f = \otimes f_v \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$ and $f' = \otimes f'_v \in \mathcal{S}(\mathrm{GL}_{2n}(\mathbf{A}))$. If for all v, f_v and f'_v are related by equation (2.1), then*

$$I_{G_3}(f : H_3, 1; N_3, \theta_3) = I_{G_1}(f' : H_1, 1; N_1, \theta_1). \tag{2.2}$$

Moreover at a $p-$adic place v, the map $f_v \mapsto f'_v$ restricts to a Hecke algebra homomorphism.

The statement follows from expanding the two distributions into sums of the orbital integrals and a detailed study of the relevant orbits.

2.1. **Orbital integral decomposition for $I_{G_3}(f : H_3, 1; N_3, \theta_3)$.** From the definition:

$$I_{G_3}(f : H_3, 1; N_3, \theta_3) = \int_{l \in H_3(k)\backslash H_3(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \sum_{\gamma \in \mathrm{Sp}_{2n}(k)} f(l^{-1}\gamma n)\theta_3(n)\, dn\, dl. \tag{2.3}$$

From [GRS1], as a disjoint union:

$$\mathrm{Sp}_{2n} = \cup_{d=1}^{n} H_3 \gamma_d^{-1} P_3$$

where the description of the representatives γ_d is given in [GRS1]. Define:

$$P_3^d = \gamma_d H_3 \gamma_d^{-1} \cap P_3. \tag{2.4}$$

The right hand side of (2.3) becomes:

$$\sum_{d=1}^{n} \int_{l \in H_3(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \sum_{p \in P_3^d \backslash P_3(k)} f(l^{-1}\gamma_d^{-1} p n)\theta_3(n)\, dn\, dl.$$

The above expression unwinds to:

$$\sum_{d=1}^{n} \int_{l \in H_3(\mathbf{A})} \sum_{p \in P_3^d(k)\backslash P_3(k)/N_3(k)} \int_{N_3^{d,p}(\mathbf{A})\backslash N_3(\mathbf{A})} V(p) f(l^{-1}\gamma_d^{-1} p n)\theta_3(n)\, dn\, dl \tag{2.5}$$

where $N_3^{d,p} = N_3 \cap p^{-1} P_3^d p$ and

$$V(p) = \int_{N_3^{d,p}(k)\backslash N_3^{d,p}(\mathbf{A})} \theta_3(n_1)\, dn_1. \tag{2.6}$$

We will say the double coset $p \in P_3^d(k)\backslash P_3(k)/N_3(k)$ is *relevant* if θ_3 is trivial on $N_3^{d,p}(\mathbf{A})$. It is simple to observe that (with our choice of measure)

Lemma 2.2. *If p is not a relevant double coset in $P_3^d(k)\backslash P_3(k)/N_3(k)$, then $V(p)=0$. If p is relevant, $V(p)=1$.*

We next study which double coset is relevant. In next subsection we show

Lemma 2.3. *When $d<n$, $p\in P_3^d(k)\backslash P_3(k)/N_3(k)$ is not relevant.*

From Lemma 2.2 and Lemma 2.3, we see the equation (2.5) equals:

$$\int_{l\in H_3(\mathbf{A})}\sum_{p\in P_3^n(k)\backslash P_3(k)/N_3(k)}\int_{N_3^{n,p}(\mathbf{A})\backslash N_3(\mathbf{A})}V(p)f(l^{-1}\gamma_n^{-1}pn)\theta_3(n)\,dn\,dl. \tag{2.7}$$

From [GRS1], $\gamma_n=1_{4n}$, thus $P_3^n=P_3\cap H_3$. The map $\left(\begin{smallmatrix} g & v\\ & g^*\end{smallmatrix}\right)\mapsto g$ then defines a bijection between $P_3^n\backslash P_3/N_3$ and $H_1\backslash G_1/N_1$. The expression (2.7) becomes

$$\int_{l\in H_3(\mathbf{A})}\sum_{\sigma\in H_1(k)\backslash G_1(k)/N_1(k)}\int_{N_3^\sigma(\mathbf{A})\backslash N_3(\mathbf{A})}V(m(\sigma))f(l^{-1}m(\sigma)n)\theta_3(n)\,dn\,dl \tag{2.8}$$

where $m(\sigma)=\left(\begin{smallmatrix}\sigma & \\ & \sigma^*\end{smallmatrix}\right)$ and $N_3^\sigma=N_3\cap m(\sigma)^{-1}(P_3\cap H_3)m(\sigma)$.

We will say the double coset of $\sigma\in H_1(k)\backslash G_1(k)/N_1(k)$ is *relevant* if θ_1 is trivial on $N_1^\sigma(\mathbf{A})=\sigma^{-1}H_1\sigma\cap N_1(\mathbf{A})$. In Lemma 2.5, we will see σ is relevant if and only if $m(\sigma)$ is relevant. We have

Lemma 2.4. *The distribution $I_{G_3}(f:H_3,1;N_3,\theta_3)$ equals:*

$$\sum_{\sigma\in H_1(k)\backslash G_1(k)/N_1(k)\ relevant}\int_{l\in H_3(\mathbf{A})}\int_{N_3^\sigma(\mathbf{A})\backslash N_3(\mathbf{A})}f(l^{-1}m(\sigma)n)\theta_3(n)\,dn\,dl. \tag{2.9}$$

2.2. **Proof of Lemma 2.3.** For $0\le d\le n$, let M^d be the set

$$\left\{\begin{pmatrix} A & B\\ & D\end{pmatrix}\mid A=\left(\begin{smallmatrix} g_1 & \\ & g_2\end{smallmatrix}\right)\in\mathrm{GL}_d\times\mathrm{GL}_d, D\in\mathrm{Sp}_{n-d}, B\in\mathrm{M}_{2d,2(n-d)}\right\} \tag{2.10}$$

and

$$\epsilon_d=\begin{pmatrix} 1_d & & & \\ & & 1_d & \\ & 1_{n-d} & & \\ & & & 1_{n-d}\end{pmatrix}\in\mathrm{GL}_{2n}\,. \tag{2.11}$$

From [GRS1, p.820], elements in P_3^d has the form $\begin{pmatrix} m & v \\ & m^* \end{pmatrix}$ with $m \in \epsilon_d^{-1} M^d \epsilon_d$.

Lemma 2.5. *The map* $\rho : p = \begin{pmatrix} g & v \\ & g^* \end{pmatrix} \in P_3 \mapsto \epsilon_d g$ *induces a bijection between the double cosets* $P_3^d \backslash P_3 / N_3$ *and* $M^d \backslash \mathrm{GL}_{2n} / N_1$*. Moreover,* p *is relevant if and only if* θ_1 *is trivial on* $\rho(p)^{-1} M^d \rho(p) \cap N_1$*.*

Proof. From the description of P_3^d we see the map ρ induces a map between the double cosets. As ρ is surjective, so is the induced map on the double cosets. If $\rho(p_1) = \rho(p_2)$, then $p_1 p_2^{-1} \in N_3$, thus the map is also injective on the double cosets.

From the description of P_3^d, when we write $n = \begin{pmatrix} u & v' \\ & u^* \end{pmatrix} \in N_3$, $n \in p^{-1} P_3^d p \cap N_3$ holds for some choice of v' if and only if $u \in N_1 \cap \rho(p)^{-1} M^d \rho(p)$. As $\theta_3(n) = \theta_1(u)$, we get the last statement. □

We will say the double coset of $\sigma \in M^d \backslash \mathrm{GL}_{2n} / N_1$ *relevant* if θ_1 is trivial on $\sigma^{-1} M^d \sigma \cap N_1$. We are left to show there is no relevant double coset when $d < n$.

Let P_1^d be the parabolic subgroup of GL_{2n} consisting of matrices of the form $\begin{pmatrix} A & B \\ & D \end{pmatrix}$ with $A \in \mathrm{GL}_{2d}$. Let W^d be a subset in the Weyl group GL_{2n} (identified with permutation matrices), consisting of w such that $w^{-1}(\alpha) > 0$ for all positive roots α of the Levi subgroup of P_1^d.

Lemma 2.6. *We have the double coset decomposition:*

$$\mathrm{GL}_{2n} = \cup_{w \in W^d} \cup_{\sigma_1 \in (\mathrm{GL}_d \times \mathrm{GL}_d) \backslash \mathrm{GL}_{2d}} \cup_{\sigma_2 \in \mathrm{Sp}_{n-d} \backslash \mathrm{GL}_{(2n-2d)}} M^d \sigma(\sigma_1, \sigma_2) w N_1 \tag{2.12}$$

where $\sigma(\sigma_1, \sigma_2) = \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix}$*.*

Proof. From the Bruhat decomposition, we have:

$$\mathrm{GL}_{2n} = \cup_{w \in W^d} P_1^d w N_1.$$

Clearly

$$P_1^d = \cup_{\sigma_1 \in (\mathrm{GL}_d \times \mathrm{GL}_d) \backslash \mathrm{GL}_{2d}} \cup_{\sigma_2 \in \mathrm{Sp}_{n-d} \backslash \mathrm{GL}_{(2n-2d)}} M^d \sigma(\sigma_1, \sigma_2).$$

□

From now on assume σ has the form $\sigma(\sigma_1, \sigma_2)$ as in (2.12).

Lemma 2.7. *If there is a positive simple root α of the Levi subgroup $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$ such that $w^{-1}(\alpha)$ is no longer a root of this Levi subgroup, then the double coset of $\sigma w \in M^d \backslash \mathrm{GL}_{2n} / N_1$ is not relevant.*

Proof. Let V^d be the unipotent radical of M^d. Let $X_\alpha \in N_1$ be the root vector corresponding to corresponding to α, ([JR, p.270]). Let $n_\alpha = 1 + xX_\alpha \in N_1$. Then $wn_\alpha w^{-1} \in V^d$ by our assumption, and $\sigma w n_\alpha w^{-1} \sigma^{-1} \in V^d \subset M^d$. Thus $n_\alpha \in (\sigma w)^{-1} M^d \sigma w \cap N_1$. Yet $\theta_1(n_\alpha) = \psi(x)$ is not always 1, thus σw gives a non-relevant double coset. □

Lemma 2.8. *Let $w \in W^d$. If $w^{-1}(\alpha)$ are roots of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$ for all simple roots α of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$, then $w^{-1}(\alpha)$ are simple roots of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$.*

Proof. Let β be a positive root of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$. Then $\beta = \sum_i c_i \alpha_i$ where c_i are non-negative integers and α_i are simple roots of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$. From the assumption, $w^{-1}(\beta) = \sum_i c_i' \alpha_i$ with c_i' non-negative integers. In particular, $w^{-1}(\beta)$ is a positive root of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$. Thus w permutes the positive roots of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$. Let α be any simple root and $\beta = w(\alpha)$. If β is not a simple root, then $\beta = \sum_i c_i \alpha_i$ with at least two c_i's non zero, and $\alpha = w^{-1}(\beta)$ will not be simple. Thus β is a simple root, and w permutes the simple roots of $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$. □

The above two lemmas imply that if the double coset σw in (2.12) is relevant, then w permutes simple roots in $\mathrm{GL}_{2d} \times \mathrm{GL}_{(2n-2d)}$.

Proof of Lemma 2.3: When $d = 0$, this is proved in [JR], which says there is no relevant double coset in $\mathrm{Sp}_n \backslash \mathrm{GL}_{2n} / N_1$. Equivalently, for any given $\sigma \in \mathrm{GL}_{2n}$, one can find a simple root α of GL_{2n}, such that if X_α is the corresponding root vector and $n_\alpha = 1 + xX_\alpha$, $\sigma n_\alpha \sigma^{-1} \in \mathrm{Sp}_n$.

Now assume $0 < d < n$, we show $\sigma(\sigma_1, \sigma_2)w$ is not relevant. Apply the above mentioned result of [JR] to the case of $\mathrm{GL}_{(2n-2d)}$, we see there is a simple root α of $\mathrm{GL}_{(2n-2d)}$, such that $\sigma_2 n_\alpha \sigma_2^{-1} \in \mathrm{Sp}_{n-d}$. Let $\beta = w(\alpha)$ and Y be the root vector of β and $n_\beta = 1 + xY$.

Then we have

$$n_\beta \in (\sigma w)^{-1} M^d \sigma w \cap N_1.$$

From Lemma 2.7 and Lemma 2.8, if β is not a simple root, then σw is not relevant. Yet if β is a simple root, $\theta_1(n_\beta) = \psi(x)$ is not always 1, again we get σw is not relevant. We have shown the lemma. □

2.3. **Comparison with $I_{G_1}(f' : H_1, 1; N_1, \theta_1)$.** Same argument (though much simpler) as in the decomposition of $I_{G_3}(f : H_3, 1; N_3, \theta_3)$ gives:

Lemma 2.9. *The distribution $I_{G_1}(f' : H_1, 1; N_1, \theta_1)$ equals:*

$$\sum_{\sigma \in H_1(k)\backslash G_1(k)/N_1(k) \text{ relevant}} \int_{h \in H_1(\mathbf{A})} \int_{N_1^\sigma(\mathbf{A})\backslash N_1(\mathbf{A})} f'(h^{-1}\sigma n)\theta_1(n)\, dn\, dh \tag{2.13}$$

where $N_1^\sigma = \sigma^{-1} H_1 \sigma \cap N_1$.

Thus to show the identity (2.2), we only need to show (for a compatible choice of measures):

Lemma 2.10. *For any relevant $\sigma \in \mathrm{GL}_{2n}$, any place v of k, with f'_v defined as in (2.1), we have:*

$$\begin{aligned} (2.14)\quad |\det(\sigma)|_v^{-n-1} \int_{h \in H_1(k_v)} \int_{N_1^\sigma(k_v)\backslash N_1(k_v)} f'_v(h^{-1}\sigma n)\theta_1(n)\, dn\, dh \\ = \int_{l \in H_3(k_v)} \int_{N_3^\sigma(k_v)\backslash N_3(k_v)} f_v(l^{-1} m(\sigma) n)\theta_3(n)\, dn\, dl. \end{aligned}$$

Proof. We fix a place v and drop the reference to k_v in the notations.

Write $n \in N_3$ as $vm(n_1)$ with $n_1 \in N_1$ and $v \in V$. Then $\theta_3(n) = \theta_1(n_1)$. An explicit computation shows $m(\sigma) n m(\sigma)^{-1} \in H_3 \cap P_3$ if and only if $\sigma n_1 \sigma^{-1} \in H_1$ and $m(\sigma) v m(\sigma)^{-1} \in V \cap H_3$. Thus $n \in N_3^\sigma$ if and only if $n_1 \in N_1^\sigma$ and $m(\sigma) v m(\sigma)^{-1} \in V \cap H_3$.

Change $v \mapsto m(\sigma)^{-1} v m(\sigma)$, the right hand side of (2.14) becomes:

$$\int_{l \in H_3} \int_{n_1 \in N_1^\sigma \backslash N_1} \int_{v \in (V \cap H_3)\backslash V} f(l^{-1} v m(\sigma) n_1)\theta_1(n_1) |\det(\sigma)|^{-(n+1)}\, dv\, dn_1\, dl. \tag{2.15}$$

From the Iwasawa decomposition, $H_3 = (P_3 \cap H_3)(K_3 \cap H_3)$. Write $l^{-1} \in H_3$ as $km(g)u$ where $u \in V \cap H_3$, $k \in K_3 \cap H_3$ and $g \in H_1$. Combining the integration over v and u, the above equation becomes:

$$\int_{k \in K_3 \cap H_3} \int_{g \in H_1} \int_{n_1 \in N_1^\sigma \backslash N_1} \int_{u \in V} f(km(g)um(\sigma n_1))\theta_1(n_1) |\det(g\sigma^{-1})|^{(n+1)}\, du\, dn_1\, dg\, dk.$$

After a change of variable $u \mapsto m(\sigma n_1)^{-1} u m(\sigma n_1)$, we get the above integral equals:

$$\int_{k \in K_3 \cap H_3} \int_{g \in H_1} \int_{n_1 \in N_1^\sigma \backslash N_1} \int_{u \in V} f(km(g\sigma n_1)u)\theta_1(n_1) |\det(g)|^{(n+1)}\, du\, dn_1\, dg\, dk.$$

By the definition of f' (equation (2.1)), this integral is:

$$|\det(\sigma)|^{-n-1} \int_{g \in H_1} \int_{n_1 \in N_1^\sigma \backslash N_1} f'(g\sigma n_1)\theta_1(n_1)\, dn_1\, dg$$

which is the left hand side of the equation (2.14). □

Proof of Theorem 2.1: As the product over all places of $|\det(\sigma)|_v$ equals 1, we get the equation (2.2) from the above Lemma, the equations (2.13) and (2.9).

Consider v a $p-$adic place. For $z \in \mathbf{C}^{2n}$ let χ_z be an unramified character on the subgroup of diagonal matrices of GL_{2n} such that

$$\chi_z(\mathrm{diag}[a_1, \ldots, a_{2n}]) = \prod_{i=1}^{2n} |a_i|_v^{z_i}. \tag{2.16}$$

Define a Hecke algebra homomorphism $\lambda_{1,v}$ from $\mathcal{H}(G_{3,v}, K_{3,v})$ to $\mathcal{H}(G_{1,v}, K_{1,v})$ so that when $f'_v = \lambda_{1,v}(f_v)$,

$$\widehat{f'_v}(z - \frac{1}{2}) = \widehat{f_v}(z) \tag{2.17}$$

where:

$$\begin{aligned}
\widehat{f_v}(z) &= \int_{\mathbf{a} \in T_{2n}(k_v)} \int_{n \in N_3(k_v)} f_v(m(\mathbf{a})n)\chi_z(\mathbf{a})\delta_3^{\frac{1}{2}}(m(\mathbf{a}))\, dn\, d\mathbf{a}, \\
\widehat{f'_v}(z) &= \int_{\mathbf{a} \in T_{2n}(k_v)} \int_{n \in N_1(k_v)} f'_v(\mathbf{a}n)\chi_z(\mathbf{a})\delta_1^{\frac{1}{2}}(\mathbf{a})\, dn\, d\mathbf{a}.
\end{aligned}$$

Here δ_3 and δ_1 are the modulus functions of the Borel subgroups of $\mathrm{Sp}_{2n}(k_v)$ and $\mathrm{GL}_{2n}(k_v)$ respectively. We will let $\epsilon'_{1,v}$ to be the map on $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ defined by (2.1). Then when $f_v \in \mathcal{H}(G_{3,v}, K_{3,v})$,

$$\epsilon'_{1,v}(f_v)(g) = \int_{u\in V} f_v(m(g)u)|\det(g)|_v^{n+1}\, du.$$

From Iwasawa decomposition, we get immediately $\widehat{f_v}(z) = \widehat{\epsilon'_{1,v}(f_v)}(z-\frac{1}{2})$. Thus we get $\epsilon'_{1,v}(f_v) = \lambda_{1,v}(f_v)$; $\epsilon_{1,v}$ restricts to a Hecke algebra homomorphism. □

2.4. **Conclusion.** From Theorem 2.1 we get immediately a map from $f \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$ to $f' \in \mathcal{S}(\mathrm{GL}_{2n}(\mathbf{A}))$, so that the pull back of the distribution $I_{G_1}(f' : H_1, 1; N_1, \theta_1)$ equals $I_{G_3}(f : H_3, 1; N_3, \theta_3)$. We also need to construct a map in the opposite direction, which is given in the following corollary.

Corollary 2.11. *For any place v there exists maps $\epsilon_{1,v} : \mathcal{S}(\mathrm{GL}_{2n}(k_v)) \mapsto \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$, such that equation (2.2) holds for $f = \otimes f_v$ and $f' = \otimes f'_v$ when*

(1) $f'_v = \lambda_{1,v}(f_v)$ for $v \notin S$ a finite set of places containing bad places.

(2) $f_v = \epsilon_{1,v}(f'_v)$ for $v \in S$.

Proof. Given $f'_v \in \mathcal{S}(\mathrm{GL}_{2n}(k_v))$, define $f_{1,v}(p)$ on $P_3(k_v)$ by setting $f_{1,v}(m(\sigma)u) = f'_v(\sigma)\phi(u)$ where $\sigma \in \mathrm{GL}_{2n}(k_v)$, $u \in V(k_v)$ and $\phi(u)$ is a Schwartz function on $V(k_v)$ such that $\int_{V(k_v)} \phi(u)\, du = 1$. Define

$$f_{2,v}(p) = \int_{k\in K_3\cap H_3\cap P_3(k_v)} f_{1,v}(kp)\, dk.$$

Then $f_{2,v}$ is left $K_3 \cap H_3 \cap P_3(k_v)$ invariant. We can extend it to a function $f_{3,v}$ on $H_3P_3(k_v)$ as follows: using the Iwasawa decomposition, any element in $H_3P_3(k_v)$ has the form kp with $k \in K_3 \cap H_3(k_v)$ and $p \in P_3(k_v)$; we let $f_{3,v}(kp) = f_{2,v}(p)$.

As H_3P_3 is a closed subset of Sp_{2n}, the restriction map from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ to $\mathcal{S}(H_3P_3(k_v))$ is surjective. There is a function $f_v \in \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ that restricts to $f_{3,v}$. We will let $f_v = \epsilon_{1,v}(f'_v)$.

We now check that equation (2.2) holds under the conditions in the corollary. For the given f'_v, define a function on $\mathrm{GL}_{2n}(k_v)$:

$$f''_v(g) = \int_{K_1 \cap H_1(k_v)} f'_v(kg)\, dk.$$

Then $f'' = \otimes f''_v \in \mathcal{S}(\mathrm{GL}_{2n}(\mathbf{A}))$ and

$$I(f'' : H_1, 1; N_1, \theta_1) = I(f' : H_1, 1; N_1, \theta_1). \tag{2.18}$$

When $v \notin S$, $f'_v = \lambda_{1,v}(f_v)$; since f'_v is in $\mathcal{H}(G_{1,v}, K_{1,v})$, we have $f''_v = f'_v$. Thus from the last statement of Theorem 2.1 (and its proof), we have f_v and f''_v satisfy equation (2.1). When $v \in S$, $f_v = \epsilon_{1,v}(f'_v)$; it is easy to check that f_v and f''_v again satisfy equation (2.1). It follows from Theorem 2.1, $I(f : H_3, 1; N_3, \theta_3)$ equals $I(f'' : H_1, 1; N_1, \theta_1)$. From (2.18) we get the claim of the corollary. □

3. Some global identities on Sp_{2n}

The purpose of this section is to relate the distributions $I_{G_3}(f : H_3, 1; N_3, \theta_3)$ with $I_{G_3}(f : H_3, 1; N_3, \theta_4 \Theta^{\Phi}_{\psi^{-1}})$ on Sp_{2n}. We will introduce two more distributions $I_1(f)$ and $I_2(f)$ on Sp_{2n}, then get identity (1.6) as the result of three global identities. Here we rely heavily on results from [GRS2].

For $f \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$, define

$$\Psi_f(g) = \int_{l \in H_3(k)\backslash H_3(\mathbf{A})} K_f(l, g)\, dl. \tag{3.1}$$

Then $\Psi_f(g)$ is a left $\mathrm{Sp}_{2n}(k)$ invariant form on $\mathrm{Sp}_{2n}(\mathbf{A})$ satisfying the moderate growth condition: $|\Psi_f(g)|$ is bounded by a polynomial in $||g||$ where

$$||g|| = \prod_v ||g_v||_v = \prod_v (\max_{i,j}\{|g_{i,j,v}|_v, |g^{-1}_{i,j,v}|_v\}). \tag{3.2}$$

Clearly $f \mapsto \Psi_f(g)$ is a linear map. When $f_{g'}(g) = f(gg')$, we have $\Psi_{f'_g}(g) = \Psi_f(gg')$.

3.1. **Definition of $I_1(f)$.** We recall the definition of sets $\mathcal{X}_0$ and $Y^*_{n-1,n}$, elements ν_0 and ω in [GRS2, §4].

Let $\widetilde{\omega}$ be a permutation matrix in GL_{2n} such that

$$\widetilde{\omega}_{2i,i} = 1, \;\; \widetilde{\omega}_{2i-1,n+i} = 1, \;\; i = 1, \dots, n.$$

Recall m is map from GL_{2n} to Sp_{2n}: $m(g) = \left(\begin{smallmatrix} g & \\ & g^* \end{smallmatrix}\right)$. Let $\omega = m(\widetilde{\omega})$. Let

$$a = \mathrm{diag}[b, \dots, b, b^*, \dots, b^*] \in \mathrm{Sp}_{2n}, \;\; b = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right),$$

and ν be the Weyl element in Sp_{2n} such that

$$\nu_{i,2i-1} = \nu_{n+i,2n+2i-1} = \nu_{3n+i,2n+2i} = 1, \;\; \nu_{2n+i,2i} = -1, \;\; i = 1, \dots, n.$$

Let $\nu_0 = \nu a$. We only need to note here that ν_0 and ω are elements in $\mathrm{Sp}_{2n}(k)$; and over a $p-$adic place v where p is odd, ν_0 and ω lie in the maximal compact subgroup of $\mathrm{Sp}_{2n}(k_v)$.

Recall (from §1.4) the element $\sigma_n \in \mathrm{GL}_n$ is the longest Weyl element, and the set $\mathcal{S}_n$ is the set of matrices $g \in \mathrm{GL}_n$ satisfying $\sigma_n g$ is a symmetric matrix. Let

$$\mathcal{X}_0 = \{x \in \mathcal{S}_{2n} \,|\, x \text{ is nilpotent and upper triangular}\}. \tag{3.3}$$

For $x \in \mathcal{X}_0$, let

$$\bar{l}(x) = \begin{pmatrix} 1_{2n} & \\ x & 1_{2n} \end{pmatrix}. \tag{3.4}$$

Let $T(n) \subset \mathrm{GL}_{2n}$ be defined as in [GRS2, (4.34),(4.35)], then $T(n) = \widetilde{\omega} N_{\backslash n} \widetilde{\omega}^{-1}$ where $N_{\backslash n}$ denotes the subgroup of N_1 consisting of matrices whose $n-$th row has only one nonzero entry. Let $Y^*_{n-1,n}$ be the set

$$\{m(T) \,|\, T \in T(n),\, T \text{ is lower triangular}\} \subset \mathrm{Sp}_{2n}. \tag{3.5}$$

Define:

$$I_1(f) = \int_{y^* \in Y^*_{n-1,n}(\mathbf{A})} \int_{x \in \mathcal{X}_0(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \Psi_f(n\bar{l}(x)\nu_0 y^* \omega)\theta_3(n)\, dn\, dx\, dy^*. \tag{3.6}$$

This definition is motivated by the Corollary on p.895 of [GRS2]. The integral over X_0 and $Y^*_{n-1,n}$ are absolutely convergent, which is clear from another expression (3.20) after applying Dixmier-Malliavin Theorem.

3.2. **Definition of $I_2(f)$.** Let j be the injection from Sp_n to Sp_{2n}:

$$j:\ g \mapsto j(g) = \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix}. \tag{3.7}$$

We define some subgroups of N_3. Let Z_i be the maximal unipotent subgroup of GL_i consisting of upper triangular matrices with unit diagonal. Let

$$\hat{N}^k = \{v = \begin{pmatrix} z & * & * \\ & 1_{4n-2k+2} & * \\ & & z^* \end{pmatrix} \in N_3 |\, z \in Z_{k-1}\}. \tag{3.8}$$

Then $\hat{N}^k$ is a normal subgroup of $\hat{N}^j$ whenever $k < j \leq 2n+1$.

Define a subgroup U^n of N_3:

$$U^n = \{\eta(\mathbf{x},\mathbf{y},t) = \begin{pmatrix} 1_{n-1} & & & & & \\ & 1 & \mathbf{x} & \mathbf{y} & t & \\ & & 1_n & 0 & * & \\ & & & 1_n & * & \\ & & & & 1 & \\ & & & & & 1_{n-1} \end{pmatrix}\}. \tag{3.9}$$

Then U^n is a Heisenberg group and is isomorphic to $\hat{N}^n \backslash \hat{N}^{n+1}$. Let U^n_0 be the normal subgroup of U^n consisting of $\eta(\mathbf{0},\mathbf{y},t)$.

Define $\tilde{N}^n$ to be $U^n_0 \hat{N}^n$. Define a character $\tilde{\chi}_n$ on $\tilde{N}^n(\mathbf{A})$, such that for $n = \eta(\mathbf{0},\mathbf{y},t)n'$ with $n' \in \hat{N}^n$:

$$\tilde{\chi}_n(\eta(\mathbf{0},\mathbf{y},t)n') = \psi(\sum_{i=1}^{n-1} n'_{i,i+1} + t). \tag{3.10}$$

Note that $j(N_2)\tilde{N}^n$ is a group with $\tilde{N}^n$ being a normal subgroup. Define

$$I_2(f) = \int_{n_2 \in N_2(k)\backslash N_2(\mathbf{A})} \int_{v \in \tilde{N}^n(k)\backslash \tilde{N}^n(\mathbf{A})} \Psi_f(vj(n_2))\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, dv\, dn_2. \tag{3.11}$$

This expression can be rewritten as $I_{G_3}(f : H_3, 1; j(N_2)\tilde{N}^n, \theta_2\tilde{\chi}^{-1})$, which is absolutely convergent.

3.3. **Global identity 1: between $I_1(f)$ and $I_2(f)$.**

Proposition 3.1. *The equation $I_1(f) = I_2(f)$ holds for any $f \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$.*

Before prove the Proposition, we recall two more notations from [GRS2]. Let $E_{2n} \subset N_3$ consisting of u with $u_{i,i+1} = 0$ when i is odd; let ψ^{2n} be a character of E_{2n} such that (see [GRS2, p. 879])

$$\psi^{2n}(u) = \psi(\sum_{i=1}^{2n-2} u_{i,i+2} + u_{2n-1,2n+2} - u_{2n,2n+1}).$$

Proof. We apply [GRS2, Theorem 5.2], with Ψ_f in place of ξ in that Theorem. (Note the Theorem clearly applies to Ψ_f). Then theorem states:

$$I_2(f) = \int_{Y^*_{n-1,n}(\mathbf{A})} \int_{u \in E_{2n}(k)\backslash E_{2n}(\mathbf{A})} \Psi_f(uy^*\omega)\psi^{2n}(u)\, du\, dy^*. \tag{3.12}$$

Meanwhile [GRS2, Theorem 5.1] gives an expression for

$$\int_{u \in E_{2n}(k)\backslash E_{2n}(\mathbf{A})} \Psi_f(u)\psi^{2n}(u)\, du \tag{3.13}$$

if we consider $\Psi_f(g)$ in the place of $\mathrm{Res}_{s=1} E(g, \phi_{\eta,s})$ in that Theorem. Again all the steps in the proof of [GRS2, Theorem 5.1] carries through for $\Psi_f(g)$ until we reach equation (5.16) of [GRS2]. From [GRS2, p.890], the left hand side of [GRS2, (5.16)] is just the expression (3.13). From [GRS2, (5.16)] we get

$$\int_{E_{2n}(k)\backslash E_{2n}(\mathbf{A})} \Psi_f(u)\psi^{2n}(u)\, du = \int_{x \in \mathcal{X}_0(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \Psi_f(n_3\bar{l}(x)\nu_0)\theta_3(n_3)\, dn_3\, dx. \tag{3.14}$$

From the equations (3.12), (3.14) and the definition of $I_1(f)$ in (3.6), we get the equation in Proposition. □

3.4. **Global identity 2: between $I_1(f)$ and $I_{G_3}(f : H_3, 1; N_3, \theta_3)$.** Recall:

$$I_{G_3}(f : H_3, 1; N_3, \theta_3) = \int_{N_3(k)\backslash N_3(\mathbf{A})} \Psi_f(n)\theta_3(n)\, dn. \tag{3.15}$$

Theorem 3.2. *There exist maps $\epsilon_{2,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ to itself, such that*

(1) the equation

$$I_{G_3}(f : H_3, 1; N_3, \theta_3) = I_1(f') \tag{3.16}$$

holds for $f = \otimes f_v$, $f' = \otimes f'_v$ when $f'_v = \epsilon_{2,v}(f_v)$.

(2) for v a good place, $\epsilon_{2,v}$ restricts to identity map on Hecke algebra $\mathcal{H}(G_2, K_2)_v$.

Similarly there exist maps $\epsilon'_{2,v}$ satisfying condition (2) such that (3.16) holds when $f_v = \epsilon'_{2,v}(f'_v)$.

Proof. We prove the existence of f_v corresponding to f'_v (the map $\epsilon'_{2,v}$) in four steps. First if we set $f^1_v(g) = f'_v(g\omega)$ for all v, then we have

$$I_1(f') = \int_{y^*\in Y^*_{n-1,n}(\mathbf{A})} \int_{x\in\mathcal{X}_0(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \Psi_{f^1}(n\bar{l}(x)\nu_0 y^*)\theta_3(n)\, dn\, dx\, dy^*. \tag{3.17}$$

When f'_v is a Hecke function at a good place v, we have $f^1_v(g) = f'_v(g)$.

Next we state a Lemma. Note $Y^*_{n-1,n}$ is an abelian group and can be written as a direct product $\prod_{i=1}^{n-1} K_i$ where

$$K_i = \{(k_i(t_1, \dots, t_i) = m(1_{2n} + \sum_{j=1}^{i} t_j e_{2i,2j-1})\}.$$

Let $K^i = \prod_{l=1}^{i} K_i$. Then $K^0 = \{1_{4n}\}$ and $K^{n-1} = Y^*_{n-1,n}$.

Define a homomorphism r_i from $\mathbf{A}^{i-1}$ to $\mathrm{Sp}_{2n}(\mathbf{A})$:

$$r_i(t_1, \dots, t_{i-1}) = m(1_{2n} + \sum_{j=1}^{i-1} t_j e_{2j-1,2i}).$$

Let L be an $\mathrm{Sp}_{2n}(\mathbf{A})$ invariant space of smooth functions on $\mathrm{Sp}_{2n}(\mathbf{A})$ such that when $f(g) \in L$, $f(ug) = \psi^{2n}(u^{-1})f(g)$ for $u \in E_{2n}$.

From the proof of [GRS2, Lemma 5.1], we get the following:

Lemma 3.3. *For fixed i and a function $h_i(g) \in L$, such that $h_i(g)$ equals*

$$\sum_\alpha \int_{\mathbf{A}^i} \phi_\alpha(x_1, \dots, x_i) h_\alpha(g r_i(x_1, \dots, x_i))\, d(x_1, \dots, x_i)$$

for some $h_\alpha \in L$ and $\phi_\alpha \in \mathcal{S}(\mathbf{A}^i)$, we have

$$\int_{K^i(\mathbf{A})} h_i(y)\, dy = \int_{K^{i-1}(\mathbf{A})} h_{i-1}(y)\, dy, \tag{3.18}$$

where

$$h_{i-1}(g) = \sum_\alpha \int_{\mathbf{A}^i} \widehat{\phi_\alpha}(x_1, \dots, x_i) h_\alpha(g r_i(x_1, \dots, x_i))\, d(x_1, \dots, x_i),$$

$\widehat{\phi_\alpha}$ *is the Fourier transform of ϕ_α:*

$$\widehat{\phi_\alpha}(x_1, \dots, x_i) = \int \phi_\alpha(t_1, \dots, t_i) \psi(\sum_{j=1}^i x_i t_i)\, d(t_1, \dots, t_i).$$

Let

$$h_f(g) = \int_{x \in \mathcal{X}_0(\mathbf{A})} \int_{N_3(k)\backslash N_3(\mathbf{A})} \Psi_f(n \bar{l}(x) \nu_0 g) \theta_3(n)\, dn\, dx.$$

From equation (3.14) we get $h_f(g) \in L$. Moreover the righthand side of (3.17) is

$$\int_{K^{n-1}(\mathbf{A})} h_{f^1}(y) dy.$$

Assume now $f^1 = \otimes f_v^1$. From the Theorem of Dixmier-Malliavin [DMa], any f_v^1 can be expressed as

$$f_v^1(g) = \sum_{\alpha_v} \int_{k_v^i} \phi_{\alpha_v}(x_1, \dots, x_i) f_{\alpha_v}(g r_i(x_1, \dots, x_i))\, d(x_1, \dots, x_i). \tag{3.19}$$

for some $f_{\alpha_v} \in \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ and $\phi_{\alpha_v} \in \mathcal{S}(k_v^i)$. Moreover at good places, f_v^1 is a Hecke function, and can be expressed as above with a single α_v, with ϕ_{α_v} being the characteristic function of the integer lattice, and $f_{\alpha_v} = f_v^1$.

We have

$$h_{f^1}(g) = \sum_\alpha \int_{\mathbf{A}^i} \phi_\alpha(x_1, \dots, x_i) h_{f_\alpha}(g r_i(x_1, \dots, x_i))\, d(x_1, \dots, x_i) \tag{3.20}$$

for some $f_\alpha \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$ and $\phi_\alpha \in \mathcal{S}(\mathbf{A}^i)$. Thus from the equation (3.18), we get a $f^1_{n-2} = \otimes f^1_{n-2,v} \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$ defined by

$$f^1_{n-2,v}(g) = \sum_{\alpha_v} \int_{k_v^i} \widehat{\phi_{\alpha_v}}(x_1, \dots, x_i) f_{\alpha_v}(g r_i(x_1, \dots, x_i))\, d(x_1, \dots, x_i), \tag{3.21}$$

satisfying

$$\int_{K^{n-1}(\mathbf{A})} h_{f^1}(y) dy = \int_{K^{n-2}(\mathbf{A})} h_{f^1_{n-2}}(y)\, dy. \tag{3.22}$$

Note that from (3.21), $f^1_{n-2,v} = f^1_v$ when f^1_v is a Hecke function at a good place v. Continue the procedure we get eventually $f^2 = f^1_0 \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$, satisfying $f^2_v = f^1_v$ when f^1_v is a Hecke function at a good place v, and

$$\int_{K^{n-1}(\mathbf{A})} h_{f^1}(y) dy = \int_{K^0(\mathbf{A})} h_{f^1_0}(y)\, dy = h_{f^2}(1_{4n}).$$

We get righthand side of (3.17) equals:

$$\int_{x \in \mathcal{X}_0(\mathbf{A})} \int_{N_3(k) \backslash N_3(\mathbf{A})} \Psi_{f^2}(n \bar{l}(x) \nu_0) \theta_3(n)\, dn\, dx. \tag{3.23}$$

Moreover $f^2_v = f^1_v$ when f^1_v is a Hecke function at a good place v.

By letting $f^3(g) = f^2(g\nu_0)$, we get (3.23) equals

$$\int_{x \in \mathcal{X}_0(\mathbf{A})} \int_{N_3(k) \backslash N_3(\mathbf{A})} \Psi_{f^3}(n \bar{l}(x)) \theta_3(n)\, dn\, dx. \tag{3.24}$$

Clearly $f^3_v = f^2_v$ when f^2_v is a Hecke function at a good place v.

Consider now

$$\bar{h}_f(g) = \int_{N_3(k) \backslash N_3(\mathbf{A})} \Psi_f(ng) \theta_3(n)\, dn.$$

It lies in the space $\bar{L}$ consisting of functions satisfying $\phi(ng) = \theta_3(n^{-1})\phi(g)$. Similar to the situation of Lemma 3.3, we have the equation

$$\int_{\bar{K}^i(\mathbf{A})} \bar{h}_i(y)\, dy = \int_{\bar{K}^{i-1}(\mathbf{A})} \bar{h}_{i-1}(y)\, dy \tag{3.25}$$

holds for $\bar{h}_i, \bar{h}_{i-1} \in \bar{L}$ related as in Lemma 3.3, (with homomorphism r replaced by a different homomorphism $\bar{r}_i$). Here the subgroups $\bar{K}^i$ are defined as on [GRS2, p.897]; we note $\bar{K}^{2n} = \bar{l}(\mathcal{X}_0)$ and $\bar{K}^1 = \{1_{4n}\}$. Similar to the above argument, using (3.25) we get a function $f^4 \in \mathcal{S}(\mathrm{Sp}_{2n}(\mathbf{A}))$, with $f^4_v = f^3_v$ when f^3_v is a Hecke function at a good place v, such that (3.24) equaling $\bar{h}_{f^4}(1_{4n})$. Since $\bar{h}_{f^4}(1_{4n})$ is just $I_{G_3}(f^4 : H_3, 1; N_3, \theta_3)$, we can set $f = \epsilon'_2(f') = f^4$. Then $f_v = f'_v$ when f^1_v is a Hecke function at a good place v, and the equality (3.16) holds.

As each of the steps above can be reversed, given f, we can find $f' = \epsilon_2(f)$ to make the equality (3.16) hold. □

3.5. **Heisenberg representation and definition of $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$.** Recall the definition of the map η in (3.9). The character ψ determines an irreducible unitary representation of Heisenberg group U^n acting on $\mathcal{S}(k^n)$, denoted again by ω_ψ; then:

$$\omega_\psi(\eta(\mathbf{x},0,0))\Phi(X) = \Phi(X+\mathbf{x}). \tag{3.26}$$

$$\omega_\psi(\eta(0,\mathbf{y},t))\Phi(X) = \psi(t + 2\,\mathrm{tr}(\mathbf{y}\sigma_n X))\Phi(X). \tag{3.27}$$

$$\omega_\psi(\widetilde{g})\omega_\psi(j(g)^{-1}uj(g))\Phi(X) = \omega_\psi(u)\omega_\psi(\widetilde{g})\Phi(X),\ g \in \mathrm{Sp}_n,\ u \in U^n. \tag{3.28}$$

For $n \in N_3$, we will use $\mathrm{pr}(n)$ to denote the middle $2n+2 \times 2n+2$ block of n. Then $\mathrm{pr}(n) = j(n_2)\eta(\mathbf{x},\mathbf{y},t)$ for some $\eta(\mathbf{x},\mathbf{y},t)$ in the Heisenberg group and $n_2 \in N_2$. We define:

$$\omega_{\psi^{-1}}(n)\Phi(X) = \omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(\eta(\mathbf{x},\mathbf{y},t))\Phi(X),\ \mathrm{pr}(n) = j(n_2)\eta(\mathbf{x},\mathbf{y},t). \tag{3.29}$$

Clearly the above defines an action of N_3 on the space $\mathcal{S}(\mathbf{A}^n)$. Define Theta function

$$\Theta^{\Phi}_{\psi^{-1}}(n) = \sum_{X \in k^n} \omega_{\psi^{-1}}(n)\Phi(X). \tag{3.30}$$

Define a character θ_4 on N_3 by setting

$$\theta_4(n) = \psi(\sum_{i=1}^{n-1} -n_{i,i+1})\theta_2(n_2),\ \text{if}\ \mathrm{pr}(n) = j(n_2)\eta(\mathbf{x},\mathbf{y},t). \tag{3.31}$$

Define $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ to be:

$$\int_{l \in H_3(k)\backslash H_3(\mathbf{A})} \int_{n \in N_3(k)\backslash N_3(\mathbf{A})} K_f(l,n)\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n)\,dn\,dl. \tag{3.32}$$

3.6. **Global identity 3: between $I_2(f)$ and $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$.**

Theorem 3.4. *We have $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}) = I_2(f')$ when*

$$f'(g) = \int_{X\in\mathbf{A}^n} \Phi(X) f(g\eta(X, \mathbf{0}, 0))\, dX. \tag{3.33}$$

Proof. The distribution $I_2(f')$ equals

$$\int_{N_2(k)\backslash N_2(\mathbf{A})} \int_{\tilde{N}^n(k)\backslash\tilde{N}^n(\mathbf{A})} \int_{X\in\mathbf{A}^n} \Psi_f(vj(n_2)\eta(X, 0, 0))\Phi(X)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, dX\, dv\, dn_2.$$

From the definition of Weil representation (1.7), (1.8) and (3.26), we get:

$$\Phi(X) = \omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(\eta(X, \mathbf{0}, 0))\Phi(0).$$

Thus if we identify the set of $\{\eta(X, 0, 0)\}$ with $U_0^n\backslash U^n$, we get the above distribution is

$$\int_{N_2(k)\backslash N_2(\mathbf{A})} \int_{\tilde{N}^n(k)\backslash\tilde{N}^n(\mathbf{A})} \int_{u\in U_0^n\backslash U^n(\mathbf{A})} \Psi_f(vj(n_2)u) \\ \omega_{\psi^{-1}}(\widetilde{n_2})\omega_{\psi^{-1}}(u)\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, du\, dv\, dn_2.$$

As $j(N_2)$ acts on U^n by conjugation and stabilizes U_0^n, a change of variable $u \mapsto j(n_2)^{-1}uj(n_2)$ along with (3.28) gives

$$\int_{N_2(k)\backslash N_2(\mathbf{A})} \int_{\tilde{N}^n(k)\backslash\tilde{N}^n(\mathbf{A})} \int_{u\in U_0^n\backslash U(\mathbf{A})} \Psi_f(vuj(n_2)) \\ \omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{n_2})\Phi(0)\theta_2(n_2)\tilde{\chi}_n^{-1}(v)\, du\, dv\, dn_2.$$

Let $f_{n_2}(g) = f(gj(n_2))$ and $\Phi_{n_2} = \omega_{\psi^{-1}}(\widetilde{n_2})\Phi$, then we get

$$I_2(f) = \int_{N_2(k)\backslash N_2(\mathbf{A})} \Xi(f_{n_2}, \Phi_{n_2})\theta_2(n_2)\, dn_2, \tag{3.34}$$

where

$$\Xi(f, \Phi) = \int_{u\in U_0^n\backslash U^n(\mathbf{A})} \int_{\tilde{N}^n(k)\backslash\tilde{N}^n(\mathbf{A})} \Psi_f(vu)\omega_{\psi^{-1}}(u)\Phi(0)\tilde{\chi}_n^{-1}(v)\, dv\, du. \tag{3.35}$$

As $\tilde{N}^n = U_0^n \hat{N}^n$, we get $\Xi(f,\Phi)$ equals

$$\int_{u\in U_0^n(k)\backslash U^n(\mathbf{A})} \int_{\hat{N}^n(k)\backslash \hat{N}^n(\mathbf{A})} \Psi_f(vu)\omega_{\psi^{-1}}(u)\Phi(0)\tilde{\chi}_n^{-1}(v)\,dv\,du$$

$$= \int_{u\in U^n(k)\backslash U^n(\mathbf{A})} \sum_{u'\in U_0^n(k)\backslash U^n(k)} \int_{\hat{N}^n(k)\backslash \hat{N}^n(\mathbf{A})} \Psi_f(vu'u)\omega_{\psi^{-1}}(u'u)\Phi(0)\tilde{\chi}_n^{-1}(v)\,dvd\,u.$$

Make a change of variable $v \mapsto u'v(u')^{-1}$ and use the fact $\tilde{\chi}$ is stabilized under the conjugation of U^n, we get the above is:

$$\int_{u\in U^n(k)\backslash U^n(\mathbf{A})} \int_{\hat{N}^n(k)\backslash \hat{N}^n(\mathbf{A})} \Psi_f(vu) \sum_{u'\in U_0^n(k)\backslash U^n(k)} \omega_{\psi^{-1}}(u'u)\Phi(0)\tilde{\chi}_n^{-1}(v)\,dv\,du.$$

Notice:

$$\sum_{u'\in U_0^n(k)\backslash U^n(k)} \omega_{\psi^{-1}}(u'u)\Phi(0) = \sum_{X\in k^n} \omega_{\psi^{-1}}(u)\Phi(X) = \Theta^{\Phi}_{\psi^{-1}}(u).$$

We get:

$$\Xi(f,\Phi) = \int_{u\in U^n(k)\backslash U^n(\mathbf{A})} \int_{\hat{N}^n(k)\backslash \hat{N}^n(\mathbf{A})} \Psi_f(vu)\Theta^{\Phi}_{\psi^{-1}}(u)\tilde{\chi}_n^{-1}(v)\,dv\,du. \tag{3.36}$$

On the other hand since $\hat{N}^{n+1}\backslash N_3 \cong j(N_2)$ and $\hat{N}^n\backslash \hat{N}^{n+1} \cong U^n$, the right hand side of (3.32) is given by

$$I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}) = \int_{N_2(k)\backslash N_2(\mathbf{A})} \Xi'(f_{n_2}, \Phi_{n_2})\theta_2(n_2)\,dn_2, \tag{3.37}$$

where

$$\Xi'(f,\Phi) = \int_{u\in U^n(k)\backslash U^n(\mathbf{A})} \int_{\hat{N}^n(k)\backslash \hat{N}^n(\mathbf{A})} \Psi_f(vu)\Theta^{\Phi}_{\psi^{-1}}(u)\theta_4(v)\,dv\,du. \tag{3.38}$$

As $\theta_4(v) = \tilde{\chi}_n^{-1}(v)$ when $v \in \hat{N}^n$, compare with (3.36) we get $\Xi'(f,\Phi) = \Xi(f,\Phi)$. Therefore from (3.34) and (3.37) we get the identity in the Theorem. □

We note that the above identity between $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ and $I_2(f')$ (which is $I_{G_3}(f' : H_3, 1; \tilde{N}^n j(N_2), \tilde{\chi}_n^{-1}\theta_2)$ using the notation in introduction) can not be proven through matching orbital integrals. Even in the case of $n = 1$, the orbits do not match.

Corollary 3.5. *There exist maps $\epsilon_{3,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ to $\mathcal{S}(\mathrm{Sp}_{2n}(k_v)) \otimes \mathcal{S}(k_v^n)$, such that:*

(1) at a good place v, $\epsilon_{3,v}(f_v) = f_v \otimes \Phi_{0,v}$ when f_v is a Hecke function and $\Phi_{0,v}$ is the characteristic function of $\mathcal{O}_v^n$.

(2) when $\epsilon_3 = \otimes \epsilon_{3,v}$ and $f \otimes \Phi = \epsilon_3(f')$:

$$I_2(f') = I_{G_3}(f : H_3, 1; N_3, \theta_4 \Theta^{\Phi}_{\psi^{-1}}). \tag{3.39}$$

Proof. From Theorem 3.4, to define $\epsilon_{3,v}(f'_v)$ so that (3.39) holds, we only need to find f and Φ so that (3.33) holds. The map $(f, \Phi) \mapsto f'$ defined by (3.33) is a convolution, clearly factors into local maps. The existence of f_v and Φ_v follows from the result of Dixmier-Malliavin [DMa]. For v a good place, it is clear that when f_v is a Hecke function and $\Phi_v = \Phi_{0,v}$,

$$\int_{X \in k_v^n} f_v(g\eta(X, 0, 0)) \Phi_{0,v}(X) \, dX = f_v(g). \tag{3.40}$$

Thus at good place v, we can choose $f_v = f'_v$ and Φ_v to be $\Phi_{0,v}$. □

We remark that equation (3.33) defines the map $\epsilon'_{3,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v)) \otimes \mathcal{S}(k_v^n)$ to $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$, with the property that at good places $\epsilon'_{3,v}(f_v \otimes \Phi_{0,v}) = f_v$ when f_v is a Hecke function, and equation (3.39) holds when $f' = \epsilon'_3(f \otimes \Phi)$.

3.7. **Conclusion.** Combining the three global identities on Sp_{2n}, we get:

Corollary 3.6. *There exist maps $\epsilon_{4,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ to $\mathcal{S}(\mathrm{Sp}_{2n}(k_v)) \otimes \mathcal{S}(k_v^n)$ such that:*

(1) at a good place v, $\epsilon_{4,v}(f_v) = f_v \otimes \Phi_{0,v}$ when f_v is a Hecke function and $\Phi_{0,v}$ is the characteristic function of $\mathcal{O}_v^n$.

(2) when $\epsilon_4 = \otimes \epsilon_{4,v}$ and $f \otimes \Phi = \epsilon_4(f')$:

$$I_{G_3}(f' : H_3, 1; N_3, \theta_3) = I_{G_3}(f : H_3, 1; N_3, \theta_4 \Theta^{\Phi}_{\psi^{-1}}). \tag{3.41}$$

Proof. Define $\epsilon_{4,v} = \epsilon_{3,v}\epsilon_{2,v}$. The claim follows from Proposition 3.1, Theorem 3.2 and Corollary 3.5 □

We remark that one can also define the maps $\epsilon'_{4,v} = \epsilon'_{2,v}\epsilon'_{3,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v)) \otimes \mathcal{S}(k_v^n)$ to $\mathcal{S}(\mathrm{Sp}_{2n}(k_v))$, such that at a good place $\epsilon'_{4,v}(f_v \otimes \Phi_{0,v}) = f_v$ when f_v is a Hecke function, and equation (3.41) holds when $f' = \epsilon'_4(f \otimes \Phi)$.

4. Orbital integral decompositions

We turn to the third step discussed in the introduction, which is the comparison between two distributions $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ and $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$. In this section, we study the decomposition of the distributions into sums of orbital integrals. The results are stated in Propositions 4.9 and 4.10.

4.1. **Symmetric space Y.** Define matrices:

$$E' = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}, \; E'_1 = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}, \; E = m(E'), \; E_1 = m(E'_1). \tag{4.1}$$

Define for $g \in \mathrm{Sp}_{2n}$,

$$\theta(g) = EgE^{-1} = EgE.$$

Then θ is an involution. The centralizer of E in Sp_{2n} is H_3. Let

$$Y = \{g^{-1}\theta(g)E \mid g \in \mathrm{Sp}_{2n}\}.$$

Then $Y \cong H_3\backslash G_3$.

The distribution $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ is given by (3.32). We can unwind it as:

$$\sum_{\gamma \in H_3(k)\backslash G_3(k)/N_3(k)} \int_{n \in N'_{3,\gamma}(k)\backslash N_3(\mathbf{A})} \int_{l \in H_3(\mathbf{A})} f(l^{-1}\gamma n)\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n)\, dl\, dn.$$

where

$$N'_{3,\gamma} = \gamma^{-1}H_3\gamma \cap N_3.$$

Define:

$$F(g^{-1}Eg) = \int_{l \in H_3(\mathbf{A})} f(l^{-1}g)\, dl. \tag{4.2}$$

Then F is a well defined function on $Y(\mathbf{A})$. The above distribution can be written as

$$\sum_{\gamma\in H_3(k)\backslash G_3(k)/N_3(k)} \int_{n\in N'_{3,\gamma}(k)\backslash N_3(\mathbf{A})} F(n^{-1}\gamma^{-1}E\gamma n)\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n)\,dn. \tag{4.3}$$

4.2. N_3**-orbits on** Y**.** Recall [GRS1, Proposition 18]:

Proposition 4.1. *Each $y \in Y$ can be written in the form*

$$y = n^{-1}w\mathbf{a}n,$$

where $n \in N_3$, $w \in W(\mathrm{Sp}_{2n})$, $\mathbf{a}$ a diagonal matrix, satisfying $w\mathbf{a} = (w\mathbf{a})^{-1}$.

With the conjugation action, the N_3 orbits of Y are represented by $w\mathbf{a}$ as above. Thus each representative $\gamma \in H_3\backslash G_3/N_3$ corresponds to a representative $w\mathbf{a}$ as in Proposition 4.1, such that $\gamma^{-1}E\gamma$ and $w\mathbf{a}$ are in the same N_3 orbits of Y. Define for $y \in Y$,

$$N_{3,y} = \{n \in N_3 \,|\, n^{-1}yn = y\}.$$

Lemma 4.2. *When $\gamma^{-1}E\gamma = w\mathbf{a}$, we have $N'_{3,\gamma} = N_{3,w\mathbf{a}}$.*

Proof. An element $n \in N_3$ lies in $N'_{3,\gamma}$ if and only if $\gamma n\gamma^{-1} \in H_3$. It lies in $N_{3,w\mathbf{a}}$ if and only if

$$n^{-1}\gamma^{-1}E\gamma n = \gamma^{-1}E\gamma,$$

which is equivalent to $\gamma n\gamma^{-1}$ centralizes E. As the centralizer of E is H_3, we get the Lemma. □

The distribution (4.3) can now be written as:

$$\sum_{w\mathbf{a}\in \mathrm{Sp}_{2n}(k),\, w\mathbf{a}=(w\mathbf{a})^{-1}} \int_{n\in N_{3,w\mathbf{a}}(k)\backslash N_3(\mathbf{A})} F(n^{-1}w\mathbf{a}n)\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n)\,dn. \tag{4.4}$$

4.3. **Relevant orbits.** Given a representative $w\mathbf{a}$, the summand in (4.4) is

$$\int_{n\in N_{3,w\mathbf{a}}(\mathbf{A})\backslash N_3(\mathbf{A})} F(n^{-1}w\mathbf{a}n)\int_{n'\in N_{3,w\mathbf{a}}(k)\backslash N_{3,w\mathbf{a}}(\mathbf{A})} \theta_4(n'n)\Theta^{\Phi}_{\psi^{-1}}(n'n)\,dn'\,dn. \tag{4.5}$$

We introduce a subgroup $N^{(n)}$ of $\tilde{N}^n\subset N_3$: an element of $\tilde{N}^n$ can be written as $\eta(\mathbf{0},\mathbf{y},t)n$ with $n\in\hat{N}^n$; $N^{(n)}$ consists of elements with $\mathbf{y}=\mathbf{0}$. Then for $n'\in N^{(n)}$:

$$\theta_4(n'n)\Theta^{\Phi}_{\psi^{-1}}(n'n)=\tilde{\chi}_n^{-1}(n')\theta_4(n)\Theta^{\Phi}_{\psi^{-1}}(n).$$

For the inner integral in (4.5) to be nonzero, $\tilde{\chi}_n$ must be trivial on $N_{3,w\mathbf{a}}\cap N^{(n)}$.

We use a special case of [GRS1, Lemma 22].

Lemma 4.3. *If $\tilde{\chi}_n$ is trivial on $N^{(n)}\cap N_{3,w\mathbf{a}}$, then w has the form:*

$$w=\begin{pmatrix}0&*&*&0\\ *&0&0&*\\ *&0&0&*\\ 0&*&*&0\end{pmatrix}.$$

Here each entry represents a $n\times n$ block.

Proof. [GRS1, Lemma 21] says w has the form:

$$w=\begin{pmatrix}0&A&B&0\\ C&D&E&F\\ G&H&I&J\\ 0&K&L&0\end{pmatrix}$$

where each entry represents a $n\times n$ block. Since w^2 is a diagonal matrix from Proposition 4.1, we get the above form

$$\begin{pmatrix}A&B\\ K&L\end{pmatrix}\begin{pmatrix}D&E\\ H&I\end{pmatrix}=0,\ \begin{pmatrix}A&B\\ K&L\end{pmatrix}\begin{pmatrix}C&F\\ G&J\end{pmatrix}\text{ is invertible.}$$

Thus $D=E=H=I=0$ and w is of the form in the Lemma. $\square$

Recall given an element $g \in \mathrm{Sp}_n$, we defined $j(g)$ an element in Sp_{2n} by (3.7). Recall the element E_1 defined in (4.1). For $g \in \mathrm{Sp}_n$, we will define:

$$\mathrm{P}(g) = j(g)^{-1} E_1 j(g) \in Y. \tag{4.6}$$

Lemma 4.4. *The map* P *defines a bijection from the set of* $w'\mathbf{a}'$ *where* $w' \in W(\mathrm{Sp}_n)$ *and* $\mathbf{a}'$ *is a diagonal matrix in* Sp_n*, to the set of* $w\mathbf{a}$ *as in Proposition 4.1, with* w *as in Lemma 4.3.*

Proof. As $E_1^2 = 1_{4n}$, we get $\mathrm{P}(w'\mathbf{a}') = w\mathbf{a}$ satisfies the condition of Proposition 4.1. The form of w is clearly as in Lemma 4.3.

For $g \in H_3$, we denote by $\rho'(g)$ its middle $2n \times 2n$ block. It is clear that

$$\rho'(E_1\, \mathrm{P}(w'\mathbf{a}')) = w'\mathbf{a}'.$$

Thus P is an injection.

On the other hand, given $w\mathbf{a}$ as in the Lemma, $E_1 w\mathbf{a} \in H_3$ and $\rho'(E_1 w\mathbf{a})$ has the form $w'\mathbf{a}'$ where $w' \in W(\mathrm{Sp}_n)$ and $\mathbf{a}'$ is a diagonal matrix in Sp_n. The fact P is a surjection follows from the equality

$$\mathrm{P}(\rho'(E_1 w\mathbf{a})) = w\mathbf{a}. \tag{4.7}$$

We now show this identity. Let

$$s = E_1 j(w'\mathbf{a}') w\mathbf{a} j(w'\mathbf{a}')^{-1}.$$

As $\rho'(E_1 j(w'\mathbf{a}') E_1^{-1}) = 1_{2n}$ and $\rho'(E_1 w\mathbf{a} j(w'\mathbf{a}')^{-1}) = 1_{2n}$, we have $\rho'(s) = 1_{2n}$. Since s satisfies $E_1 s = (E_1 s)^{-1}$ (by Proposition 4.1), we get $s = 1_{4n}$ and the identity (4.7). □

From Lemma 4.4, we see (4.4) can be written as

$$I_{G_3}(f : H_3, 1; N_3, \theta_4 \Theta^{\Phi}_{\psi^{-1}}) = \sum_{w'\mathbf{a}' \in \mathrm{Sp}_n(k), w\mathbf{a} = \mathrm{P}(w'\mathbf{a}')} I_{w\mathbf{a}}(F, \Phi) \tag{4.8}$$

where $I_{w\mathbf{a}}(F, \Phi)$ equals (4.5).

4.4. **Inner integral of (4.5).** In what follows we will assume $w\mathbf{a} = \mathrm{P}(w'\mathbf{a}')$ with $w'\mathbf{a}' \in \mathrm{Sp}_n$. Our goal in this subsection is to show

Proposition 4.5.

$$\int_{n' \in N_{3,w\mathbf{a}}(k)\backslash N_{3,w\mathbf{a}}(\mathbf{A})} \theta_4(n')\Theta^{\Phi}_{\psi^{-1}}(n')\,dn' = c(w'\mathbf{a}')\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0) \tag{4.9}$$

where $c(w'\mathbf{a}')$ is a constant defined in (4.17).

4.4.1. *Description of $N_{3,w\mathbf{a}}$.* When $w\mathbf{a} = E_1$, we get from the definition:

$$N_{3,E_1} = \{u(n,B,T) = \begin{pmatrix} n & & nB & nT \\ & n & nT & nB \\ & & n^* & \\ & & & n^* \end{pmatrix} \in N_3 \,|\, n \in Z_n, B, T \in \mathcal{S}_n\}. \tag{4.10}$$

We use V_{E_1} to denote the intersection of N_{3,E_1} with Siegel unipotent. It consists of $u(1_n, B, T)$. Define $U^1_{E_1}$ to be the subgroup consisting of $u(1_n, B, 0)$.

Lemma 4.6. *When $w\mathbf{a} = \mathrm{P}(w'\mathbf{a}')$ with $w'\mathbf{a}' \in \mathrm{Sp}_n$, we have $N_{3,w\mathbf{a}} = j(w'\mathbf{a}')^{-1}N_{3,E_1}j(w'\mathbf{a}') \cap N_3$.*

Proof. From definition, $N_{3,w\mathbf{a}}$ consists of n satisfying:

$$n^{-1}j(w'\mathbf{a}')^{-1}E_1 j(w'\mathbf{a}')n = j(w'\mathbf{a}')^{-1}E_1 j(w'\mathbf{a}')^{-1}.$$

It is clear that $j(w'\mathbf{a}')^{-1}N_{3,E_1}j(w'\mathbf{a}') \cap N_3 \subset N_{3,w\mathbf{a}}$. We show given $n \in N_{3,w\mathbf{a}}$, $j(w'\mathbf{a}')nj(w'\mathbf{a}')^{-1} \in N_{3,E_1}$. The element $j(w'\mathbf{a}')nj(w'\mathbf{a}')^{-1}$ fixes E_1 through conjugation, and it has the form:

$$\begin{pmatrix} n_1 & * & * \\ 0 & * & * \\ 0 & 0 & n_1^* \end{pmatrix},$$

where n_1 is in Z_n. Any element of the above form fixing E_1 through conjugation must lie in N_3, thus $j(w'\mathbf{a}')nj(w'\mathbf{a}')^{-1} \in N_{3,E_1}$. $\square$

4.4.2. *Integration over a normal subgroup.* Define the group

$$U^1_{w\mathbf{a}} = j(w'\mathbf{a}')^{-1} U^1_{E_1} j(w'\mathbf{a}'). \tag{4.11}$$

The group $U^1_{w\mathbf{a}}$ is a normal subgroup of $N_{3,w\mathbf{a}}$ as $U^1_{E_1}$ is a normal subgroup of N_{3,E_1}.

Lemma 4.7. *We have*

$$\int_{u\in U^1_{w\mathbf{a}}(k)\backslash U^1_{w\mathbf{a}}(\mathbf{A})} \theta_4(u)\Theta^{\Phi}_{\psi^{-1}}(u)\,du = \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0).$$

Proof. From the definitions, $\theta_4(u) = 1$, while using the Poisson summation formula:

$$\begin{aligned} \Theta^{\Phi}_{\psi^{-1}}(u) &= \sum_{X\in k^n} \omega_{\psi^{-1}}(u)\Phi(X) \\ &= \sum_{X\in k^n} \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\omega_{\psi^{-1}}(u)\Phi(X). \end{aligned}$$

Thus the integral in the Lemma equals

$$\int_{u\in U^1_{w\mathbf{a}}(k)\backslash U^1_{w\mathbf{a}}(\mathbf{A})} \sum_{X\in k^n} \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\omega_{\psi^{-1}}(u)\Phi(X)\,du.$$

Write $u = j(w'\mathbf{a}')^{-1}u(1_n, B, 0)j(w'\mathbf{a}')$; the above integral is

$$\begin{aligned} &\int_{\mathcal{S}_n(k)\backslash\mathcal{S}_n(\mathbf{A})} \sum_{X\in k^n} \omega_{\psi^{-1}}(u(1_n, B, 0))\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(X)\,dB \\ = &\int_{\mathcal{S}_n(k)\backslash\mathcal{S}_n(\mathbf{A})} \sum_{X\in k^n} \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(X)\psi^{-1}(2\langle B, X\rangle)\,dB \end{aligned}$$

where $\langle B, X\rangle$ denotes the inner product of the last row of B with X. The integral over B would be 0 unless $X = 0$, in which case the integration over B gives

$$\int_{\mathcal{S}_n(k)\backslash\mathcal{S}_n(\mathbf{A})} \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0)\,dB = \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0).$$

Thus the identity in the Lemma holds. □

4.4.3. *Description of* $U^1_{w\mathbf{a}}\backslash N_{3,w\mathbf{a}}$. Let $U^2_{E_1}$ be the group of elements $u(n,0,T)$. Note it is isomorphic to N_2 through the embedding $i : N_2 \mapsto H_3$:

$$i\begin{pmatrix} n & nT \\ & n^* \end{pmatrix} = u(n,0,T). \tag{4.12}$$

Let

$$U^2_{w\mathbf{a}} = j(w'\mathbf{a}')^{-1}U^2_{E_1}j(w'\mathbf{a}') \cap N_3. \tag{4.13}$$

Then from Lemma 4.6, $U^2_{w\mathbf{a}} \cong U^1_{w\mathbf{a}}\backslash N_{3,w\mathbf{a}}$.

Lemma 4.8. *Define for* $g \in \mathrm{Sp}_n$

$$N'_{2,g} = gN_2g^{-1} \cap N_2,$$

then $U^2_{w\mathbf{a}} = j(w'\mathbf{a}')^{-1}i(N'_{2,w'\mathbf{a}'})j(w'\mathbf{a}')$.

Proof. Since $U^2_{E_1} = i(N_2)$, the group $j(w'\mathbf{a}')U^2_{w\mathbf{a}}j(w'\mathbf{a}')^{-1}$ consists of all elements $i(n)$ with $n \in N_2$ such that $j(w'\mathbf{a}')^{-1}i(n_2)j(w'\mathbf{a}')$ is in N_3. A simple calculation shows the condition is equivalent to $(w'\mathbf{a}')^{-1}n_2(w'\mathbf{a}') \in N_2$, or $n \in N'_{2,w'\mathbf{a}'}$. □

4.4.4. *Proof of Proposition 4.5.* From Lemma 4.7, we get the integral of Proposition 4.5 is

$$\int_{n' \in U^2_{w\mathbf{a}}(k)\backslash U^2_{w\mathbf{a}}(\mathbf{A})} \theta_4(n')\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'}\cdot\widetilde{n'})\Phi(0)\,dn'. \tag{4.14}$$

From Lemma 4.8, it becomes:

$$\int_{n' \in N'_{2,w'\mathbf{a}'}(k)\backslash N'_{2,w'\mathbf{a}'}(\mathbf{A})} \theta_4(j(w'\mathbf{a}')^{-1}i(n')j(w'\mathbf{a}'))\omega_{\psi^{-1}}(i(n'))\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0)\,dn'. \tag{4.15}$$

Let $n' = \left(\begin{smallmatrix} n'' & n''T \\ & n''^* \end{smallmatrix}\right) \in N_2$, from the formulas (1.7), (1.8), (3.27) and (3.29):

$$\omega_{\psi^{-1}}(i(n'))\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0) = \omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0)\psi^{-1}(T_{n,1}).$$

Here $T_{n,1}$ is the lower left entry of T. From the definition

$$\theta_4(j(w'\mathbf{a}')^{-1}i(n')j(w'\mathbf{a}')) = \psi^{-1}(\sum_{i=1}^{n-1} n''_{i,i+1})\theta_2((w'\mathbf{a}')^{-1}n'(w'\mathbf{a}')).$$

Observe that $\theta_2^{-1}(n') = \psi^{-1}(\sum_{i=1}^{n-1} n''_{i,i+1} + T_{n,1})$. Thus (4.15) is:

$$\int_{n'\in N'_{2,w'\mathbf{a}'}(k)\backslash N'_{2,w'\mathbf{a}'}(\mathbf{A})} \theta_2((w'\mathbf{a}')^{-1}n'(w'\mathbf{a}'))\theta_2^{-1}(n')\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\Phi(0)\, dn'. \tag{4.16}$$

We will denote by $c(w'\mathbf{a}')$ the integral:

$$c(w'\mathbf{a}') = \int_{n'\in N'_{2,w'\mathbf{a}'}(k)\backslash N'_{2,w'\mathbf{a}'}(\mathbf{A})} \theta_2((w'\mathbf{a}')^{-1}n'(w'\mathbf{a}'))\theta_2^{-1}(n')\, dn'. \tag{4.17}$$

From the equation (4.16) we get equation (4.9) thus Proposition 4.5.

4.5. **The orbital integral decomposition of $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$.** From (4.5) and Proposition 4.5, we get

Proposition 4.9. *When $f = \otimes f_v$, and $\Phi = \otimes \Phi_v$, the distribution $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$ equals*

$$\sum_{w'\mathbf{a}'\in \mathrm{Sp}_n(k)} c(w'\mathbf{a}') \prod_v I_{w'\mathbf{a}'}(F_v, \Phi_v)$$

with $w' \in W(\mathrm{Sp}_n)$, $\mathbf{a}'$ a diagonal matrix in Sp_n, $F = \otimes F_v$ defined by (4.2) and $I_{w'\mathbf{a}'}(F_v, \Phi_v)$ equals

$$\int_{n\in N_{3,\mathrm{P}(w'\mathbf{a}')}(k_v)\backslash N_3(k_v)} F_v(n^{-1}\,\mathrm{P}(w'\mathbf{a}')n)\theta_4(n)\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'}\cdot n)\Phi_v(0)\, dn. \tag{4.18}$$

4.6. **The orbital integral decomposition of $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$.** From the definition of $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$ and the Bruhat decomposition of Sp_n, we get the distribution equals:

$$\begin{aligned} &\int_{N_2(k)\backslash N_2(\mathbf{A})}\int_{N_2(k)\backslash N_2(\mathbf{A})} \sum_{\gamma\in N_2(k)\backslash \mathrm{Sp}_n(k)/N_2(k)} \tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{\gamma}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n_2)\, dn_1\, dn_2 \\ &= \sum_{w'\mathbf{a}'\in\mathrm{Sp}_n(k)} \int_{n_2\in N_{2,w'\mathbf{a}'}(k)\backslash N_2(\mathbf{A})}\int_{n_1\in N_2(\mathbf{A})} \tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n_2)\, dn_1\, dn_2, \end{aligned} \tag{4.19}$$

with

$$N_{2,w'\mathbf{a}'} = (w'\mathbf{a}')^{-1}N_2(w'\mathbf{a}') \cap N_2.$$

The integral in (4.19) can be written as:

$$\int_{n_2\in N_{2,w'\mathbf{a}'}(\mathbf{A})\backslash N_2(\mathbf{A})}\int_{n'\in N_{2,w\mathbf{a}}(k)\backslash N_{2,w\mathbf{a}}(\mathbf{A})}\int_{n_1\in N_2(\mathbf{A})} \tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n'}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n'n_2)dn_1dn'dn_2.$$

As for $n'\in N_{2,w'\mathbf{a}'}$, $(w'\mathbf{a}')n'(w'\mathbf{a}')^{-1}\in N_2$, we can make a change of variable $n_1\mapsto (w'\mathbf{a}')n'(w'\mathbf{a}')^{-1}n_1$. Using the fact that when $n\in N_2$ we have $\widetilde{n}\cdot\widetilde{g}=\widetilde{ng}$ and $\widetilde{g}\cdot\widetilde{n}=\widetilde{gn}$, we get:

$$\begin{aligned}\left((w'\mathbf{a}')\widetilde{n'(w'\mathbf{a}')^{-1}n_1}\right)^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n'} &= \widetilde{n_1}^{-1}\cdot(w'\mathbf{a}')\widetilde{(n')^{-1}(w'\mathbf{a}')^{-1}w'\mathbf{a}'}\cdot\widetilde{n'}\\ &= \widetilde{n_1}^{-1}\cdot\widetilde{(w'\mathbf{a}')}\cdot\widetilde{(n')^{-1}}\cdot\widetilde{n'}=\widetilde{n_1}^{-1}\cdot\widetilde{(w'\mathbf{a}')}.\end{aligned}$$

The above integral becomes:

$$\int_{n_2\in N_{2,w'\mathbf{a}'}(\mathbf{A})\backslash N_2(\mathbf{A})}\int_{n'\in N_{2,w\mathbf{a}}(k)\backslash N_{2,w\mathbf{a}}(\mathbf{A})}\int_{n_1\in N_2(\mathbf{A})} \tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n'n_2)\theta_2^{-1}((w'\mathbf{a}')n'(w'\mathbf{a}')^{-1})\,dn_1\,dn'\,dn_2.$$

Clearly $n'\in N_{2,w'\mathbf{a}'}$ iff $(w'\mathbf{a}')n'(w'\mathbf{a}')^{-1}\in N'_{2,w'\mathbf{a}'}$; from the definition of $c(w'\mathbf{a}')$ in (4.17), we get the above integral equals:

$$c(w'\mathbf{a}')\int_{n_2\in N_{2,w'\mathbf{a}'}(\mathbf{A})\backslash N_2(\mathbf{A})}\int_{n_1\in N_2(\mathbf{A})}\tilde{f}(\widetilde{n_1}^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n_2)\,dn_1\,dn_2. \tag{4.20}$$

From (4.19) and (4.20), we get

Proposition 4.10. *When $\tilde{f}=\otimes\tilde{f}_v$, the distribution $I_{G_2}(\tilde{f}:N_2,\theta_2^{-1};N_2,\theta_2)$ equals:*

$$\sum_{w'\mathbf{a}'\in\mathrm{Sp}_n(k)}c(w'\mathbf{a}')\prod_v J_{w'\mathbf{a}'}(\tilde{f}_v)$$

with $w'\in W(\mathrm{Sp}_n)$, $\mathbf{a}'$ a diagonal matrix in Sp_n, and

$$J_{w'\mathbf{a}'}(\tilde{f}_v)=\int_{n_2\in N_{2,w'\mathbf{a}'}(k_v)\backslash N_2(k_v)}\int_{n_1\in N_2(k_v)}\tilde{f}_v(\widetilde{n_1}^{-1}\cdot\widetilde{w'\mathbf{a}'}\cdot\widetilde{n_2})\theta_2(n_1^{-1}n_2)\,dn_1\,dn_2. \tag{4.21}$$

5. Comparison of orbital integrals

5.1. **Definition of $\Psi_{F,\Phi}$.** From the Propositions 4.10 and 4.9, to compare the distributions $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$ and $I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}})$, we only need to compare the local orbital integrals $I_{w'\mathbf{a}'}(F_v, \Phi_v)$ and $J_{w'\mathbf{a}'}(\tilde{f}_v)$. In the rest of this section, we will fix a place v and omit v in the notation..

Recall we defined the groups U^n and $\hat{N}^n$ in section 3.2. $\hat{N}^{n+1} = U^n\hat{N}^n$ is a normal subgroup of N_3 with $N_3/\hat{N}^{n+1} \cong N_2$. Note also $N_{3,E_1} \cap \hat{N}^{n+1}$ is just the group $U^1_{E_1}$. Given $F \in \mathcal{S}(Y(k))$, and $\Phi \in \mathcal{S}(k^n)$, we define a genuine function on $\widetilde{\mathrm{Sp}}_n(k)$:

$$\Psi_{F,\Phi}(\widetilde{g}) = \int_{u\in U^1_{E_1}\backslash \hat{N}^{n+1}} F(j(g)^{-1}u^{-1}E_1uj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\, du. \tag{5.1}$$

5.2. **Comparison of $I_{w'\mathbf{a}'}(F,\Phi)$ and $J_{w'\mathbf{a}'}(\tilde{f})$.** For a compatible choice of measures, we have:

Lemma 5.1.

$$I_{w'\mathbf{a}'}(F,\Phi) = \int_{n\in N_{2,w'\mathbf{a}'}\backslash N_2} \Psi_{F,\Phi}(\widetilde{w'\mathbf{a}'}\cdot \widetilde{n})\theta_2(n)\, dn. \tag{5.2}$$

Proof. As in §4, we let $U^1_{\mathrm{P}(g)} = j(g)^{-1}U^1_{E_1}j(g) \subset N_3$. Then $N_{3,\mathrm{P}(w'\mathbf{a}')} \cap \hat{N}^{n+1} = U^1_{\mathrm{P}(w'\mathbf{a}')}$. From (4.18), we get $I_{w'\mathbf{a}'}(F,\Phi)$ equals

$$\int_{n\in N_{3,\mathrm{P}(w'\mathbf{a}')}\hat{N}^{n+1}\backslash N_3} \int_{u\in U^1_{\mathrm{P}(w'\mathbf{a}')}\backslash \hat{N}^{n+1}} F(n^{-1}u^{-1}\,\mathrm{P}(w'\mathbf{a}')un)$$
$$\theta_4(un)\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\omega_{\psi^{-1}}(un)\Phi(0)\, du\, dn.$$

As $N_3 = j(N_2)\hat{N}^{n+1}$ and $j(N_2) \cap \hat{N}^{n+1}$ consists just of identity element, we can write an element in N_3 in a unique way as $j(n)u$ such that $n \in N_2$ and $u \in \hat{N}^{n+1}$.

Since $N_{3,\mathrm{P}(w'\mathbf{a}')} = U^1_{\mathrm{P}(w'\mathbf{a}')}U^2_{\mathrm{P}(w'\mathbf{a}')}$, with $U^1_{\mathrm{P}(w'\mathbf{a}')} \subset \hat{N}^{n+1}$, we have

$$N_{3,\mathrm{P}(w'\mathbf{a}')}\hat{N}^{n+1} = U^2_{\mathrm{P}(w'\mathbf{a}')}\hat{N}^{n+1}.$$

As $U^2_{\mathrm{P}(w'\mathbf{a}')} = j(w'\mathbf{a}')^{-1}i(N'_{2,w'\mathbf{a}'})j(w'\mathbf{a}')$ from Lemma 4.8,

$$U^2_{\mathrm{P}(w'\mathbf{a}')}\hat{N}^{n+1} = j((w'\mathbf{a}')^{-1}N'_{2,w'\mathbf{a}'}(w'\mathbf{a}'))\hat{N}^{n+1} = j(N_{2,w'\mathbf{a}'})\hat{N}^{n+1}.$$

Thus we can choose the representatives in $N_{3,\mathrm{P}(w'\mathbf{a}')}\hat{N}^{n+1}\backslash N_3$ as $j(n)$ with $n \in N_{2,w'\mathbf{a}'}\backslash N_2$. The above integral equals:

$$\int_{n\in N_{2,w'\mathbf{a}'}\backslash N_2}\int_{u\in U^1_{\mathrm{P}(w'\mathbf{a}')}\backslash \hat{N}^{n+1}} F(j(n)^{-1}u^{-1}\,\mathrm{P}(w'\mathbf{a}')uj(n))$$
$$\theta_4(uj(n))\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'})\omega_{\psi^{-1}}(uj(n))\Phi(0)\,du\,dn.$$

As $j(w\mathbf{a}')$ stabilizes $\hat{N}^{n+1}$ through conjugation, we can make a change of variable $u \mapsto j(w\mathbf{a}')^{-1}uj(w\mathbf{a}')$. Notice that $\theta_4(j(w\mathbf{a}')^{-1}uj(w\mathbf{a}')) = \theta_4(u)$, and from (3.28), (3.29) and (4.11), the above integral is the same as:

$$\int_{n\in N_{2,w'\mathbf{a}'}\backslash N_2}\int_{u\in U^1_{E_1}\backslash \hat{N}^{n+1}} F(j(w'\mathbf{a}'n)^{-1}u^{-1}E_1uj(w'\mathbf{a}'n))$$
$$\theta_4(j(n)u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{w'\mathbf{a}'n})\Phi(0)\,du\,dn.$$

From the definition of $\Psi_{F,\Phi}$ in (5.1), the above integral equals:

$$\int_{n\in N_{2,w'\mathbf{a}'}\backslash N_2}\Psi_{F,\Phi}(\widetilde{w'\mathbf{a}'n})\theta_4(j(n))dn.$$

Since $\theta_4(j(n)) = \theta_2(n)$ for $n \in N_2$, we get the Lemma. □

It follows immediately from the Lemma and equation (4.21) that

Corollary 5.2. *If $\tilde{f} \in \mathcal{S}(\widetilde{\mathrm{Sp}}_n(k))$, $f \in \mathcal{S}(\mathrm{Sp}_{2n}(k))$ and $\Phi \in \mathcal{S}(k^n)$ satisfying (for F defined by (4.2))*

$$\Psi_{F,\Phi}(\widetilde{g}) = \int_{N_2}\tilde{f}(\widetilde{n}^{-1}\cdot\widetilde{g})\theta_2^{-1}(n)dn, \tag{5.3}$$

then $I_{w'\mathbf{a}'}(F,\Phi) = J_{w'\mathbf{a}'}(\tilde{f})$.

5.3. **Properties of $\Psi_{F,\Phi}$ and matching.** The function $\Psi_{F,\Phi}(\widetilde{g})$ has the following equivariance property:

Lemma 5.3. *The function $\Psi_{F,\Phi}(\widetilde{g})$ satisfies for all $n \in N_2$:*

$$\Psi_{F,\Phi}(\widetilde{n}\cdot\widetilde{g}) = \theta_2(n)^{-1}\Psi_{F,\Phi}(\widetilde{g}). \tag{5.4}$$

We delay the proof to a later subsection. We also need to consider the behavior of the function $\Psi_{F,\Phi}(\widetilde{g})$ when $g = \mathrm{diag}[\mathbf{a}, \mathbf{a}^*]$ where $\mathbf{a} = \mathrm{diag}[a_1, \ldots, a_n]$ is a diagonal matrix.

Lemma 5.4. *When $F \in \mathcal{S}(Y)$ and $\Phi \in \mathcal{S}(k^n)$, as function of $\mathbf{a}$, $\Psi_{F,\Phi}(\widetilde{\mathrm{diag}[\mathbf{a}, \mathbf{a}^*]})$ is a Schwartz function on $(k^*)^n$.*

From Lemmas 5.3, 5.4 and [MR2, Lemma 5.6], we get

Corollary 5.5. *Given any $f \in \mathcal{S}(\mathrm{Sp}_{2n})$ and $\Phi \in \mathcal{S}(k^n)$, there is $\tilde{f} \in \mathcal{S}(\widetilde{\mathrm{Sp}}_n)$ such that equation (5.3) holds.*

From Corollary 5.2, we get:

Corollary 5.6. *There is a map ϵ_5 from $\mathcal{S}(\mathrm{Sp}_{2n}) \otimes \mathcal{S}(k^n)$ to $\mathcal{S}(\widetilde{\mathrm{Sp}}_n)$ such that when $\tilde{f} = \epsilon_5(F \otimes \Phi)$, $I_{w'\mathbf{a}'}(F, \Phi) = J_{w'\mathbf{a}'}(\tilde{f})$.*

5.4. **Proof of Lemma 5.3.** We only need to establish the identity in the case g is identity, which we now assume. Recall the definition of $u(n, B, T) \in N_{3,E_1}$ in (4.10).

From (5.1) $\Psi_{F,\Phi}(\widetilde{n})$ equals:

$$\int_{u \in U^1_{E_1} \backslash \hat{N}^{n+1}} F(j(n)^{-1} u^{-1} E_1 u j(n)) \theta_4(u) \omega_{\psi^{-1}}(u j(n)) \Phi(0)\, du. \tag{5.5}$$

Write n as $\left(\begin{smallmatrix} n' & n'T \\ & (n')^* \end{smallmatrix}\right)$.

Case (1): when n' is identity. We observe $u(1_n, 0, T) \in N_{3,E_1}$ and $j(n)u(1_n, 0, T)^{-1} \in \hat{N}^{n+1}$. Since $j(N_2)$ and $u(1_n, 0, T)$ fix the group $U^1_{E_1}$ by conjugation, we can make a change of variable $u \mapsto u(1_n, 0, T) u j(n)^{-1}$ in $\hat{N}^{n+1}$. Notice $u(1_n, 0, T)$ fixes E_1; the above integral becomes:

$$\int_{u \in U^1_{E_1} \backslash \hat{N}^{n+1}} F(u^{-1} E_1 u) \theta_4(u(1_n, 0, T) u j(n)^{-1}) \omega_{\psi^{-1}}(u(1_n, 0, T) u) \Phi(0)\, du.$$

Clearly $\theta_4(u(1_n, 0, T) u j(n)^{-1}) = \theta_4(u)$. From (3.27) and (3.29)

$$\omega_{\psi^{-1}}(u(1_n, 0, T) u) \Phi(0) = \psi^{-1}(T_{n,1}) \omega_{\psi^{-1}}(u) \Phi(0)$$

where $T_{n,1}$ is the lower left entry of T. The above integral becomes:

$$\psi^{-1}(T_{n,1})\int_{u\in U^1_{E_1}\backslash\hat{N}^{n+1}} F(u^{-1}E_1u)\theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\,du$$

which is $\theta_2(n)^{-1}\Psi_{F,\Phi}(\widetilde{1_{2n}})$ in the case n' is identity.

Case (2): when $T'=0$. Now $u(n',0,0)\in N_{3,E_1}$ and $u(n',0,0)j(n)^{-1}\in\hat{N}^{n+1}$. Since $u(n',0,0)$ and $j(n)$ fixes the group $U^1_{E_1}$ by conjugation, we can change u to $u(n',0,0)uj(n)^{-1}$; the integration (5.5) becomes:

$$\int_{u\in U^1_{E_1}\backslash\hat{N}^{n+1}} F(u^{-1}E_1u)\theta_4(u(n',0,0)uj(n)^{-1})\omega_{\psi^{-1}}(u(n',0,0)u)\Phi(0)\,du.$$

Clearly $\theta_4(u(n',0,0)uj(n)^{-1})=\theta_4(u)\theta_2^{-1}(n)$ (in our case $T=0$). From (1.7) and (3.29) we get

$$\omega_{\psi^{-1}}(u(n',0,0)u)\Phi(0)=\omega_{\psi^{-1}}(u)\Phi(0).$$

Thus the above integral is just

$$\theta_2^{-1}(n)\int_{u\in U^1_{E_1}\backslash\hat{N}^{n+1}} F(u^{-1}E_1u)\theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\,du$$

which is $\theta_2(n)^{-1}\Psi_{F,\Phi}(\widetilde{1_{2n}})$.

From the above two cases, identity (5.4) holds for any $n\in N_2$ when g is identity, thus holds in general.

5.5. **Proof of Lemma 5.4.**

5.5.1. *Reduction to a problem on* GL_{2n}. There is an element $\epsilon\in\mathrm{GL}_{2n}$ such that $m(\epsilon)^{-1}Em(\epsilon)=E_1$. Recall F is defined through a function $f\in\mathcal{S}(\mathrm{Sp}_{2n})$ by (4.2). We get

$$F(g^{-1}E_1g)=\int_{l\in H_3} f(l^{-1}m(\epsilon)g)\,dl. \tag{5.6}$$

Then $\Psi_{F,\Phi}(\widetilde{g})$ becomes:

$$\int_{u\in U^1_{E_1}\backslash\hat{N}^{n+1}}\int_{l\in H_3(k)} f(l^{-1}m(\epsilon)uj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\,dl\,du.$$

Recall V is the unipotent radical of Siegel parabolic P_3 in $G_3 = \mathrm{Sp}_{2n}$. Use Iwasawa decomposition in $H_3 = V'm(H_1)K_3'$ where $V' = V \cap H_3$ and $K_3' = K_3 \cap H_3$, and integrate over K_3', we get the above is (for some $f' \in \mathcal{S}(P_3)$):

$$\int_{u\in U_{E_1}^1\backslash \hat{N}^{n+1}} \int_{v\in V'} \int_{h\in H_1} f'(m(h)^{-1}m(v)^{-1}m(\epsilon)uj(g)) \\ \theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\,dh\,dv\,du.$$

Change $v \mapsto m(\epsilon)v^{-1}m(\epsilon)^{-1}$; observe that $m(\epsilon)V'm(\epsilon)^{-1} = V \cap N_{3,E_1}$ contains $U_{E_1}^1$, thus combine the integral of v and u we get the above is:

$$\int_{u\in V\hat{N}^{n+1}} \int_{h\in H_1} f'(m(h)^{-1}m(\epsilon)uj(g))\theta_4(u)\omega_{\psi^{-1}}(u)\omega_{\psi^{-1}}(\widetilde{g})\Phi(0)\,dh\,du.$$

Now consider the case $g = \mathrm{diag}[\mathbf{a}, \mathbf{a}^*]$ where $\mathbf{a} = \mathrm{diag}[a_1, \ldots, a_n]$; then $j(g) = m(\mathbf{b})$ where $\mathbf{b} = \mathrm{diag}[1, \mathbf{a}]$. Change $u \mapsto j(g)uj(g)^{-1}$; the above becomes:

$$\int_{u\in V\hat{N}^{n+1}} \int_{h\in H_1} f'(m(h)^{-1}m(\epsilon)j(g)u)\theta_4(u_g)\omega_{\psi^{-1}}(\widetilde{g})\omega_{\psi^{-1}}(u)\Phi(0)\,dh\,du.$$

Here $u_g = j(g)uj(g)^{-1}$.

Write $u \in V\hat{N}^{n+1}$ as u_1u_2 where $u_2 \in V$ and $u_1 = m(u) \in \hat{N}^{n+1}$ for some $u \in N_1 \subset \mathrm{GL}_{2n}$. The element u has the form $\left(\begin{smallmatrix} * & * \\ & 1_n \end{smallmatrix}\right)$; we denote the subgroup of such u in N_1 by N_1'. Then $\theta_4(u_g) = \theta_4(u_1)\psi(a_n^2\check{u}_2)$ where $\check{u}_2$ is the $(2n, 2n+1)-$th entry in u_2.

From (1.7),

$$\omega_{\psi^{-1}}(\widetilde{g})\omega_{\psi^{-1}}(u)\Phi(0) = \det(\mathbf{a})^{\frac{1}{2}}\frac{\gamma(1,\psi^{-1})}{\gamma(\det\mathbf{a},\psi^{-1})}\omega_{\psi^{-1}}(u)\Phi(0).$$

Clearly $\omega_{\psi^{-1}}(u)\Phi(0)$ is a smooth and bounded function on $V\hat{N}^{n+1}$. Write $f'(m(h)v) = f_1'(h)f_2'(v)$ with $v \in V$ and $h \in \mathrm{GL}_{2n}$, then the above integral has the form:

$$\det(\mathbf{a})^{\frac{1}{2}}\frac{\gamma(1,\psi^{-1})}{\gamma(\det\mathbf{a},\psi^{-1})}\int_{u_2\in V} f_2''(u_2)\psi(a_n^2\check{u}_2)\,du_2 \int_{u\in N_1'}\int_{h\in H_1} f_1''(h^{-1}\epsilon\mathbf{b}u)\theta_4(m(u))\,dh\,du.$$

Here f_1'' is in $\mathcal{S}(\mathrm{GL}_{2n})$ and $f_2'' \in \mathcal{S}(V)$. The integration over V gives a function $\phi(a_n) \in \mathcal{S}(k)$. Since $\gamma(\det\mathbf{a},\psi^{-1})$ is a smooth function of $\mathbf{a}$ taking finitely many nonzero values, to prove Lemma 5.4, we are reduced to show:

Lemma 5.7. *For $f \in \mathcal{S}(\mathrm{GL}_{2n})$, $\mathbf{b} = \mathrm{diag}[1, \mathbf{a}]$, let*

$$\varphi_f(\mathbf{a}) = \int_{u \in N_1'} \int_{h \in H_1} f(h^{-1}\epsilon \mathbf{b} u)\theta_4(m(u))\, dh\, du.$$

Then for any Schwartz function $\phi(a_n)$, $\phi(a_n)\varphi_f(\mathbf{a})$ is a Schwartz function on $(k^*)^n$.

5.5.2. *Proof of Lemma 5.7.* We prove the Lemma in the case when v is an archimedean place. The case v is non-archimedean can be proved in the same manner, and the argument is simpler.

First observe $\mathbf{b}u = u'\mathbf{b}$ with $u' \in N_1'$ and $\theta_4(m(u')) = \theta_4(m(u))$. Thus a change of variable gives $\varphi_f(\mathbf{a}) = |\det \mathbf{a}|^n \varphi_f'(\mathbf{a})$ where

$$\varphi_f'(\mathbf{a}) = \int_{u \in N_1'} \int_{h \in H_1} f(h^{-1}\epsilon u \mathbf{b})\theta_4(m(u))\, dh\, du.$$

Thus any derivative of φ_f is a combination of polynomial functions of $|a_i|$ times φ_{Xf}' where Xf is a suitable derivative of f. To prove the Lemma, we only need to prove $\phi(a_n)\varphi_f'(\mathbf{a})$ is bounded by $||\mathbf{a}||^{-r}$ for any real number r as $||\mathbf{a}|| \mapsto \infty$.

Let $Y' = \{g^{-1}E'g \,|\, g \in \mathrm{GL}_{2n}\}$ where $E' = \mathrm{diag}[1_n, -1_n]$ is defined in (4.1). Then $Y' \cong H_1 \backslash \mathrm{GL}_{2n}$. For $f \in \mathcal{S}(\mathrm{GL}_{2n})$, define $F_f'(y)$ on Y' by

$$F_f'(g^{-1}E'g) = \int_{l \in H_1} f(l^{-1}g)\, dl. \tag{5.7}$$

Then (recall the definition of E_1' from (4.1))

$$\varphi_f'(\mathbf{a}) = \int_{u \in N_1'} F_f'((u\mathbf{b})^{-1}E_1' u\mathbf{b})\theta_4(m(u))\, du. \tag{5.8}$$

From result of Dixmier-Malliavin [DMa], we can write $f(g)$ as a finite sum of convolutions of $f_\alpha'(g) \in \mathcal{S}(\mathrm{GL}_{2n})$ and $\phi_\alpha \in \mathcal{S}(k^{n-1})$:

$$f(g) = \sum_\alpha \int f_\alpha'(gu(\mathbf{x}))\phi_\alpha(\mathbf{x})\, d\mathbf{x},$$

where $\mathbf{x} \in k^{n-1}$ and $u(\mathbf{x})$ is the image of the exponential map (from $\mathrm{M}_{2n,2n}$ to GL_{2n}) of $\sum_{i=1}^{n-1} \mathbf{x}_i e_{n+i,n+i+1}$. Then

$$\varphi_f'(\mathbf{a}) = \sum_\alpha \int_{u \in N_1'} \int F_{f_\alpha'}'((u\mathbf{b}u(\mathbf{x}))^{-1}E_1' u\mathbf{b}u(\mathbf{x}))\theta_4(m(u))\phi_\alpha(\mathbf{x})\, d\mathbf{x}\, du.$$

Note

$$(u\mathbf{b}u(\mathbf{x}))^{-1}E_1'u\mathbf{b}u(\mathbf{x}) = (u'\mathbf{b})^{-1}E_1'u'\mathbf{b}$$

with $u' \in N_1$ and $\theta_4(m(u')) = \theta_4(m(u))\psi(\sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}}\mathbf{x}_i)$. Thus a change of variable gives:

$$\varphi_f'(\mathbf{a}) = \sum_\alpha \varphi_{f_\alpha'}'(\mathbf{a}) \int \psi(-\sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}}\mathbf{x}_i)\phi_\alpha(\mathbf{x})\, d\mathbf{x}.$$

We get

$$\phi(a_n)\varphi_f'(\mathbf{a}) = \phi(a_n)\sum_\alpha \widehat{\phi_\alpha}(-\frac{a_i}{a_{i+1}})\varphi_{f_\alpha'}'(\mathbf{a})$$

where $\widehat{\phi_\alpha}$ is the Fourier transform of ϕ_α and is a Schwartz function of k^{n-1}. It is now clear that for $1 \leq i \leq n$, $\phi(a_n)\varphi_f'(\mathbf{a})$ is bounded by $|a_i|^{-r}$ for any real number r as $|a_i| \mapsto \infty$.

An element u in N_1' can be written in the form $\left(\begin{smallmatrix} n & v \\ & 1_n \end{smallmatrix}\right)$ where n is in Z_n the maximal unipotent subgroup of GL_n. Observe

$$||(u\mathbf{b})^{-1}E_1'u\mathbf{b}|| = ||\begin{pmatrix} * & * \\ \mathbf{a}^{-1}n & \mathbf{a}^{-1}v \end{pmatrix}|| \geq \max\{|a_i^{-1}|, |a_i^{-1}n_{i,j}|, |a_i^{-1}v_{i,j}|\}. \tag{5.9}$$

Lemma 5.8. *When $f \in \mathcal{S}(\mathrm{GL}_{2n})$, $|F_f'(y)|$ is bounded by $||y||^{-r}$ for any r as $||y|| \mapsto \infty$.*

Proof. Use the Iwasawa decomposition of GL_{2n}, any element $g \in \mathrm{GL}_{2n}$ can be written as hvk where $h \in H_1$, $v = \left(\begin{smallmatrix} 1_n & \mathbf{v} \\ & 1_n \end{smallmatrix}\right)$ and k is in the maximal compact subgroup K_1 of GL_{2n}. For such a g, the corresponding element y in Y' has the form:

$$y = g^{-1}E'g = k^{-1}\begin{pmatrix} 1_n & 2\mathbf{v} \\ & -1_n \end{pmatrix} k.$$

In particular $||y|| \leq 2||v||$. Thus as $||y|| \mapsto \infty$ so does $||v||$, the integral in (5.7) with $g = hvk$ is bounded by $||v||^{-r}$ and thus by $||y||^{-r}$. □

From the Lemma, (5.8) and (5.9), we see for any $r > 0$ (for some constant c)

$$\varphi_f'(\mathbf{a}) \leq \int\int (\max\{c, |a_i^{-1}|, |a_i^{-1}n_{i,j}|, |a_i^{-1}v_{i,j}|\})^{-r}\, dn\, dv.$$

Clearly as $|a_i^{-1}| \mapsto \infty$, $\varphi_f'(\mathbf{a})$ is bounded by $|a_i^{-r}|$ for any r.

We get the conclusion in Lemma 5.7.

6. Fundamental Lemma

6.1. **Statement of the result.** Let v be a nonarchimedean place with odd residue characteristic, and where ψ is unramified. We will omit v in the notations.

Let $\mathcal{O}$ be the ring of integers in k. Let $K_3 = G_3(\mathcal{O})$ and K_2 be image of an embedding of $\mathrm{Sp}_n(\mathcal{O})$ in G_2, (the covering splits over $\mathrm{Sp}_n(\mathcal{O})$). Let $\mathcal{H}(G_3, K_3)$ (and $\mathcal{H}(G_2, K_2)$) be the algebra of Hecke functions on G_3 (and G_2 respectively). For $\tilde{f} \in \mathcal{H}(G_3, K_3)$, define

$$\widehat{\tilde{f}}(z) = \int_{\mathbf{a}\in T_{2n}} \int_{n\in N_3} \tilde{f}(\widetilde{m(\mathbf{a})} \cdot \widetilde{n})\gamma(\det \mathbf{a}, \psi^{-1})^{-1}\chi_z(\mathbf{a})\delta_3^{\frac{1}{2}}(m(\mathbf{a}))\, dn\, d\mathbf{a}$$

where δ_3 is the modulus functions of the Borel subgroup of Sp_{2n} and χ_z is the character defined by (2.16). We can define a homomorphism $f \mapsto \tilde{f}$ between $\mathcal{H}(G_3, K_3)$ and $\mathcal{H}(G_2, K_2)$ so that:

$$\widehat{f}(z_1 - \frac{1}{2}, z_1 + \frac{1}{2}, \ldots, z_n - \frac{1}{2}, z_n + \frac{1}{2}) = \widehat{\tilde{f}}(z_1, \ldots, z_n). \tag{6.1}$$

We prove

Proposition 6.1. *If $f \in \mathcal{H}(G_3, K_3)$ and $\tilde{f} \in \mathcal{H}(G_2, K_2)$ are related by (6.1), then when Φ_0 is the characteristic function of $\mathcal{O}^n$, we have*

$$\Psi_{F,\Phi}(\widetilde{g}) = \int_{N_2} \tilde{f}(n^{-1} \cdot \widetilde{g})\theta_2^{-1}(n)\, dn, \tag{6.2}$$

where F is defined by (4.2).

From Corollary 5.2, we get:

Corollary 6.2. *When $f \in \mathcal{H}(G_3, K_3)$ and $\tilde{f} \in \mathcal{H}(G_2, K_2)$ are such that (6.1) holds, then when Φ_0 is the characteristic function of $\mathcal{O}^n$, we have $I_{w'a'}(F, \Phi_0) = J_{w'a'}(\tilde{f})$ where F is defined by (4.2).*

This identity of orbital integrals is the fundamental lemma for the case at hand. The rest of the section gives the proof of Proposition 6.1.

6.2. **Unit element case.** We prove the Proposition first in the case when both f and $\tilde{f}$ are unit elements. In this case we denote the functions by f_0 and $\tilde{f}_0$ respectively. Then f_0 is the characteristic functions of K_3, while $\tilde{f}_0$ takes value 1 over K_2, and vanishes outside the inverse image of $\mathrm{Sp}_n(\mathcal{O})$ in G_2. Let F_0 be the function associated to f_0 by (4.2).

Lemma 6.3. *The function F_0 is the characteristic function of $Y \cap K_3$.*

Proof. Clearly F_0 is a K_3-invariant function on Y. As $lg \in K_3$ for $l \in H_3$ implies $g^{-1}Eg \in K_3$, we get F_0 vanishes outside $Y \cap K_3$. As $Y \cap K_3$ is a single K_3-orbit of E (see [MR1, Lemma 3.1]), we get F_0 is constant on $Y \cap K_3$. Plug in $g = 1_{4n}$ in (4.2) shows that F_0 is the characteristic function of $Y \cap K_3$. □

Denote the right hand side of (6.2) by $\Psi_{\tilde{f}}(\widetilde{g})$. Then

$$\Psi_{\tilde{f}_0}(\widetilde{n} \cdot \widetilde{g}) = \theta_2^{-1}(n)\Psi_{\tilde{f}_0}(\widetilde{g}),\ n \in N_2.$$

Lemma 5.3 shows that Ψ_{F_0,Φ_0} satisfies the same left N_2-equivariance condition. Both functions $\Psi_{\tilde{f}_0}$ and Ψ_{F_0,Φ_0} are clearly right K_2-invariant. Thus to show the identity (6.2), we only need to show it holds when $\widetilde{g} = \widetilde{\mathbf{a}}$ where

$$\mathbf{a} = \mathrm{diag}[a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}]$$

is a diagonal matrix.

It is easy to see that $\Psi_{\tilde{f}_0}(\widetilde{\mathbf{a}}) = 1$ when $|a_i| = 1$ for $i = 1, \ldots, n$, and $\Psi_{\tilde{f}_0}(\widetilde{\mathbf{a}}) = 0$ otherwise. Thus the Proposition in this case follows from

Lemma 6.4. *When $|a_i| = 1$ for $i = 1, \ldots, n$, $\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}}) = 1$. Otherwise $\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}}) = 0$.*

Proof. For $u \in \hat{N}^{n+1}$, it has the form $\left(\begin{smallmatrix} n_1 & * \\ & n_1^* \end{smallmatrix}\right)$ with n_1 having the form $\left(\begin{smallmatrix} n_2 & v \\ & 1_n \end{smallmatrix}\right)$, where $n_2 \in Z_n$ the maximal unipotent subgroup of GL_n. The matrix $j(\mathbf{a})^{-1}u^{-1}E_1uj(\mathbf{a})$ lies in the Siegel parabolic subgroup; it has the form $\left(\begin{smallmatrix} A & * \\ & A^* \end{smallmatrix}\right)$ where

$$A = \begin{pmatrix} 1_n & \\ & \mathbf{b}^{-1} \end{pmatrix} \begin{pmatrix} n_2^{-1} & v' \\ & 1_n \end{pmatrix} \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \begin{pmatrix} n_2 & v \\ & 1_n \end{pmatrix} \begin{pmatrix} 1_n & \\ & \mathbf{b} \end{pmatrix}$$

with $\mathbf{b} = \mathrm{diag}[a_1, \dots, a_n]$, and $v' = -n_2^{-1}v$. A computation shows:

$$A = \begin{pmatrix} v'n_2 & n_2^{-1}\mathbf{b} + v'v\mathbf{b} \\ \mathbf{b}^{-1}n_2 & \mathbf{b}^{-1}v\mathbf{b} \end{pmatrix}.$$

If $|a_i| < 1$ for some i, we get from looking at the lower left block of A that

$$F_0(j(\mathbf{a})^{-1}u^{-1}E_1uj(\mathbf{a})) = 0.$$

Thus in this case $\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}}) = 0$.

On the other hand, since

$$\Psi_{F_0,\Phi_0}(\widetilde{n} \cdot \widetilde{g} \cdot \bar{k}) = \theta_2^{-1}(n)\Psi_{F_0,\Phi_0}(\widetilde{g})$$

when $n \in N_2$ and $\bar{k} \in K_2$, we get $\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}}) = 0$ whenever $|a_i| > |a_{i+1}|$ for some i or $|a_n| > |a_n|^{-1}$.

Therefore, $\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}})$ is nonzero only when $|a_n|^{-1} \geq |a_n| \geq \dots \geq |a_1|$ and $|a_1| \geq 1$. This condition is only satisfied when $|a_i| = 1$ for all i, in which case using the K_3-invariance of F_0 and K_2-invariance of Φ_0, we get

$$\Psi_{F_0,\Phi_0}(\widetilde{\mathbf{a}}) = \Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}).$$

Consider now the case $\mathbf{a} = 1_{2n}$. Use the computation of the matrix A again. From the lower left block and lower right block of A, we see in this case if $u^{-1}E_1u \in K_3 \cap Y$, n_2 is in $\mathrm{GL}_n(\mathcal{O})$ and v has integral entries. Thus $u = \begin{pmatrix} n_1 & v_1 \\ & n_1^* \end{pmatrix}$ with $n_1 \in \mathrm{GL}_{2n}(\mathcal{O})$.

Write u as $m(n_1)v$ with v in the Siegel unipotent subgroup of Sp_{2n}. Since $n_1 \in \mathrm{GL}_{2n}(\mathcal{O})$, we get $v^{-1}E_1v \in K_3 \cap Y$. Write $v = \begin{pmatrix} 1_{2n} & \mathbf{v} \\ & 1_{2n} \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} B_1 & T \\ & B_2 \end{pmatrix}$, then the condition is equivalent to all entries in T and $B_1 - B_2$ are integers. Since $U_{E_1}^1$ consists of v of the above form with $T = 0$ and $B_1 = B_2$, we see over the subdomain where $F_0(u^{-1}E_1u) \neq 0$, the representatives of $U_{E_1}^1 \backslash \hat{N}^{n+1}$ can be chosen to be in K_3. Thus:

$$\Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}) = \int_{u \in U_{E_1}^1(\mathcal{O}) \backslash \hat{N}^{n+1}(\mathcal{O})} \theta_4(u)\omega_{\psi^{-1}}(u)\Phi(0)\, du.$$

Over the domain $\theta_4(u) = 1$ and $\omega_{\psi^{-1}}(u)\Phi(0) = \Phi(0) = 1$. Thus we get $\Psi_{F_0,\Phi_0}(\widetilde{1_{2n}}) = 1$. We get the claim in the Lemma. □

6.3. **General Hecke element case.** The derivation of Proposition 6.1 for general Hecke element from the unit element case is explained in [MR1], and the result is given as Proposition 8.1 (along with Proposition 4.1) in [MR1]. We sketch the argument used there.

Assume f and $\tilde{f}$ match through equation (6.1).

F defined by (4.2) is a K_3 invariant function on Y, thus through Plancherel formula proved in [MR1]:

$$F(x) = \int_{D_n} \widehat{\tilde{f}}(z)\Xi_z(x)\, d_1 z. \tag{6.3}$$

Here $\Xi_z(x)$ are spherical functions on Y parameterized by $z \in \mathbf{C}^n$ and D_n is a compact subset of $\mathbf{C}^n$ and $d_1 z$ is a Plancherel measure on D_n.

On the other hand, let $\Psi_{\tilde{f}}(\widetilde{g})$ be the right hand side of (6.2); it is a right K_2-invariant and left N_2-equivariant function on G_2. Use another Plancherel formula for space of such functions ([MR1]):

$$\Psi_{\tilde{f}}(\widetilde{g}) = \int_{D_n} \widehat{\tilde{f}}(z) W_z(\widetilde{g})\, d_2 z. \tag{6.4}$$

Here $W_z(x)$ are unramified Whittaker functions ([BFH]) on G_2 and $d_2 z$ is another Plancherel measure on D_n.

From (6.3) we get

$$\Psi_{F,\Phi_0}(\widetilde{g}) = \int_{D_n} \widehat{\tilde{f}}(z)\Psi_{\Xi_z,\Phi_0}(\widetilde{g})\, d_1 z$$

where formally Ψ_{Ξ_z,Φ_0} is defined by (5.1) with F replaced by Ξ_z (precise definition in [MR1]). Using a Jacquet module computation (see the equation right after [MR1, Lemma 8.7]), we get $\Psi_{\Xi_z,\Phi_0} = c(z)W_z$ for some constant $c(z)$. Thus we get

$$\Psi_{F,\Phi_0}(\widetilde{g}) - \Psi_{\tilde{f}}(\widetilde{g}) = \int_{D_n} \widehat{\tilde{f}}(z) W_z(\widetilde{g})\, d_3 z \tag{6.5}$$

where $d_3 z = c(z) d_1 z - d_2 z$. Now when $\widehat{\tilde{f}} = 1$, from the unit Hecke element case, we have shown the above expression equals 0 for all g. We have

$$\int_{D_n} W_z(\widetilde{g})\, d_3 z = 0, \;\; \forall g \in \mathrm{Sp}_n\,. \tag{6.6}$$

For arbitrary $\tilde{f}$ and fixed g, $\widehat{\tilde{f}}(z)W_z(\widetilde{g})$ as a function of z is in $\mathbf{C}[q^z, q^{-z}]^{W(\mathrm{Sp}_n)}$ the space of polynomials $\phi(z)$ in variables q^{z_i} and q^{-z_i} satisfying $\phi(z) = \phi(wz)$ where $w \in W(\mathrm{Sp}_n)$ acts on z in a natural way. (Here q is the size of residue field of k_v). Since $\{W_z(\widetilde{g}) \,|\, g \in \mathrm{Sp}_n\}$ spans $\mathbf{C}[q^z, q^{-z}]^{W(\mathrm{Sp}_n)}$, we get $\widehat{\tilde{f}}(z)W_z(\widetilde{g})$ is a linear combination of $W_z(\widetilde{g_i})$ for some choices of g_i. Thus from (6.6) the expression in (6.5) equals 0 for all $g \in \mathrm{Sp}_n$.

Therefore we get Proposition 6.1. □

7. Proof of Theorem 1.1

From the previous sections, we get the following trace identity:

Theorem 7.1. *There exists maps $\epsilon_{5,v}$ from $\mathcal{S}(\mathrm{Sp}_{2n}(k_v)) \otimes \mathcal{S}(k_v^n)$ to $\mathcal{S}(\widetilde{\mathrm{Sp}}_n(k_v))$, such that*

$$I_{G_3}(f : H_3, 1; N_3, \theta_4\Theta^{\Phi}_{\psi^{-1}}) = I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2) \tag{7.1}$$

when

(1) at $v \notin S$ where S is a finite set of places containing all bad places, f_v is a Hecke function and Φ_v is the characteristic function of the lattice $\mathcal{O}_v^n$, $\tilde{f}_v$ is the Hecke function associated to f_v by (6.1).

(2) at $v \in S$, $\tilde{f}_v = \epsilon_{5,v}(f_v \otimes \Phi_v)$.

Proof. : Given f_v, Φ_v, we find $\tilde{f}_v$ through Corollary 5.6. The identity follows from Propositions 4.9, 4.10, and Corollary 6.2. □

Proof of Theorem 1.1:

Given $f_v \in \mathcal{S}(\mathrm{GL}_{2n}(k_v))$, find $f'_v \in \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ through Corollary 2.11, then find a pair $f''_v \in \mathcal{S}(\mathrm{Sp}_{2n}(k_v))$ and $\Phi_v \in \mathcal{S}(k_v^n)$ through Corollary 3.6, then find $\tilde{f}_v \in \mathcal{S}(\widetilde{\mathrm{Sp}}_n(k_v))$ through Theorem 7.1. This gives the maps ϵ_v which is $\epsilon_{5,v}\epsilon_{4,v}\epsilon_{1,v}$.

At a good place v, the Hecke algebra homomorphism $\lambda_v : f_v \mapsto \tilde{f}_v$ from $\mathcal{H}(G_{1,v}, K_{1,v})$ to $\mathcal{H}(G_{2,v}, K_{2,v})$ is defined so that

$$\widehat{f_v}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n) = \widehat{\tilde{f}_v}(z_1, \ldots, z_n).$$

Let S be a finite set of places containing archimedean places and even places and places where ψ is not unramified. Assume $f = \otimes f_v$ and $\tilde{f} = \otimes_{v\in S}\epsilon_v(f_v) \otimes_{v\notin S} \lambda_v(f_v)$. We need to show (1.3) holds.

For $v \notin S$, there is $f'_v \in \mathcal{H}(G_{3,v}, K_{3,v})$ such that equation (6.1) holds. From Theorem 7.1, equation (7.1) holds when we replace $f \otimes \Phi$ by

$$\otimes_{v\in S}\epsilon_{4,v}\epsilon_{1,v}(f_v) \otimes_{v\notin S} (f'_v \otimes \Phi_{0,v}).$$

From Corollary 3.6, we get $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$ equals $I_{G_3}(f' : H_3, 1; N_3, \theta_3)$ when $f' = \otimes_{v\in S}\epsilon_{1,v}(f_v) \otimes_{v\notin S} f'_v$.

From Corollary 2.11, we get $I_{G_2}(\tilde{f} : N_2, \theta_2^{-1}; N_2, \theta_2)$ equals $I_{G_1}(f_1 : H_1, 1; N_1, \theta_1)$ where

$$f_1 = \otimes_{v\in S} f_v \otimes_{v\notin S} f_{1,v},\ \ f_{1,v} = \lambda_{1,v}(f'_v).$$

We have the following relationship between f_v and $f_{1,v}$:

Lemma 7.2. *For all $z \in \mathbf{C}^n$:*

$$\widehat{f_v}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n) = \widehat{f_{1,v}}(z_1, -z_1, z_2, -z_2, \ldots, z_n, -z_n). \tag{7.2}$$

Proof. The left hand side of the equation is $\widehat{\tilde{f}}(z_1, \ldots, z_n)$, which equals

$$\widehat{f'_v}(z_1 + \frac{1}{2}, z_1 - \frac{1}{2}, \ldots, z_n + \frac{1}{2}, z_n - \frac{1}{2}).$$

Use the invariance under Weyl group of Sp_{2n}, the above equals:

$$\widehat{f'_v}(z_1 + \frac{1}{2}, -z_1 + \frac{1}{2}, \ldots, z_n + \frac{1}{2}, -z_n + \frac{1}{2}).$$

Using the relation (2.17) we get the above equals the right hand side of the equation. □

To complete the proof of identity (1.3), we only need to show

$$I_{G_1}(f_1 : H_1, 1; N_1, \theta_1) = I_{G_1}(f : H_1, 1; N_1, \theta_1).$$

From the orbital integral decomposition (Lemma 2.9), this will follow from

Lemma 7.3. *When equation (7.2) holds for all $z \in \mathbf{C}^n$,*

$$\int_{H_1(k_v)} f_v(hg)\, dh = \int_{H_1(k_v)} f_{1,v}(hg) dh.$$

Proof. We use a result in [O]. Let $C_0(H_{1,v}\backslash G_{1,v}, K_{1,v})$ be the space of right $K_{1,v}$ and left $H_{1,v}$ invariant functions compactly supported on $G_1(k_v)$. Then the Hecke algebra $\mathcal{H}(G_{1,v}, K_{1,v})$ acts on this space by:

$$f * \phi(g) = \int \phi(gh) f(h^{-1})\, dh,\ \ \phi \in C_0(H_{1,v}\backslash G_{1,v}, K_{1,v}),\ f \in \mathcal{H}(G_{1,v}, K_{1,v}).$$

[O, Proposition 4.9] described $C_0(H_{1,v}\backslash G_{1,v}, K_{1,v})$ as $\mathcal{H}(G_{1,v}, K_{1,v})$-module. In particular the action of f is determined by the values of $\widehat{f}(z_1, -z_1, \ldots, z_n, -z_n)$. Thus we have $f_v * \phi = f_{1,v} * \phi$ for all $\phi \in C_0(H_{1,v}\backslash G_{1,v}, K_{1,v})$, when f_v and $f_{1,v}$ satisfy (7.2).

Let $f_{0,v}$ be the unit Hecke function on G_1, and define

$$\Xi_{0,v}(g) = \int_{H_1(k_v)} f_{0,v}(hg) dh.$$

Then $\Xi_{0,v} \in C_0(H_{1,v}\backslash G_{1,v}, K_{1,v})$. It is clear the two sides of the equation in Lemma are $f_v * \Xi_{0,v}$ and $f_{1,v} * \Xi_{0,v}$, thus we get the equation in Lemma. □

This completes the proof of Theorem 1.1. □

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