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Compositio Math. **145** (2009), 501–527.

[doi:10.1112/S0010437X08003904](https://doi.org/10.1112/S0010437X08003904)



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A Plancherel formula for $\mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ and its application

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ABSTRACT

We compute the spherical functions on the symmetric space $\mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ and derive a Plancherel formula for functions on the symmetric space. As an application of the Plancherel formula, we prove an identity which amounts to the fundamental lemma of a relative trace identity between Sp_{2n} and $\widetilde{\mathrm{Sp}}_n$.

1. Introduction

Let F be a p -adic field with p odd. Let \mathcal{O} be the ring of integers in F . Let π be a prime in \mathcal{O} and let $q = |\pi|^{-1}$. Let \mathbf{G} be a reductive group over F and $G = \mathbf{G}(F)$. Let K be a maximal compact subgroup of G . Unless otherwise specified, the measures on a subgroup G' of G are normalized so that the volume of $K \cap G'$ is one.

Let H be a closed unimodular subgroup of G such that there is a Borel subgroup B of G with BH being open in G . Let χ be a unitary character on H that is trivial on $H \cap K$. Denote by $C_K^\infty(H \backslash G, \chi)$ the space of all complex functions $f(g)$ on G satisfying $f(hgk) = \chi(h)f(g)$; denote by $\mathcal{S}_K(H \backslash G, \chi)$ the subspace consisting of all functions compactly supported modulo H in $C_K^\infty(H \backslash G, \chi)$.

In this paper we study two such spaces $\mathcal{S}_{K_i}(H_i \backslash G_i, \chi_i)$ ($i = 1, 2$) and the linear maps between them. In the case we consider, $G_1 = \mathrm{Sp}_{2n}$, $H_1 \cong \mathrm{Sp}_n \times \mathrm{Sp}_n$, $K_1 = \mathrm{Sp}_{2n}(\mathcal{O})$ and χ_1 is a trivial character; $G_2 = \widetilde{\mathrm{Sp}}_n$, the double cover of Sp_n , H_2 is the maximal unipotent subgroup of G_2 (which is just two copies of the maximal unipotent subgroup of Sp_n), $K_2 = \mathrm{Sp}_n(\mathcal{O})$ embedded in G_2 and χ_2 is a non-degenerate character on H_2 (see § 7 for precise description of χ_2). In the case of G_2 we let $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$ be a subspace of *genuine* functions on G_2 . The study of linear maps is used in proving the relative trace identity in [MR08].

1.1 Spherical functions in $C_K^\infty(H \backslash G, \chi)$

Denote by $\mathcal{H}(G, K)$ the Hecke algebra of G with respect to K . It is the algebra of all compactly supported functions on G satisfying $f(k_1 g k_2) = f(g)$ for any $k_1, k_2 \in K$ and $g \in G$. The multiplication in $\mathcal{H}(G, K)$ is given by the convolution product

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) dg.$$

Received 14 February 2008, accepted in final form 8 September 2008, published online 19 February 2009.

2000 Mathematics Subject Classification 43A90 (primary), 11F72 (secondary).

Keywords: spherical function, Macdonald polynomial, Plancherel formula.

The first author was partially supported by a NSF grant DMS0700030 and a Israel–US BSF grant #2004200.

The second author was partially supported by a NSF grant DMS0500392.

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There is an action of the Hecke algebra $\mathcal{H}(G, K)$ on $C_K^\infty(H \backslash G, \chi)$ through

$$(f * \phi)(g) = \int_G f(h)\phi(gh) dh, \quad f \in \mathcal{H}(G, K), \quad \phi \in C_K^\infty(H \backslash G, \chi). \quad (1.1)$$

A *spherical function* in $C_K^\infty(H \backslash G, \chi)$ is by definition an eigenfunction $\Psi(g)$ in $C_K^\infty(H \backslash G, \chi)$ under the action by $\mathcal{H}(G, K)$ such that Ψ equals one at the identity. We first classify all spherical functions in $C_{K_1}^\infty(H_1 \backslash G_1, \chi_1)$ and $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$.

We call an element g in G *relevant* if χ is trivial on $gKg^{-1} \cap H$. Clearly if g is not relevant, then $\phi(g) = 0$ for all ϕ in $C_K^\infty(H \backslash G, \chi)$. Let G^{rel} be the subset of relevant elements in G .

LEMMA 1.1. *Let*

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbf{Z}^n; \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}.$$

For $i = 1, 2$, there are injective maps $\Pi_i : \Lambda_n^+ \mapsto G_i$, such that $G_i^{\text{rel}} = \bigcup_{\lambda \in \Lambda_n^+} H_i \Pi_i(\lambda) K_i$ as a disjoint union. Moreover $\Pi_i(\mathbf{0}) \in K_i$, where $\mathbf{0} = (0, \dots, 0)$.

Thus, functions ϕ in $C_{K_i}^\infty(H_i \backslash G_i, \chi_i)$ are determined by the values of $\phi(\Pi_i(\lambda))$ for $\lambda \in \Lambda_n^+$.

Before describing values of spherical functions at $\Pi_i(\lambda)$, we recall the definition of the Macdonald polynomial. Let S be the reduced root system of type C_n , let R be the root system of type BC_n . The root systems R and S are inside the same vector space identified with \mathbf{C}^n . Let $\epsilon_i, i = 1, \dots, n$, be the standard basis of \mathbf{C}^n , then

$$\begin{aligned} S &= \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, 1 \leq i \leq n, i < j \leq n\}, \\ R &= \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm 2\epsilon_i, 1 \leq i \leq n, i < j \leq n\}. \end{aligned}$$

The root systems R and S have the same Weyl group W which is the Weyl group of Sp_n . There is a natural action of W on \mathbf{C}^n .

Let $\mathbf{C}[q^z, q^{-z}]^W$ be the space of functions $F(z)$, with $z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n$, that are polynomials in variables q^{z_i} and q^{-z_i} , satisfying $F(z) = F(wz)$ for all $w \in W$. Macdonald defined sets of basis for this space. A Macdonald polynomial has the form [Mac00, (10.1)]

$$Q_\lambda^t(z) = P_\lambda^t(e^{\epsilon_i}) = V_\lambda(t)^{-1} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in R^+} \frac{1 - t_\alpha t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}}{1 - t_{2\alpha}^{\frac{1}{2}} e^{-\alpha}} \right). \quad (1.2)$$

Here $\lambda \in \Lambda_n^+$ is identified with the dominant weights of R , R^+ is the set of positive roots, and e^{ϵ_i} are the independent variables of the polynomial P_λ^t ; Q_λ^t and P_λ^t are related through the equation $e^{\epsilon_i} = q^{-z_i}$. The data t_α are parameters such that when α is not a root in R , $t_\alpha = t_\alpha^{\frac{1}{2}} = 1$. Thus, the parameters t are determined by values of t_α when α is a long root in S , and t_α with $t_{2\alpha}^{\frac{1}{2}}$ when α is a short root in S . Here $V_\lambda(t)$ are nonzero constants independent of variables e^{ϵ_i} ; they are defined in [Mac00] (and denoted by $W_\lambda(t)$ there).

Then $\{Q_\lambda^t(z) \mid \lambda \in \Lambda_n^+\}$ forms a basis of $\mathbf{C}[q^z, q^{-z}]^W$.

THEOREM 1.2. For $i = 1, 2$, there are choices of real number parameters t_α^i for α roots in R , and nonzero values $a_i(\lambda)$ for $\lambda \in \Lambda_n^+$, such that for all $z = (z_1, \dots, z_n) \in \mathbf{C}^n$,

$$\Psi_z^i(\Pi_i(\lambda)) = a_i(\lambda) Q_\lambda^{t^i}(z), \quad \lambda \in \Lambda_n^+, \quad (1.3)$$

determines a spherical function Ψ_z^i in $C_{K_i}^\infty(H_i \backslash G_i, \chi_i)$.

The exact descriptions of parameters t_α^i and functions $a_i(\lambda)$ are given in Theorems 3.2 and 7.1, respectively. We make the remark that, when $i = 2$, the above is a restatement of the main theorem in [BFH91].

1.2 Consequences of Theorem 1.2

For $f \in \mathcal{H}(G, K)$, let $\tilde{f}(z)$ be the eigenvalue of $f^*(g) = f(g^{-1})$ acting on $\overline{\Psi_z}$ through (1.1):

$$\tilde{f}(z) = \int f(g^{-1}) \overline{\Psi_z(g)} dg. \quad (1.4)$$

Then $f \mapsto \tilde{f}(z)$ is an algebra homomorphism from $f \in \mathcal{H}(G, K)$ to $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$. We can consider $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$ as a $\mathcal{H}(G, K)$ module through the multiplication by $\tilde{f}(z)$. It is easy to verify the following.

LEMMA 1.3. *For $i = 1, 2$, the map $f \mapsto \tilde{f}(z)$ is onto $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$ from $\mathcal{H}(G_i, K_i)$.*

Define the measure dg on $H \backslash G$ so that $H \backslash KH$ has volume one. When $\phi \in \mathcal{S}_K(H \backslash G, \chi)$, we define its Fourier transform $\hat{\phi}(z)$ by

$$\hat{\phi}(z) = \langle \phi, \Psi_z \rangle = \int_{H \backslash G} \phi(g) \overline{\Psi_z(g)} dg. \quad (1.5)$$

Clearly $\hat{\phi}(z) \in \mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$. Recall that $\mathcal{S}_K(H \backslash G, \chi)$ is a $\mathcal{H}(G, K)$ module through (1.1).

THEOREM 1.4. *Let (G, H, χ, K) be such that Lemmas 1.1, 1.3 and Theorem 1.2 hold.*

- (a) *If Ψ is a spherical function in $C_K^\infty(H \backslash G, \chi)$, then $\Psi = \Psi_z$ for some $z = (z_1, \dots, z_n) \in \mathbf{C}^n$.*
- (b) *(Isomorphism between $\mathcal{S}_K(H \backslash G, \chi)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$) The map $S: \phi \mapsto \hat{\phi}$ gives an isomorphism between $\mathcal{H}(G, K)$ modules $\mathcal{S}_K(H \backslash G, \chi)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$.*

If, moreover, the property

$$f * \Phi = f^* * \Phi \quad \text{for all } \Phi \in C_K^\infty(H \backslash G, \chi) \text{ and } f \in \mathcal{H}(G, K), \text{ where } f^*(g) = f(g^{-1}) \quad (*)$$

holds, then we have the following.

- (c) *(Volume formula) The volume of $H \backslash H\Pi(\lambda)K$ is $a(\mathbf{0})^2 V_\lambda(t) / a(\lambda)^2 V_{\mathbf{0}}(t)$.*
- (d) *(Plancherel formula) Assume further that the parameters t_α satisfy condition (5.1.1) in [Mac71]. Define $\Delta^t(z)$ as in (2.1). Denote by D_n the direct product of n copies of $\sqrt{-1}\mathbb{R} / ((2\pi / \log q)\mathbb{Z})$, then for $\phi \in \mathcal{S}_K(H \backslash G, \chi)$,*

$$\phi(g) = \int_{D_n} \hat{\phi}(z) \Psi_z(g) d_\mu(z) \quad (1.6)$$

with measure $d_\mu(z) = (1/|W|) V_{\mathbf{0}}(t) \Delta^t(z) dz$.

Obviously condition (*) holds in the case of G_1 since $f(g) = f(g^{-1})$ in our case. We state condition (*) in this form since it is also satisfied in the three cases considered in [Off04], owing to the existence of automorphism $g \mapsto g^* = w(g^t)^{-1} w^{-1}$ of G such that $\Phi(g^*) = \Phi(g)$ and $f(g^*) = f(g^{-1})$; here w is the longest Weyl element in G .

Property (*) does not hold in the case of G_2 . However, in that case the volume of $H \backslash H\Pi(\lambda)K$ is easily computed and we can again obtain the Plancherel formula from the volume formula (see § 7).

COROLLARY 1.5. For $i = 1, 2$, there are isomorphisms between $\mathcal{H}(G_i, K_i)$ modules $\mathcal{S}_K(H_i \backslash G_i, \chi_i)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$ given by $S_i : \phi \mapsto \hat{\phi}$.

For $i = 1, 2$, there are measures $d_\mu^i(z)$ on D_n such that

$$\phi(g) = \int_{D_n} \hat{\phi}(z) \Psi_z^i(g) d_\mu^i(z). \quad (1.7)$$

1.3 Linear maps between $\mathcal{S}_{K_i}(H_i \backslash G_i, \chi_i)$

We consider the linear maps T from $\mathcal{S}_{K_1}(H_1 \backslash G_1, \chi_1)$ to $\mathcal{S}_{K_2}(H_2 \backslash G_2, \chi_2)$ that preserve spherical functions, namely satisfying $T(\Psi_z^1) = c(z) \Psi_z^2$. Then from (1.7), formally we have

$$T(\phi_1) = \int_{D_n} \hat{\phi}_1(z) \Psi_z^2 c(z) d_\mu^1(z), \quad \forall \phi_1 \in \mathcal{S}_{K_1}(H_1 \backslash G_1, \chi_1). \quad (1.8)$$

LEMMA 1.6. Let T be a linear map satisfying:

- (1) Equation (1.8) holds for some function $c(z)$;
- (2) $T(S_1^{-1}(1)) = S_2^{-1}(1)$;

then $S_1 = S_2 T$ on $\mathcal{S}_{K_1}(H_1 \backslash G_1, \chi_1)$.

Proof. Let $\phi_i \in \mathcal{S}_{K_i}(H_i \backslash G_i, \chi_i)$ such that $S_1(\phi_1) = S_2(\phi_2)$; we show that $\phi_2 = T(\phi_1)$.

By (1.7),

$$\phi_2 = \int_{D_n} \hat{\phi}_2(z) \Psi_z^2 d_\mu^2(z).$$

Since $\hat{\phi}_1(z) = \hat{\phi}_2(z)$, we obtain a function $c'(z)$ on D_n with

$$T(\phi_1) - \phi_2 = \int_{D_n} \hat{\phi}_2(z) \Psi_z^2 c'(z) dz. \quad (1.9)$$

When $\hat{\phi}_2(z) \equiv 1$, condition (2) implies

$$\int_{D_n} \Psi_z^2(\Pi_2(\lambda)) c'(z) dz \equiv 0, \quad \forall \lambda \in \Lambda_n^+. \quad (1.10)$$

When $z \in D_n$, $\bar{z} = -z$; thus, we can consider $\hat{\phi}_2(z)$ as a polynomial in $\mathbf{C}[q^z, q^{-z}]^W$. For any $\lambda \in \Lambda_n^+$ and any ϕ_2 , $\hat{\phi}_2(z) \Psi_z^2(\Pi_2(\lambda)) \in \mathbf{C}[q^z, q^{-z}]^W$. Thus it can be written as a linear combination $\sum c_i \Psi_z^2(\Pi_2(\lambda_i))$. From (1.9) and (1.10), we obtain $(T(\phi_1) - \phi_2)(\Pi_2(\lambda)) = 0$; thus, $T(\phi_1) = \phi_2$. \square

In § 8, we construct an explicit map T from $\mathcal{S}_{K_1}(H_1 \backslash G_1, \chi_1)$ to $\mathcal{S}_{K_2}(H_2 \backslash G_2, \chi_2)$. Conditions (1) and (2) in the lemma are then verified. Thus, we obtain the conclusion $S_1 = S_2 T$. The identity (Proposition 8.1) is related to the fundamental lemma of the relative trace identities between G_1 and G_2 studied in [MR08]. In essence the identity $S_1 = S_2 T$ is the fundamental lemma for general Hecke algebra elements, and condition (2) in Lemma 1.6 is the fundamental lemma for the unit elements in Hecke algebras. Thus, from the lemma with condition (1), the fundamental lemma for general elements follows from that of the unit elements.

The proof of Theorem 1.2 follows the ideas of [Hir99]. The new ingredient we need is an idea to show the vanishing of some intertwining operators. While working on the proof we benefited a lot from discussion with Omer Offen, who was writing a thesis (under Jacquet's advice) on a similar problem, the case when $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_n \times \mathrm{GL}_n$ (see [Off04, Off08]).

The paper is organized as follows. In § 2 we prove Theorem 1.4. In § 3 we transfer the question of spherical functions in $C_{K_1}^\infty(H_1 \backslash G_1, \chi_1)$ to the equivalent question of spherical functions on a

symmetric space X . In §4 we give the integral representation of the spherical functions on X . The integral is computed in §5 where we prove case (1) of Theorem 1.2, pending a function equation that is established in §6. The spherical functions in $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$ are studied in §7. In §8 we construct a map between two spaces and establish Proposition 8.1 which is the fundamental lemma needed in [MR08].

2. Proof of Theorem 1.4

2.1 Preliminary on Macdonald polynomial

We first recall some facts about the Macdonald polynomial P_λ^t . The results are included in [Mac00], we state them in terms of the polynomials Q_λ^t in $\mathbf{C}[q^z, q^{-z}]^W$. Here $\mathbf{C}[q^z, q^{-z}]^W$ is the space of symmetric polynomials

$$\{f(q^z) \in \mathbf{C}[q^z, q^{-z}] \mid f(q^{wz}) = f(q^z), w \in W\}.$$

The space $\mathbf{C}[q^z, q^{-z}]^W$ has a basis (where we let $e^{\epsilon_i} = q^{-z_i}$)

$$\left\{ m_\lambda = \sum_{\mu \in W\lambda} e^\mu \mid \lambda \in \Lambda_n^+ \right\}.$$

Let

$$\Delta(z) = \Delta^t(z) = \prod_{\alpha \in R} \frac{1 - t_{2\alpha}^{\frac{1}{2}} e^\alpha}{1 - t_{2\alpha}^{\frac{1}{2}} t_\alpha e^\alpha}. \quad (2.1)$$

Then on $\mathbf{C}[q^z, q^{-z}]^W$ we can define a scalar product [Mac00, (3.4)]

$$\langle f, g \rangle = |W|^{-1} [f \bar{g} \Delta]_1 \quad (2.2)$$

where the notation is the same as in [Mac00, §3].

Define a partial order on Λ_n^+ : $\lambda > \mu$ if $\lambda \neq \mu$ and $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$. The following theorem is shown in [Mac00].

THEOREM 2.1. *For any choice of parameters t :*

- (1) *let $\lambda \in \Lambda_n^+$, $Q_\lambda^t = m_\lambda + \sum_{\lambda > \mu \in \Lambda_n^+} u(\lambda, \mu) m_\mu$, for some constant coefficients $u(\lambda, \mu)$;*
- (2) *$\langle Q_\lambda^t, Q_\mu^t \rangle = 0$ if $\lambda \neq \mu$, $\lambda, \mu \in \Lambda_n^+$;*
- (3) *$\langle Q_\lambda^t, Q_\lambda^t \rangle = V_\lambda(t)^{-1}$;*
- (4) *$\{Q_\lambda^t \mid \lambda \in \Lambda_n^+\}$ is the unique basis of $\mathbf{C}[q^z, q^{-z}]^W$ satisfying parts (1) and (2).*

We remark that $Q_{\mathbf{0}}^t = m_{\mathbf{0}} \equiv 1$; thus, $a(\mathbf{0}) = 1$ in Theorem 1.2. For the root systems (R, S) we consider, there is an element in $w \in W$ such that $w(z) = -z$. Thus, for $f \in \mathbf{C}[q^z, q^{-z}]^W$, $\bar{f} = f$ and we obtain

$$\langle f, g \rangle = \langle fg, 1 \rangle. \quad (2.3)$$

2.2 Hecke algebra isomorphism

For $\lambda \in \Lambda_n^+$, define ch_λ to be a function in $\mathcal{S}_K(H \backslash G, \chi)$ such that $ch_\lambda(\Pi(\lambda')) = 0$ unless $\lambda = \lambda'$, in which case $ch_\lambda(\Pi(\lambda)) = 1$. Then $\{ch_\lambda\}$ is a basis of $\mathcal{S}_K(H \backslash G, \chi)$.

Proof of part (b) of Theorem 1.4. Since $\widehat{ch_\lambda} = \text{vol}(H \backslash H\Pi(\lambda)K) \overline{a(\lambda)} Q_\lambda^t(z) \in \mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$, we obtain a bijection between the basis of $\mathcal{S}_K(H \backslash G, \chi)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$. Thus, the Fourier transform

is a bijection. We show it is also a homomorphism. Taking $f \in \mathcal{H}(G, K)$, $\phi \in \mathcal{S}_K(H \backslash G, \chi)$, then

$$\widehat{f * \phi}(z) = \int_G \int_{H \backslash G} f(g) \phi(hg) \overline{\Psi_z(h)} dh dg.$$

Changing $h \mapsto hg^{-1}$, we obtain

$$\begin{aligned} \widehat{f * \phi}(z) &= \int_G \int_{H \backslash G} f(g) \phi(h) \overline{\Psi_z(hg^{-1})} dh dg \\ &= \int_{H \backslash G} \int_G f(g^{-1}) \phi(h) \overline{\Psi_z(hg)} dh dg \\ &= \tilde{f}(z) \hat{\phi}(z). \end{aligned}$$

Thus, the Fourier transform is a homomorphism and an isomorphism. □

Proof of part (a) of Theorem 1.4. We can proceed as in the proof of [HS88, Theorem 2]. Let Ψ be a spherical function and $\omega : \mathcal{H}(G, K) \mapsto \mathbf{C}$ be the algebra homomorphism defined through the equation

$$\int f(g^{-1}) \overline{\Psi(g)} dg = \omega(f).$$

If $\tilde{f}(z) \equiv 0$, then $\widehat{f * \phi} = 0$ for all $\phi \in \mathcal{S}_K(H \backslash G, \chi)$ by the above proposition, and we obtain

$$\omega(f) \langle \phi, \Psi \rangle = \langle f * \phi, \Psi \rangle = 0$$

where $\langle \cdot, \cdot \rangle$ is defined in (1.5). We obtain $\omega(f) = 0$, in particular the homomorphism ω must be of the form $\tilde{f}(z_0)$ for some $z_0 \in \mathbf{C}^n$. It is now easy to see that Ψ must be a multiple of Ψ_{z_0} from Lemma 1.3. □

2.3 Volume formula and Plancherel formula

In this subsection we assume that property (*) in Theorem 1.4 holds.

Given any $f \in \mathcal{H}(G, K)$, since $f * ch_{\mathbf{0}} \in \mathcal{S}_K(H \backslash G, \chi)$, we have

$$f * ch_{\mathbf{0}} = \sum_{\mu \in \Lambda_n^+} c(\mu) ch_{\mu} \tag{2.4}$$

where all but finitely many constants $c(\lambda)$ equal zero. We use this equation in two ways. First we use it to compute $f * \overline{\Psi_z}(\Pi(\lambda))$:

$$\begin{aligned} f * \overline{\Psi_z}(\Pi(\lambda)) &= \sum_{\mu} \overline{\Psi_z(\Pi(\mu))} f * ch_{\mu}(\Pi(\lambda)) \\ &= \overline{\Psi_z(\Pi(\mathbf{0}))} c(\lambda) + \sum_{\mu \neq \mathbf{0}} \overline{\Psi_z(\Pi(\mu))} d(\mu) \end{aligned}$$

where $d(\mu)$ are some constants, and all but finitely many of them are zero. On the other hand, using property (*):

$$f * \overline{\Psi_z}(g) = \tilde{f}(z) \overline{\Psi_z}(g).$$

We obtain $f * \overline{\Psi_z}(\Pi(\lambda)) = \tilde{f}(z) \overline{a(\lambda) Q_{\lambda}^t(z)}$. Thus, as functions of z , we obtain

$$a(\lambda) \overline{\tilde{f}(z) Q_{\lambda}^t(z)} = \overline{c(\lambda) a(\mathbf{0}) Q_{\mathbf{0}}^t(z)} + \sum_{\mu \neq \mathbf{0}} d'(\mu) Q_{\mu}^t(z) \tag{2.5}$$

for some other constants $d'(\mu)$. Taking the scalar product (2.2) of the equation with $Q_{\mathbf{0}}^t$, we obtain

$$a(\lambda)\langle \overline{\tilde{f}(z)}Q_{\lambda}^t, Q_{\mathbf{0}}^t \rangle = \overline{c(\lambda)}a(\mathbf{0})V_{\mathbf{0}}(t)^{-1}. \quad (2.6)$$

Here we have applied Theorem 2.1. We now apply (2.4) again, this time in computing

$$\int_{H \backslash G} f * ch_{\mathbf{0}}(g)\overline{\Psi_z(g)} dg.$$

The integral equals $\tilde{f}(z)\widehat{ch_{\mathbf{0}}}(z)$ by part (b) of Theorem 1.4. Thus, this integral equals

$$\text{vol}(H \backslash HK)\tilde{f}(z)\overline{a(\mathbf{0})Q_{\mathbf{0}}^t(z)}.$$

On the other hand, from (2.4), we find that the above integral equals $\sum_{\mu} c(\mu)\widehat{ch_{\mu}}$, which is $\sum_{\mu} \text{vol}(H \backslash H\Pi(\mu)K)c(\mu)\overline{a(\mu)Q_{\mu}^t(z)}$. Thus, we obtain

$$\text{vol}(H \backslash HK)\tilde{f}a(\mathbf{0})Q_{\mathbf{0}}^t = \sum_{\mu} \text{vol}(H \backslash H\Pi(\mu)K)\overline{c(\mu)}a(\mu)Q_{\mu}^t. \quad (2.7)$$

as functions of z . Taking the scalar product with Q_{λ}^t and use Theorem 2.1, we obtain

$$\text{vol}(H \backslash HK)a(\mathbf{0})\langle \overline{\tilde{f}}Q_{\mathbf{0}}^t, Q_{\lambda}^t \rangle = \text{vol}(H \backslash H\Pi(\lambda)K)\overline{c(\lambda)}a(\lambda)V_{\lambda}(t)^{-1}. \quad (2.8)$$

From (2.6) and (2.8), we obtain

$$\text{vol}(H \backslash HK)/\text{vol}(H \backslash H\Pi(\lambda)K) = \frac{a(\lambda)^2V_{\mathbf{0}}(t)}{a(\mathbf{0})^2V_{\lambda}(t)} \frac{\langle \overline{\tilde{f}}P_{\lambda}^t, P_{\mathbf{0}}^t \rangle}{\langle \overline{\tilde{f}}P_{\mathbf{0}}^t, P_{\lambda}^t \rangle}. \quad (2.9)$$

From (2.3), we obtain

$$\text{vol}(H \backslash HK)/\text{vol}(H \backslash H\Pi(\lambda)K) = \frac{a(\lambda)^2V_{\mathbf{0}}(t)}{a(\mathbf{0})^2V_{\lambda}(t)}. \quad (2.10)$$

Note that we fixed measure so that $\text{vol}(H \backslash HK) = 1$, and thus the above equation gives the volume formula in part (c) of Theorem 1.4.

Note that from (2.10), we find that $a(\lambda)$ must be real. The proof of the Plancherel formula now follows from Theorem 2.1 and the above volume formula; see [Mac71, ch. V] for the argument.

3. Spherical functions on symmetric space X

Let w_n be the matrix in $\text{GL}_n(F)$ with ones on the antidiagonal and zeros elsewhere. Let $J_n = \begin{pmatrix} & -w_n \\ w_n & \end{pmatrix}$. Let $\text{Sp}_n(F)$ be the symplectic group:

$$\text{Sp}_n(F) = \{g \in \text{GL}_{2n}(F) \mid g^t J_n g = J_n\}.$$

We use G, H, K to denote G_1, H_1, K_1 . We explicitly define an embedding of $H = \text{Sp}_n \times \text{Sp}_n$ in $G = \text{Sp}_{2n}$. Let E be the diagonal matrix with the diagonal being $[1, \dots, 1, -1, \dots, -1, 1, \dots, 1]$ (i.e. n consecutive ones followed by $2n$ consecutive minus ones and n consecutive ones). Then we let

$$H = \{g \in G \mid g^{-1}Eg = E\}.$$

3.1 The symmetric space X

Let $J = J_{2n}$. Let X be the space of antisymmetric matrices $g \in G$. Then $g \mapsto g^{-1}EgJ$ gives a bijection from $H \backslash G$ to X . We identify X with the symmetric homogeneous space $H \backslash G$ and let X inherit the measure of $H \backslash G$. The group G acts on X by $g \cdot x = gxg^t$.

Denote the matrix $\begin{pmatrix} -\pi^a & \pi^a \end{pmatrix}$ by Π_a . For $\lambda \in \Lambda_n^+$, let $\Pi^\lambda \in X$ be the matrix with the diagonal being

$$[\Pi_{\lambda_1}, \Pi_{\lambda_2}, \dots, \Pi_{\lambda_n}, -\Pi_{-\lambda_n}, \dots, -\Pi_{-\lambda_1}]$$

and zeros elsewhere.

We prove Lemma 1.1 in this case.

LEMMA 3.1. *As a disjoint union,*

$$X = \bigcup K \cdot \Pi^\lambda, \quad \lambda \in \Lambda_n^+. \quad (3.1)$$

Proof. Let $x \in X$. Write $x = (x_{i,j})$ with $x_{i,j}$ being the entry at the i th row and the j th column. Let x_{i_0, j_0} be the entry with the largest norm. Since $i_0 \neq j_0$, there is a Weyl element $w_0 \in K$ and a diagonal matrix $a \in K$ so that $x' = w_0 a \cdot x$ has the property that $x'_{1,2} = -x'_{2,1} = \pi^{-\lambda_1}$ has the largest norm. Then there exists a lower triangular matrix $n \in K$ so that $x'' = n \cdot x'$ has the property that $x''_{1,i} = 0$ when $i \neq 2$ and $x''_{2,i} = 0$ when $i \neq 1$. Since $Jx'' = (Jx'')^{-1}$ and $x''^t = -x''$, we obtain $x''_{1,i} = x''_{2,j} = x''_{i,1} = x''_{j,2} = x''_{4n+1-j, 4n-1} = x''_{4n+1-i, 4n} = 0$ unless $i = 2$ or $j = 1$ when $x''_{4n, 4n-1} = -x''_{4n-1, 4n} = \pi^{\lambda_1}$. Thus, x'' has the form of a block diagonal matrix, with the diagonal being $\Pi_{-\lambda_1}$, y and $-\Pi_{\lambda_1}$, where y is in the space of $4(n-1) \times 4(n-1)$ antisymmetric matrices g satisfying $gJ = (gJ)^{-1}$. Moreover, the maximum norm of the entries of y is q^{λ_2} with $\lambda_2 \leq \lambda_1$. Continuing this process, we see that $x \in K \cdot \Pi^{-\lambda}$ with $\lambda \in \Lambda_n^+$. The sets $K \cdot \Pi^{-\lambda}$ are clearly disjoint for distinct $\lambda \in \Lambda_n^+$. As $J \in K$, we obtain the claim. \square

3.2 Spherical function on X

Let $C_K^\infty(X)$ be the space of all K -invariant complex functions on X and $\mathcal{S}_K(X)$ the subspace consisting of all compactly supported functions in $C_K^\infty(X)$. With the identification between $H \backslash G$ and X , the action (1.1) of $\mathcal{H}(G, K)$ on $C_K^\infty(X)$ becomes the convolution product:

$$(f * \Phi)(x) = \int_G f(g)\Phi(g^{-1} \cdot x) dg, \quad f \in \mathcal{H}(G, K), \quad \Phi \in C_K^\infty(X). \quad (3.2)$$

A spherical function on X is, by definition, an eigenfunction $\Psi(x)$ in $C_K^\infty(X)$ of all of the convolutions defined by elements in $\mathcal{H}(G, K)$ such that $\Psi(\Pi^{\mathbf{0}}) = 1$. Here $\mathbf{0}$ is the vector in Λ_n^+ where all entries are zero. The spherical functions on X are clearly in one-to-one correspondence with spherical functions on $C_{K_1}^\infty(H_1 \backslash G_1, \chi_1)$.

We prove the following result over the next three sections.

THEOREM 3.2. *For $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $\Psi_z(x)$ defined by the following equation is a spherical function on X :*

$$\Psi_z(\Pi^\lambda) = q^{b(\lambda)} \frac{V_\lambda(t^I)}{V_{\mathbf{0}}(t^I)} Q_\lambda^{t^I}(z), \quad \lambda \in \Lambda_n^+ \quad (3.3)$$

where $b(\lambda) = -\sum_{i=1}^n (2(n-i+1) - 1/2)\lambda_i$. The parameter t_α^I in the definition of $Q_\lambda^{t^I}$ is set as follows: for α is a short root in S , we let $t_\alpha^I = -q^{-1}$, $(t_{2\alpha}^I)^{\frac{1}{2}} = -q^{-\frac{1}{2}}$; when α is a long root in S , let $t_\alpha^I = q^{-2}$.

4. Integral representation of spherical functions

4.1 Definition of $\Psi_z(x)$

We construct a function $\Psi_z(x)$ on X that is an eigenfunction under the Hecke algebra action, thus a spherical function, and then we establish formula (3.3) for $\Psi_z(x)$.

For $x \in X$, denote by $\mathcal{P}f_i(x)$ ($1 \leq i \leq n$) the Pfaffian of the lower right $2i \times 2i$ block of x . The sign of the Pfaffian is irrelevant for our purpose. Define the integral

$$\zeta(x; s) = \zeta(x; s_1, \dots, s_n) = \int_K \prod_{i=1}^n |\mathcal{P}f_i(k \cdot x)|^{s_i} dk, \quad (4.1)$$

where $x \in X$ and $s \in \mathbf{C}^n$. The integral is taken over the open subset

$$\left\{ k \in K \mid \prod_{i=1}^n |\mathcal{P}f_i(k \cdot x)| \neq 0 \right\}.$$

The integral converges when $Re(s_i) \geq 0$ for all i . By the general theory of complex powers of polynomial functions, it has an analytic continuation to a rational function in q^{s_i} (see [Off04, Theorem 4.1]).

One can show that $\zeta(\Pi^0, s)$ is not identically zero using the same argument as on [Off04, p. 113]. Set

$$\Psi_z(x) = \zeta(x; s) / \zeta(\Pi^0, s), \quad x \in X, \quad (4.2)$$

where $z = (z_1, \dots, z_n)$ satisfies the relation

$$\sum_{i=j}^n s_i = -z_j - 2(n-j) - \frac{3}{2}, \quad j = 1, \dots, n. \quad (4.3)$$

4.2 Hecke algebra action

We first prove that $\Psi_z(x)$ is an eigenfunction of the Hecke algebra.

Let A be the set of diagonal matrices in G and let U be the set of upper triangular matrices with unit diagonal. Let $B = AU$ be the Borel subgroup. Given $\nu = (\nu_1, \dots, \nu_{2n}) \in \mathbf{C}^{2n}$, let $\Phi_\nu(g)$ be the K -invariant vector in the induced representation $I(\chi_\nu) = \text{ind}_B^G \chi_\nu$, where

$$\chi_\nu(au) = \prod_{i=1}^{2n} |a_i|^{\nu_i}, \quad a = \text{diag}[a_1, \dots, a_{2n}, a_{2n}^{-1}, \dots, a_1^{-1}], \quad u \in U.$$

We normalize Φ_ν so that

$$\Phi_\nu(auk) = \prod_{i=1}^{2n} |a_i|^{\nu_i + (2n-i+1)}, \quad k \in K. \quad (4.4)$$

The Satake transform $\hat{f}(\nu)$ of $f \in \mathcal{H}(G, K)$ is defined by

$$\hat{f}(\nu) = \int_G f(g) \Phi_\nu(g) dg.$$

We let

$$\omega_f(z) = \hat{f}(-z_1 + \frac{1}{2}, -z_1 - \frac{1}{2}, \dots, -z_n + \frac{1}{2}, -z_n - \frac{1}{2}). \quad (4.5)$$

The following proposition shows that Ψ_z is a spherical function.

PROPOSITION 4.1. When $f \in \mathcal{H}(G, K)$,

$$(f * \Psi_z)(x) = \omega_f(z)\Psi_z(x), \quad x \in X. \quad (4.6)$$

Proof. Let $f \in \mathcal{H}(G, K)$. Since $\zeta(x; s) = \Psi_z(x)\zeta(\Pi^0; s)$, we have

$$\begin{aligned} \zeta(\Pi^0; s)(f * \Psi_z)(x) &= \int_G f(g)\zeta(g^{-1} \cdot x; s) dg \\ &= \int_G f(g) \left\{ \int_K \prod_{i=1}^n |\mathcal{P}f_i(kg^{-1} \cdot x)|^{s_i} dk \right\} dg \\ &= \int_G f(g) \prod_{i=1}^n |\mathcal{P}f_i(g^{-1} \cdot x)|^{s_i} dg. \end{aligned}$$

For $b \in B$,

$$|\mathcal{P}f_i(b \cdot x)|^{s_i} = |d_i(b)|^{s_i} |\mathcal{P}f_i(x)|^{s_i},$$

where $d_i(b)$ is the determinant of the lower right $2i \times 2i$ block of b . Using the Iwasawa decomposition $G = KB$, we obtain

$$\zeta(\Pi^0; s)(f * \Psi_z)(x) = \int_K \prod_{i=1}^n |\mathcal{P}f_i(k \cdot x)|^{s_i} dk \int_B f(b) \prod_{i=1}^n |d_i(b^{-1})|^{s_i} d_*b.$$

Here d_*b is the right invariant measure on B . The relation of z and s is chosen so that

$$\omega_f(z) = \int_B f(b) \prod_{i=1}^n |d_i(b^{-1})|^{s_i} d_*b.$$

Therefore, we obtain

$$\zeta(\Pi^0; s)(f * \Psi_z(x)) = \zeta(x; s)\omega_f(z).$$

This equation implies Proposition 4.1. □

From the construction of ω_f , we can obtain the surjectivity claim in Lemma 1.3.

4.3 Another definition of $\Psi_z(x)$

Given $x \in X$, define

$$F_z^x(g) = \prod_{i=1}^n |\mathcal{P}f_i(g \cdot x)|^{s_i}, \quad g \in G, \quad (4.7)$$

where z and s are related by (4.3). Then $\zeta(x; s) = \int_K F_z^x(k) dk$. It is clear that

$$F_z^x(bg) = \prod_{i=1}^n |d_i(b)|^{s_i} F_z^x(g). \quad (4.8)$$

Therefore, F_z^x defines a distribution on the space of $I(\chi_{-\nu(z)})$, where

$$\nu(z) = (z_1 - \frac{1}{2}, z_1 + \frac{1}{2}, \dots, z_n - \frac{1}{2}, z_n + \frac{1}{2}).$$

The distribution is given by

$$F_z^x(\Phi) = \int_{B \setminus G} F_z^x(g)\Phi(g) dg,$$

for $\Phi \in I(\chi_{-\nu(z)})$.

When $x = EJ$, since $h \cdot EJ = EJ$ for $h \in H$, we obtain $F_z^{EJ}(gh) = F_z^{EJ}(g)$ for $h \in H \subset G$. Therefore, if $L(\Phi) = F_z^{EJ}(\Phi)$, then L is an H -invariant linear form on $I(\chi_{-\nu(z)})$.

PROPOSITION 4.2. *Let L be the linear form above, and $\Phi_{-\nu(z)}$ be the vector in $I(\chi_{-\nu(z)})$ defined by (4.4). Denote the action of G on $I(\chi_{-\nu(z)})$ by ρ . Then, when $x = g \cdot EJ$, $g \in G$,*

$$\zeta(x; s) = \langle L, \rho(g^{-1})\Phi_{-\nu(z)} \rangle. \quad (4.9)$$

Proof. The right-hand side of the equation is

$$\int_{B \backslash G} \prod_{i=1}^n |\mathcal{P}f_i(h \cdot EJ)|^{s_i} \Phi_{-\nu(z)}(hg^{-1}) dh.$$

Making a change of variable $h \mapsto hg$, this becomes

$$\int_{B \backslash G} \prod_{i=1}^n |\mathcal{P}f_i(h \cdot x)|^{s_i} \Phi_{-\nu(z)}(h) dh,$$

which is

$$\int_K \prod_{i=1}^n |\mathcal{P}f_i(k \cdot x)|^{s_i} \Phi_{-\nu(z)}(k) dk.$$

Since $\Phi_{-\nu(z)}(k) = 1$, we obtain the equation above. \square

5. Computation of $\Psi_z(\Pi^\lambda)$

We calculate $\zeta(x; s)$ using Casselman's method [Cas80]. We assume that $\nu(z)$ is in the general position, that is, the numbers $\{z_i \pm \frac{1}{2} \mid i = 1, \dots, n\}$ are all distinct. The analytic continuation would give the formula for all cases of z .

5.1 Expansion in the basis $\{f_w^z\}$

Let B_0 be the Iwahori subgroup of G , namely B_0 is the pull back of the upper triangular Borel subgroup of G over the residue field of \mathcal{O} . Define (normalizing the measure so that B_0 has volume one)

$$\xi_z^x(g) = \int_{B_0} F_z^x(gb) db. \quad (5.1)$$

Then $\xi_z^x(g)$ is right B_0 -invariant. From (4.8), we see that $\xi_z^x(g)$ is a B_0 fixed vector in $I(\chi_{\nu(z)})$. Casselman defined a basis $\{f_w^z \mid w \in W(\mathrm{Sp}_{2n})\}$ of the space of B_0 fixed vectors in $I(\chi_{\nu(z)})$ (see [Cas80]). Here $W(\mathrm{Sp}_{2n})$ is the Weyl group of Sp_{2n} . Therefore, there exist functions $\{a_w(x; z) \mid w \in W(\mathrm{Sp}_{2n})\}$, so that

$$\xi_z^x(g) = \sum_{w \in W(\mathrm{Sp}_{2n})} a_w(x; z) f_w^z(g). \quad (5.2)$$

Here, by definition,

$$a_w(x; z) = T_w^z(\xi_z^x)(1), \quad (5.3)$$

where T_w^z is the intertwining operator of w from the space $I(\chi_{\nu(z)})$ to $I(w\chi_{\nu(z)})$, defined from the analytic continuation of the following integration:

$$T_w^z(\varphi)(g) = \int_{U \cap wUw^{-1} \backslash U} \varphi(w^{-1}ug) du.$$

As $\zeta(x; s) = \int_K F_z^x(k) dk = \int_K \xi_z^x(k) dk$, we get

$$\zeta(x; s) = \sum_{w \in W(\mathrm{Sp}_{2n})} a_w(x; z) \int_K f_w^z(k) dk. \quad (5.4)$$

The integral $\int_K f_w^z(k) dk$ is computed in [Cas80]. Recall that the root system of Sp_n is given by S in the introduction; we denote by S_{2n} the root system of Sp_{2n} . Let S_{2n}^+ and S_{2n}^- be the set of positive and negative roots, respectively. We define e^α for a given $z' = (z'_1, z'_2, \dots, z'_{2n})$ as in the introduction. Let

$$c_w(z') = \prod_{\alpha \in S_{2n}^+, w\alpha \in S_{2n}^-} \frac{1 - q^{-1}e^\alpha}{1 - e^\alpha}.$$

Let w_l be the longest Weyl element in $W(\mathrm{Sp}_{2n})$. Then we have

$$\int_K f_w^z(k) dk = Q^{-1} c_{w_l}(w_l w \nu(z)) / c_w(\nu(z)). \quad (5.5)$$

Here Q is some constant independent of z defined in [Cas80].

5.2 Vanishing of $a_w(x; z)$

We now turn our attention to the computation of $a_w(x; z)$. As in [Hir99], define an intertwining operator \tilde{T}_w^z from $I(\chi_{-\nu(z)})^*$ to $I(w\chi_{-\nu(z)})^*$, where $I(\chi)^*$ is the dual of $I(\chi)$ (see [Hir99, Proposition 1.6]). Note that $F_z^x \in I(\chi_{-\nu(z)})^*$. Then, as in [Hir99], \tilde{T}_w^z extends the intertwining operator T_w^z . Similar to [Hir99, Proposition 1.7], we have

$$a_w(x; z) = T_w^z(\xi_z^x)(1) = \int_{B_0} \tilde{T}_w^z(F_z^x)(b) db. \quad (5.6)$$

From now on we assume $x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$. We then show $a_w(x; z) = 0$ for most $w \in W(\mathrm{Sp}_{2n})$. Define $Y \subset G$:

$$Y = \{g \in G \mid F_z^{EJ}(g) \neq 0\}. \quad (5.7)$$

Clearly Y is an open subset of G .

LEMMA 5.1. *Let $b \in B$, then $g \in G$ is in Y if and only if $bg \in Y$.*

Proof. This follows from (4.8). □

LEMMA 5.2. *When $g \in G$ is such that $g \cdot EJ = x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$, $F_z^{EJ}(bg) = F_z^x(b) = F_z^x(1)$ when $b \in B_0$.*

Proof. It is clear from (4.8) that when $b \in B_0 \cap B$, then $F_z^x(b) = F_z^x(1)$. We now assume $b \in B_0 \cap \bar{B}$ where \bar{B} is the set of lower triangular matrices in G . We show in this case that $|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f_i(x)|$ for $i = 1, \dots, n$.

Write x as $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ where A_2 is the lower right $2i \times 2i$ block of x . Write b as $\begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$, where b_2 is the lower right $2i \times 2i$ block of b . Thus, b_2 is a lower triangular matrix, in the Iwahori subgroup of GL_{2i} . With this notation, we see that

$$|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f(b_2 A_2 b_2^t + S A_1 S^t)|.$$

Since the entries of S are in the prime ideal $\mathcal{P} \subset \mathcal{O}$, with our choice of x , we see that

$$b_2 A_2 b_2^t + S A_1 S^t = b_2 A_2 b_2^t \pmod{P^{1-\lambda_i}}.$$

Thus,

$$|\mathcal{P}f(b_2 A_2 b_2^t + S A_1 S^t)| = |\mathcal{P}f(b_2 A_2 b_2^t)| = |\mathcal{P}f(A_2)| = |\mathcal{P}f_i(x)|.$$

We have proved $|\mathcal{P}f_i(b \cdot x)| = |\mathcal{P}f_i(x)|$, thus the identity $F_z^x(b) = F_z^x(1)$. \square

PROPOSITION 5.3. *Let $x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$. If the distribution $\tilde{T}_w^z(F_z^{EJ}) \in I(w\chi_{-\nu(z)})^*$ is supported away from Y , then $a_w(x; z) = 0$.*

Proof. Let Φ_0^w be the vector in $I(w\chi_{-\nu(z)})$, such that Φ_0^w is supported on BB_0 , and $\Phi_0^w(b) = 1$ when $b \in B_0$. Then from (5.6),

$$a_w(x; z) = \tilde{T}_w^z(F_z^x)(\Phi_0^w).$$

If $x = g \cdot EJ$, then $\tilde{T}_w^z(F_z^x) = \rho_w(g)\tilde{T}_w^z(F_z^{EJ})$, where ρ_w is the representation on the space $I(w\chi_{-\nu(z)})^*$. Denote by ρ'_w the representation on the space $I(w\chi_{-\nu(z)})$, then

$$a_w(x; z) = \tilde{T}_w^z(F_z^{EJ})(\rho'_w(g^{-1})\Phi_0^w).$$

We show that $\rho'_w(g^{-1})\Phi_0^w$ is supported on Y ; then if the assumption of the proposition holds, $a_w(x; z) = 0$.

The support of $\rho'_w(g^{-1})\Phi_0^w$ is BB_0g . Let $b_1 \in B$, $b_2 \in B_0$. From Lemma 5.2, $F_z^{EJ}(b_2g) = F_z^x(b_2) = F_z^x(1) \neq 0$, and thus, from Lemma 5.1, $F_z^{EJ}(b_1b_2g) \neq 0$. We have shown that $\rho'_w(g^{-1})\Phi_0^w$ is supported on Y , thus the proposition follows. \square

PROPOSITION 5.4. *The distribution $\tilde{T}_w^z(F_z^{EJ}) \in I(w\chi_{-\nu(z)})^*$ is supported away from Y unless $w\chi_{\nu(z)} = \chi_{\nu(w'z)}$ for some w' in W the Weyl group of Sp_n .*

Proof. Assume that $\tilde{T}_w^z(F_z^{EJ})$ is not supported away from Y . Let $I_Y(w\chi_{-\nu(z)})$ be the subspace of $I(w\chi_{-\nu(z)})$ consisting of sections supported on Y . Then $\tilde{T}_w^z(F_z^{EJ})$ defines a nontrivial H -invariant linear form on $I_Y(w\chi_{-\nu(z)})$.

Note that $Y = P_2g_0H$ for some g_0 in G , where P_2 is the parabolic group containing B and with 2×2 blocks on the diagonal. Let $H_{g_0} = g_0^{-1}P_2g_0 \cap H$ and $\sigma = \mathrm{Ind}_B^{P_2}w\chi_{-\nu(z)}$. Consider the H -module $\mathrm{Ind}_{H_{g_0}}^H(\sigma^{g_0}\delta_{P_0}^{g_0})$ consisting of modulo H_{g_0} compactly supported functions ϕ on H with values in the space of σ , satisfying

$$\phi(h_0h) = \delta_{P_0}(g_0h_0g_0^{-1})\sigma(g_0h_0g_0^{-1})\phi(h), \quad h_0 \in H_{g_0}.$$

As H -modules, we have $I_Y(w\chi_{-\nu(z)}) \cong \mathrm{Ind}_{H_{g_0}}^H(\sigma^{g_0}\delta_{P_0}^{g_0})$.

From Frobenius reciprocity, the existence of a nontrivial H -invariant linear form on $I_Y(w\chi_{-\nu(z)})$ implies that, as a representation of H_{g_0} , $\sigma^{g_0}\delta_{P_0}^{g_0}$ contains a trivial representation as a subrepresentation. Equivalently, as the representation of $g_0H_{g_0}g_0^{-1}$, $\sigma\delta_{P_0}$ contains a trivial representation. Note that $g_0H_{g_0}g_0^{-1}$ is equal to n copies of SL_2 sitting in the diagonal 2×2 blocks of P_2 , over which δ_{P_0} is trivial; we see that σ contains the trivial representation of $\mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$. With our assumption that z is in the general position, it is only possible when w is as described in the proposition. \square

For each $w' \in W$, there is a unique $w \in W(\mathrm{Sp}_{2n})$ such that the condition in Proposition 5.4 is satisfied. We write $w = \sigma(w')$ if this is the case. We can check that σ is a group homomorphism.

COROLLARY 5.5. *We have $a_w(x; z) = 0$ unless $w = \sigma(w')$ for some w' in the Weyl group of Sp_n .*

Given $w = \sigma(w')$ with w' in the Weyl group of Sp_n , then $F_{w'z}^x \in I(w\chi_{-\nu(z)})^*$. As $\tilde{T}_w^z(F_z^{EJ})$ and $F_{w'z}^{EJ}$ satisfy the same left equivariance condition over P_2 and the right H -invariance condition, over Y we obtain

$$\tilde{T}_w^z(F_z^{EJ})(g) = \delta(w', z)F_{w'z}^{EJ}(g)$$

for some number $\delta(w', z)$ independent of $g \in Y$. From (5.6), we obtain

$$a_w(x; z) = \delta(w', z) \int_{B_0} F_{w'z}^x(b) db. \quad (5.8)$$

From Lemma 5.2, we see $a_w(x; z) = \delta(w', z)F_{w'z}^x(1)$. From (4.7),

$$F_{w'z}^x(1) = e^{w'\lambda}q^{b(\lambda)} \quad (5.9)$$

when $x = \Pi^\lambda$. Here $b(\lambda)$ is as defined in Theorem 3.2, and e^λ is as defined in the introduction with $e^{\epsilon_i} = q^{-z_i}$. Note that considered as a function of z , $F_{w'z}^x(1)$ is an additive character of z .

Summarizing the results so far, we obtain the following.

LEMMA 5.6. *When $x = \Pi^\lambda$ for some $\lambda \in \Lambda_n^+$, we have*

$$\zeta(\Pi^\lambda; z) = \sum_{w \in W} Q^{-1}q^{b(\lambda)}c(w, z)e^{w\lambda} \quad (5.10)$$

where $c(w, z) = c_{w_l}(w_l\sigma(w)\nu(z))\delta(w, z)/c_{\sigma(w)}(\nu(z))$.

5.3 Function equations

When $w = e$ is the identity we see that $c(e, z)$ in the lemma is equal to $c_{w_l}(w_l\nu(z))$ which is equal to

$$(1 + q^{-1})^n \prod_{\alpha \in S^{+L}} \left(\frac{1 - q^{-2}e^{-\alpha}}{1 - e^{-\alpha}} \frac{1 - q^{-1}e^{-\alpha}}{1 - qe^{-\alpha}} \right) \prod_{\alpha \in S^{+S}} \left(\frac{1 - q^{-1}e^{-2\alpha}}{1 - e^{-2\alpha}} \frac{1 - q^{-\frac{3}{2}}e^{-\alpha}}{1 - q^{\frac{1}{2}}e^{-\alpha}} \right), \quad (5.11)$$

where S^{+L} and S^{+S} are the set of long and short positive roots. Here e^α is related to q^z as in the introduction.

We use the functional equations of $\zeta(x; s)$ to determine $c(w, z)$ for other $w \in W$.

PROPOSITION 5.7. *The function $\Psi_z(x) = \zeta(x; s)/\zeta(\Pi^0; s)$ satisfies the function equation $\Psi_{wz}(x) = \Psi_z(x)$ for all $w \in W$.*

Proof. Let $w_0 \in W$. Then from (5.10)

$$\Psi_{w_0z}(\Pi^\lambda) = \frac{\sum_{w \in W} q^{b(\lambda)}c(w, w_0z)e^{ww_0\lambda}}{\sum_{w \in W} c(w, w_0z)}.$$

Let $w_1 \in W$. We compare the coefficients of $e^{w_1w_0\lambda}$ for $\Psi_z(\Pi^\lambda)$ and $\Psi_{w_0z}(\Pi^\lambda)$. They are

$$q^{b(\lambda)}c(w_1w_0, z) \Big/ \sum_{w \in W} c(w, z) \quad (5.12)$$

and

$$q^{b(\lambda)}c(w_1, w_0z) \Big/ \sum_{w \in W} c(w, w_0z). \quad (5.13)$$

From the definition of $c(w, z)$, we see that the quotient of the two is

$$\frac{\delta(w_1 w_0, z) c_{\sigma(w_1)}(w_0 z)}{\delta(w_1, w_0 z) c_{\sigma(w_1 w_0)}(z)} \cdot \frac{\sum_{w \in W} c(w, w_0 z)}{\sum_{w \in W} c(w, z)}. \quad (5.14)$$

It is well known that

$$c_{\sigma(w w')} (z) = c_{\sigma(w)}(w' z) c_{\sigma(w')} (z).$$

From our definition of $\delta(w, z)$, it is also clear that

$$\delta(w w', z) = \delta(w, w' z) \delta(w', z).$$

Using these two relations, we find that expression (5.14) equals

$$\frac{\sum_{w \in W} c(w, w_0 z) \delta(w_0, z) / c_{\sigma(w_0)}(z)}{\sum_{w \in W} c(w, z)}.$$

Since $c(w, w_0 z) \delta(w_0, z) / c_{\sigma(w_0)}(z) = c(w w_0, z)$ by the above relations, we find that expression (5.14) equals

$$\frac{\sum_{w \in W} c(w w_0, z)}{\sum_{w \in W} c(w, z)} = 1.$$

We have shown that the coefficients of $e^{w_1 w_0 \lambda}$ for $\Psi_z(\Pi^\lambda)$ and $\Psi_{w_0 z}(\Pi^\lambda)$ are equal. Thus, $\Psi_z(x) = \Psi_{w_0 z}(x)$. \square

We need a more explicit version of the function equation.

Let

$$\Gamma_1(z) = \prod_{\alpha \in S^{+L}} \frac{1 - q e^{-\alpha}}{1 - q^{-1} e^{-\alpha}},$$

and

$$\Gamma_2(z) = \prod_{\alpha \in S^{+S}} \frac{1 - q^{\frac{1}{2}} e^{-\alpha}}{1 + q^{-\frac{1}{2}} e^{-\alpha}}.$$

PROPOSITION 5.8. *Let $\tilde{\zeta}(x; z) = \Gamma_1(z) \Gamma_2(z) \zeta(x; s)$. Then*

$$\tilde{\zeta}(x; w z) = \tilde{\zeta}(x, z) \quad (5.15)$$

for all $w \in W$.

We give the proof in the next section. From Proposition 5.8, (5.10) and (5.11), and the linear independence of the characters $F_{wz}^x(1)$ (as characters of z), we obtain, for $\lambda \in \Lambda_n^+$, that $\tilde{\zeta}(\Pi^\lambda; z)$ equals

$$Q^{-1} (1 + q^{-1})^n q^{b(\lambda)} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in S^{+L}} \frac{1 - q^{-2} e^{-\alpha}}{1 - e^{-\alpha}} \prod_{\alpha \in S^{+S}} \frac{(1 - q^{-\frac{1}{2}} e^{-\alpha})(1 - q^{-\frac{3}{2}} e^{-\alpha})}{1 - e^{-2\alpha}} \right). \quad (5.16)$$

Comparing this with the definition of the Macdonald polynomial $Q_\lambda^t(z)$, we see that

$$\tilde{\zeta}(\Pi^\lambda; z) = Q^{-1} (1 + q^{-1})^n q^{b(\lambda)} V_\lambda(t^I) Q_\lambda^{t^I}(z) \quad (5.17)$$

when $\lambda \in \Lambda_n^+$ and t^I is the parameter defined in Theorem 3.2. Since

$$\tilde{\zeta}(x; z) / \tilde{\zeta}(\Pi^0; z) = \Psi_z(x),$$

we obtain

$$\Psi_z(\Pi^\lambda) = q^{b(\lambda)} V_\lambda(t^I) Q_\lambda^{t^I}(z) / (Q_0^{t^I}(z) V_0(t^I)) \quad (5.18)$$

when $\lambda \in \Lambda_n^+$. From [Mac00], we see that $Q_0^{t^I}(z) = 1$. Thus, the function Ψ_z defined in (4.2) is given by (3.3). We have proved Theorem 3.2. \square

6. Proof of Proposition 5.8

We prove (5.15) here. The Weyl group W is generated by elements σ_i ($1 \leq i \leq n-1$) and r_n , where σ_i fixes ϵ_j if $j \neq i, i+1$, and switches ϵ_i with ϵ_{i+1} ; r_n fixes ϵ_j if $j < n$ and maps ϵ_n to $-\epsilon_n$.

6.1 Function equation for σ_i

Assume $n > 1$. We fix an $i \leq n-1$. Recall the relation between s and z in (4.3). The following is clear from (4.3).

LEMMA 6.1. *The ordered set of complex numbers $(s_1, \dots, s_{i-2}, s_{i-1} + \frac{1}{2}s_i, \frac{1}{2}s_i + s_{i+1}, s_{i+2}, \dots, s_n)$ is invariant under the map $z \mapsto \sigma_i z$.*

Let $Y' = Y \cdot EJ$, then by definition of Y in (5.7), we see that $Y' = \{x \in X \mid F_s^x(1) \neq 0\}$. Let P_2 be the parabolic subgroup of G whose Levi subgroup is a product of GL_2 and whose unipotent subgroup consists of upper triangular matrices. The following lemma is clear from reduction theory.

LEMMA 6.2. *The set Y' is transitive under P_2 . Any $x \in Y'$ has a decomposition $x = p \cdot \Pi^{\lambda(x)}$ where $\lambda(x) \in \mathbf{Z}^n$ and the Levi part of $p \in P_2$ lies in products of $\mathrm{GL}_2(\mathcal{O})$.*

We embed $K_4 = \mathrm{GL}_4(\mathcal{O})$ in Sp_{2n} as follows. Denote by $\{e_j \mid j = 1, \dots, 4n\}$ the standard basis of the vector space where Sp_{2n} acts, then $k \in K_4$ acts trivially on the space generated by $\{e_1, \dots, e_{2i-2}, e_{2i+3}, \dots, e_{4n-2i-2}, e_{4n-2i+3}, \dots, e_{4n}\}$ and acts by multiplication of k on the space generated by $\{e_{2i+j} \mid j = -1, 0, 1, 2\}$ and by multiplication of k^* on the space generated by $\{e_{4n-2i+j} \mid j = -1, 0, 1, 2\}$. Then

$$\zeta(x; s) = \int_{k \in K} \int_{k' \in K_4} \prod_{j=1}^n |\mathcal{P}f_j(k'k \cdot x)|^{s_j} dk' dk. \quad (6.1)$$

LEMMA 6.3. *Given any $x \in X$, the expression*

$$\frac{1 - q^{z_i - z_{i+1} + 1}}{1 - q^{z_i - z_{i+1} - 1}} \int_{k \in K_4} \prod_{j=1}^n |\mathcal{P}f_j(k \cdot x)|^{s_j} dk \quad (6.2)$$

is invariant under the action $z \mapsto \sigma_i z$.

Proof. It is clear from the embedding that $|\mathcal{P}f_j(k \cdot x)| = |\mathcal{P}f_j(x)|$ when $j \neq i$. Given $y \in X$, denote by \tilde{y} the lower right $2i+2 \times 2i+2$ block. We can restrict the integration in (6.2) to the subset K'_4 where $k \cdot x \in Y'$. For $y \in Y'$, denote by \tilde{y}_2 the upper left 2×2 block of \tilde{y}^{-1} and \tilde{y}_4 the upper left 4×4 block of \tilde{y}^{-1} .

When $y \in Y'$, we see that $y = p \cdot \Pi^{\lambda(y)}$ as in Lemma 6.2. Clearly

$$|\mathcal{P}f(\tilde{y}_2)| = q^{-\lambda_{i+1}(y)} = |\mathcal{P}f_i(y)| / |\mathcal{P}f_{i+1}(y)|.$$

Similarly $|\mathcal{P}f(\tilde{y}_4)| = |\mathcal{P}f_{i-1}(y)|/|\mathcal{P}f_{i+1}(y)|$. (When $i = 1$, we set $|\mathcal{P}f_0(y)| = 1$.) Thus, the integral in (6.2) becomes the product of

$$|\mathcal{P}f_{i-1}(x)|^{s_{i-1} + \frac{1}{2}s_i} |\mathcal{P}f_{i+1}(x)|^{\frac{1}{2}s_i + s_{i+1}} \prod_{j=1, j \neq i-1, i, i+1}^n |\mathcal{P}f_j(x)|^{s_j} \quad (6.3)$$

and

$$\frac{1 - q^{z_i - z_{i+1} + 1}}{1 - q^{z_i - z_{i+1} - 1}} \int_{K'_4} |\mathcal{P}f((\widetilde{k \cdot x})_2)|^{s_i} |\mathcal{P}f((\widetilde{k \cdot x})_4)|^{-\frac{1}{2}s_i} dk.$$

Define $\widetilde{\mathcal{P}f}_i(y)$ ($i = 1, 2$) to be the Pfaffians of the upper left $2i \times 2i$ block of a skew symmetric matrix y in GL_4 . Note that $(\widetilde{k \cdot x})_4 = k' \cdot \tilde{x}_4$ where $k' = w_n k w_n$; the integration in the above expression has the form $\zeta'_4(\tilde{x}_4, s)$, where

$$\zeta'_4(y, s) = \int_{K'_4} |\widetilde{\mathcal{P}f}_1(k \cdot y)|^s |\widetilde{\mathcal{P}f}_2(k \cdot y)|^{-\frac{1}{2}s} dk. \quad (6.4)$$

Thus, our expression (6.2) is a product of (6.3) and

$$\frac{1 - q^{z_i - z_{i+1} + 1}}{1 - q^{z_i - z_{i+1} - 1}} \zeta'_4(\tilde{x}_4, s). \quad (6.5)$$

From Lemma 6.1, we see that (6.3) is invariant under $z \mapsto \sigma_i z$. The formula for $\zeta'_4(y, s)$ can be found in [HS88] (it is a special case of [HS88, Theorem 6]). From the formula, the relation between s and z (Equation (4.3) and [HS88, (2.3)]), it is clear that the expression (6.5) is invariant under $z \mapsto \sigma_i z$. \square

From the lemma, after multiplying by $(1 - q^{z_i - z_{i+1} + 1})/(1 - q^{z_i - z_{i+1} - 1})$ the inner integral in (6.1) is invariant under $z \mapsto \sigma_i z$. Thus, we obtain the following.

LEMMA 6.4. *The expression $((1 - q^{z_i - z_{i+1} + 1})/(1 - q^{z_i - z_{i+1} - 1}))\zeta(x; s)$ is invariant under $z \mapsto \sigma_i z$.*

6.2 Function equation for r_n

We first consider the case $n = 1$. Here $z \in \mathbf{C}$ and $r_n z = -z$. From Proposition 5.7, to obtain an explicit function equation, we only need to compute $\zeta(\Pi^0; s)$.

LEMMA 6.5. *When $n = 1$,*

$$\zeta(\Pi^0; s) = \frac{1 - q^{-1}}{1 + q^{-2}} \frac{1 + q^{z - \frac{1}{2}}}{1 - q^{z + \frac{1}{2}}}.$$

Proof. Let K^m be the set of $k \in K$ such that $|\mathcal{P}f_1(k \cdot \Pi^0)| = q^{-m}$. Then

$$\zeta(\Pi^0; s) = \sum_{m=0}^{\infty} \text{vol}(K^m) q^{-sm}. \quad (6.6)$$

The set $K \cdot \Pi^0$ is given by the elements in X whose entries are all in \mathcal{O} . This set can be described as

$$\left\{ \left(\begin{array}{cccc} a & b_1 & b_2 & \\ -a & & -b_2 & b_3 \\ -b_1 & b_2 & & d \\ -b_2 & -b_3 & -d & \end{array} \right) \middle| b_2^2 + b_1 b_3 = 1 + ad, a, d, b_1, b_2, b_3 \in \mathcal{O} \right\}.$$

Let X_m be the subset with $|d| = q^{-m}$. Choose a $g_0 \in K$ such that $g_0 \cdot EJ = \Pi^0$, and let $H' = g_0 H g_0^{-1}$. Then $H' \cdot \Pi^0 = \Pi^0$.

LEMMA 6.6. *Let $m > 1$ and let K_m be the set of $k \in K$ with $k \equiv 1 \pmod{P^m}$. Then $\rho : k \mapsto k \cdot \Pi^0$ induces a bijection between $K_m \backslash K/K \cap H'$ and $K \cdot \Pi^0 \pmod{P^m}$.*

Proof. The map is well defined. To show the surjectivity, we only need to show that if $b_2^2 + b_1 b_3 = 1 + ad \pmod{P^m}$, there exists $(b'_1, b'_2, b'_3, a', d') = (b_1, b_2, b_3, a, d) \pmod{P^m}$ such that $b_2'^2 + b'_1 b'_3 = 1 + a' d'$. This follows from Hensel's lemma. To show injectivity, we count the number of double cosets $K_m \backslash K/K \cap H'$ and the number of cosets $K \cdot \Pi^0 \pmod{P^m}$.

First count the number of double cosets. It is the number of $K \pmod{P^m}$ divided by the number of $K \cap H' \pmod{P^m}$. Let N be the number of $\text{SL}_2(\mathcal{O}) \pmod{P^m}$. Let $k \in K$, and $(f_1, f_2, f_3, f_4) \in \mathcal{O}^4$ be the last row of K . As one of the f_i has norm one, there are $q^{4m}(1 - q^{-4})$ possibilities $\pmod{P^m}$. Using reduction we see that the number of $K \pmod{P^m}$ is $Nq^{7m}(1 - q^{-4})$. Meanwhile the same argument gives $N = q^{3m}(1 - q^{-2})$.

Clearly the number of $K \cap H' \pmod{P^m}$ equals the number of $K \cap H \pmod{P^m}$, which is N^2 . Therefore, the number of double cosets $K_m \backslash K/K \cap H'$ is $q^{4m}(1 + q^{-2})$.

The number of cosets $K \cdot \Pi^0 \pmod{P^m}$ is the number of solutions of $b_2^2 + b_1 b_3 = 1 + ad$ in the ring $(\mathcal{O}/P)^5$. In the case when b_1 is a unit, we have $q^{4m}(1 - q^{-1})$ solutions. When b_1 is not a unit but d is a unit, we have $q^{4m}(q^{-1} - q^2)$ solutions. When both d and b_1 in P , given a, d, b_1, b_3 there are two solutions for b_2 , so we have a total of $2q^{4m-2}$ solutions. Adding up, the number of solutions is $q^{4m}(1 + q^{-2})$, which matches the number of double cosets. Thus, the map in the lemma is a bijection. \square

From the above lemma we see that the map ρ induces a bijection between $K_{m+1} \backslash K^m/K \cap H'$ and $X_m \pmod{P^m}$, when $m > 0$. Thus, the volume of K^m equals $\text{vol}(K_{m+1})$ times the number of cosets $X_m \pmod{P^m}$, and times the number of cosets $K \cap H' \pmod{P^{m+1}}$. We already counted $K \pmod{P^{m+1}}$. With our assumption that $\text{vol}(K) = 1$, we obtain

$$\text{vol}(K_{m+1}) = q^{-3(m+1)}(1 - q^{-2})^{-1}q^{-7(m+1)}(1 - q^{-4})^{-1}.$$

The number of cosets of $K \cap H' \pmod{P^{m+1}}$ is also given in the proof of the above lemma, it equals $q^{6(m+1)}(1 - q^{-2})^2$. The number of cosets of X_m can be counted as in the proof of the above lemma. Since $|d| = q^{-m} < 1$, we have two cases, $|b_1| = 1$ and $|b_1| < 1$; the two cases give us $q^{3(m+1)}(q - 1)(1 - q^{-1})$ and $2q^{3(m+1)-1}(q - 1)$, respectively. Therefore, when $m > 0$,

$$\text{vol}(K^m) = q^{-m} \frac{1 - q^{-2}}{1 + q^{-2}}.$$

Thus,

$$\text{vol}(K^0) = 1 - \sum_{m=1}^{\infty} \text{vol}(K^m) = \frac{1 - q^{-1}}{1 + q^{-2}}.$$

From (6.6), we obtain

$$\zeta(\Pi^0; s) = \sum_{m=1}^{\infty} \frac{1 - q^{-2}}{1 + q^{-2}} q^{(-1-s)m} + \frac{1 - q^{-1}}{1 + q^{-2}},$$

which equals

$$\frac{1 - q^{-1}}{1 + q^{-2}} \frac{1 + q^{-2-s}}{1 - q^{-1-s}}.$$

Our lemma follows from (4.3) which says $s = -z - \frac{3}{2}$. \square

From Proposition 5.7, we have the following corollary.

COROLLARY 6.7. *When $n = 1$, $s = -z - \frac{3}{2}$, then $((1 - q^{z+\frac{1}{2}})/(1 + q^{z-\frac{1}{2}}))\zeta(x; s)$ is invariant under $z \mapsto -z$.*

Now assume $n > 1$. Let K_2 be $\mathrm{Sp}_2(\mathcal{O})$, and embed it into K with the embedding of the F^4 into the subspace of F^{4n} generated by part of the standard basis e_j with $j = 2n - 1, 2n, 2n + 1, 2n + 2$. Then

$$\zeta(x; s) = \int_{k \in K} \int_{k' \in K_2} \prod_{j=1}^n |\mathcal{P}f_j(k'k \cdot x)|^{s_j} dk' dk. \quad (6.7)$$

LEMMA 6.8. *Given any $x \in X$, the expression*

$$\frac{1 - q^{z_n + \frac{1}{2}}}{1 + q^{z_n - \frac{1}{2}}} \int_{k \in K_2} \prod_{j=1}^n |\mathcal{P}f_j(k \cdot x)|^{s_j} dk \quad (6.8)$$

is invariant under the action $z \mapsto r_n z$.

Proof. Without loss of generality, we can assume that $x \in Y'$ and it has the form $x = p \cdot \Pi^\lambda(x)$ with $p \in P_2$ as in Lemma 6.2. Letting P'_2 be the subgroup of P_2 where the middle 4×4 block is the identity, a change of variable shows that the integral in (6.8) remains the same with x replaced by $p \cdot x$ with $p \in P'_2$. Since $P_2 \subset P'_2 K_2$, we only need to consider the case $x = \Pi^\lambda(x)$, which we now assume.

Let \tilde{x} be the middle 4×4 block of x . Then clearly $|\mathcal{P}f_j(k \cdot x)| = |\mathcal{P}f_j(x)|$ when $j \neq n$, and $|\mathcal{P}f_n(k \cdot x)| = |\mathcal{P}f_{n-1}(x)| |\mathcal{P}f(k \cdot \tilde{x})|$. Thus (6.8) is equal to

$$\begin{aligned} & \frac{1 - q^{z_n + \frac{1}{2}}}{1 + q^{z_n - \frac{1}{2}}} \left(\prod_{j=1}^{n-2} |\mathcal{P}f_j(x)|^{s_j} \right) |\mathcal{P}f_{n-1}(x)|^{s_{n-1} + s_n} \int_{K'_2} |\mathcal{P}f(k \cdot \tilde{x})|^{s_n} dk \\ &= \frac{1 - q^{z_n + \frac{1}{2}}}{1 + q^{z_n - \frac{1}{2}}} \left(\prod_{j=1}^{n-2} |\mathcal{P}f_j(x)|^{s_j} \right) |\mathcal{P}f_{n-1}(x)|^{s_{n-1} + s_n} \zeta(\tilde{x}; s_n). \end{aligned}$$

From (4.3), the ordered set $(s_1, \dots, s_{n-2}, s_{n-1} + s_n)$ is invariant under $z_n \mapsto -z_n$. Also from (4.3), $s_n = -z_n - \frac{3}{2}$, and thus our lemma follows from Corollary 6.7. \square

From the lemma, after multiplying by $(1 - q^{z_n + \frac{1}{2}})/(1 + q^{z_n - \frac{1}{2}})$ the inner integral in (6.7) is invariant under $z \mapsto r_n z$. Thus, we obtain the following.

LEMMA 6.9. *The expression $((1 - q^{z_n + \frac{1}{2}})/(1 + q^{z_n - \frac{1}{2}}))\zeta(x; s)$ is invariant under $z \mapsto r_n z$.*

6.3 Proof of Proposition 5.8

Proof. Since r_n reflects the long positive roots to long positive roots, it leaves $\Gamma_1(z)$ invariant. The reflection r_n also fixes all positive short roots except when $\alpha = \epsilon_n$, in which case

$$\frac{1 - q^{\frac{1}{2}} e^{-\alpha}}{1 + q^{-\frac{1}{2}} e^{-\alpha}} = \frac{1 - q^{z_n + \frac{1}{2}}}{1 + q^{z_n - \frac{1}{2}}}.$$

Thus, from Lemma 6.9, $\Gamma_2(z)\zeta(x; s)$ is invariant under r_n . Thus, $\tilde{\zeta}(x; z)$ is invariant under r_n .

Since σ_i maps the short positive roots to short positive roots, it leaves $\Gamma_2(z)$ invariant. It also acts as a permutation of $S^{+L} - \alpha_i$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Thus,

$$\Gamma_1(z) \left(\frac{1 - qe^{-\alpha_i}}{1 - q^{-1}e^{-\alpha_i}} \right)^{-1}$$

is invariant under σ_i . Since

$$\frac{1 - qe^{-\alpha_i}}{1 - q^{-1}e^{-\alpha_i}} = \frac{1 - q^{z_i - z_{i+1} + 1}}{1 - q^{z_i - z_{i+1} - 1}},$$

from Lemma 6.4, we see that $\Gamma_1(z)\zeta(x; s)$ is invariant under σ_i . Thus, $\tilde{\zeta}(x; z)$ is invariant under σ_i .

As the Weyl group W is generated by σ_i and r_n , we obtain that $\tilde{\zeta}(x; z)$ is invariant under W . \square

7. Unramified Whittaker function on metaplectic group

Now we consider the situation where $G_2 = \widetilde{\text{Sp}}_n$. Denote an element in G_2 by (g, ζ) with $g \in \text{Sp}_n$ and $\zeta = \pm 1$. Recall that the functions in $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$ are *genuine*; namely, for $f(g, \zeta)$ in this space, it satisfies $f(g, \zeta) = \zeta f(g, 1)$.

Let H_2 be a subgroup of G_2 , consisting of elements $(u, \pm 1)$ where u is an upper triangular matrix with unit diagonal. The character χ_2 is defined on H_2 as follows:

$$\chi_2((u, \zeta)) = \zeta \psi^{-1}(u_{12} + u_{2,3} + \cdots + u_{n,n+1}), \tag{7.1}$$

where ψ is an additive character trivial on \mathcal{O} but nontrivial on $\pi^{-1}\mathcal{O}$.

The unramified ψ -Whittaker functions on $\widetilde{\text{Sp}}_n$ are spherical functions in $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$. They are computed in [BFH91]. The results there were stated using an alternator; however, from their proof it is not difficult to obtain another statement in terms of Macdonald polynomials.

Let $\lambda \in \Lambda_n^+$, and let $\tilde{\Pi}^\lambda$ be the diagonal matrix

$$\text{diag}[\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}].$$

It is well known that $G_2^{\text{rel}} = \bigcup_{\lambda \in \Lambda_n^+} H_2(\tilde{\Pi}^\lambda, 1)K_2$ as a disjoint union. Thus, if we let Π_2 be such that $\Pi_2(\lambda) = (\tilde{\Pi}^\lambda, 1)$, Lemma 1.1 holds.

For $z = (z_1, \dots, z_n)$ in \mathbf{C}^n , let $\chi = \chi_z$ be an unramified character on the group A_2 of diagonal matrices given by

$$\chi(\tilde{\Pi}^\lambda) = \prod_{i=1}^n q^{z_i \lambda_i}. \tag{7.2}$$

This character extends to a genuine character $\tilde{\chi}$ of \tilde{A}_2 , the double cover of A_2 :

$$\tilde{\chi}(\mathbf{a}, \zeta) = \chi(\mathbf{a})\zeta\gamma_\psi(\mathbf{a})^{-1}, \quad \mathbf{a} \in A_2$$

where $\gamma_\psi(\mathbf{a})$ is a fourth root of unity (see [BFH91]). The unramified Whittaker function $W_{\tilde{\chi}}(g)$ in the principal series representation $I(\tilde{\chi})$ will be normalized so that $W_{\tilde{\chi}}(1_{2n}, 1) = 1$. Then $W_{\tilde{\chi}}(g) = W_{w\tilde{\chi}}(g)$ for all $w \in W(\text{Sp}_n)$, where $w\tilde{\chi}(\mathbf{a}) = \tilde{\chi}(\tilde{w}^{-1}\mathbf{a}\tilde{w})$, with \tilde{w} being the inverse image of w in $\widetilde{\text{Sp}}_n$.

The proof in [BFH91] shows that $W_{\tilde{\chi}}(\tilde{\Pi}^\lambda, 1)$ equals

$$\gamma_\psi(\tilde{\Pi}^\lambda)^{-1} \delta^{\frac{1}{2}}(\tilde{\Pi}^\lambda) \sum_{w \in W(\text{Sp}_n)} w \left(\prod_{i=1}^n \alpha_i^{-\lambda_i} \frac{1 - q^{-1}\alpha_i^2}{(1 + (p, p)q^{-\frac{1}{2}}\alpha_i)(1 - \alpha_i^2)} \prod_{i>j} \frac{1}{(1 - \alpha_i\alpha_j)(1 - \alpha_j\alpha_i^{-1})} \right).$$

Here (p, p) is a Hilbert symbol taking value ± 1 and δ is the modulus function of the Borel subgroup of H_2 . We rewrite the above formula in terms of the Macdonald polynomial.

THEOREM 7.1. *For $z = (z_1, \dots, z_n)$ in \mathbf{C}^n , let $\chi = \chi_z$ as in (7.2), then*

$$\Psi_z^2(\tilde{\Pi}^\lambda, 1) = W_{\tilde{\chi}}(\tilde{\Pi}^\lambda, 1) = V_\lambda(t^{II})\gamma_\psi(\tilde{\Pi}^\lambda)^{-1}\delta^{\frac{1}{2}}(\tilde{\Pi}^\lambda)Q_\lambda^{t^{II}}(z) \quad (7.3)$$

is a spherical function in $C_{K_2}^\infty(H_2 \backslash G_2, \chi_2)$. Here when $\beta \in S$, we have $t_\beta^{II} = 0$; when β is a short root in S , we have $(t_{2\beta}^{II})^{\frac{1}{2}} = -(p, p)q^{-\frac{1}{2}}$.

Lemma 1.3 is well known in this case. Since $t_\beta^{II} = 0$ for all $\beta \in S$, from the definition of $V_\lambda(t^{II})$ (see [Mac00]) we have

$$V_\lambda(t^{II}) = 1 \quad \text{for all } \lambda \in \Lambda_n^+.$$

It is clear that the volume of $H_2 \backslash H_2(\tilde{\Pi}^\lambda, 1)K_2$ equals $\delta^{-1}(\tilde{\Pi}^\lambda)$, and thus the argument in [Mac71] again gives the Plancherel formula stated in Corollary 1.5, with $d_\mu^2(z) = (1/|W|)\Delta^{t^{II}}(z)$.

We remark that in this case Lemma 1.3 can be strengthened, namely the map $f \mapsto \tilde{f}$ defined by (1.4) is indeed an algebra isomorphism between the Hecke algebra $\mathcal{H}(G_2, K_2)$ and $\mathbf{C}[q^{\bar{z}}, q^{-\bar{z}}]^W$.

8. Explicit map

We construct explicitly a linear map from $\mathcal{S}_{K_1}(H_1 \backslash G_1)$ to $\mathcal{S}_{K_2}(H_2 \backslash G_2, \chi_2)$.

Let $\epsilon = \begin{pmatrix} 1_n & 1_n \\ -1_n & 1_n \end{pmatrix}$, and $\epsilon_0 = \begin{pmatrix} \epsilon & \\ & \epsilon^* \end{pmatrix}$; $E_1 = \epsilon_0^{-1}E\epsilon_0$.

Let U be a unipotent subgroup of Sp_{2n} :

$$U = \left\{ \begin{pmatrix} u & * & * \\ & 1_{2n} & * \\ & & u^* \end{pmatrix} \right\},$$

where u is upper triangular with unit diagonal. Let V be the unipotent of the Siegel parabolic subgroup of Sp_{2n} . Define a character θ on UV so that if $u = (u_{i,j}) \in UV$, then

$$\theta(u) = \psi(u_{1,2} + u_{2,3} + \dots + u_{n-1,n} + u_{n,3n+1} - u_{2n,2n+1}).$$

Let V_{E_1} be the subgroup of elements u in V such that $u^{-1}E_1u = E_1$; then

$$V_{E_1} = \left\{ v(X, Y) = \begin{pmatrix} 1_{2n} & v \\ & 1_{2n} \end{pmatrix}, v = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \mid X, Y \in \mathcal{S}_n \right\}.$$

Here \mathcal{S}_n is the set of $n \times n$ matrices symmetric along the anti-diagonal.

Recall that the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_n$ acts on the space $\mathcal{S}(F^n)$ of Schwartz functions on F^n . For $\Phi \in \mathcal{S}(F^n)$, we define for $g \in \mathrm{Sp}_n$ and $\zeta \in \{\pm 1\}$,

$$T_\Phi(F)((g, \zeta)) = \int_{u \in V \backslash UV} \int_{v \in V_{E_1} \backslash V} F(\epsilon_0 v u j(g)) \theta(uv) \omega_\psi((g, \zeta)) \Phi(\underline{u}) dv du, \quad (8.1)$$

where \underline{u} is the vector $(u_{n,n+1}, u_{n,n+2}, \dots, u_{n,2n})$ for $u = (u_{i,j})$ and $j(g)$ is the matrix

$$j(g) = \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix}.$$

Then T_Φ is a linear map from $\mathcal{S}_{K_1}(H_1 \backslash G_1)$ to the set of genuine functions on $\widetilde{\mathrm{Sp}}_n$.

Recall the definition of ch_λ in § 2.2. We use ch_0^1 to denote the function on $H_1 \backslash G_1$ corresponding to the function ch_0 on X (through the identification in § 3.1), and let ch_0^2 be the function ch_0 on G_2 . Let Φ_0 be the characteristic function of \mathcal{O}^n . We prove in this section the following.

PROPOSITION 8.1. *For $f_i \in \mathcal{H}(G_i, K_i)$ such that $\tilde{f}_1(z) \equiv \tilde{f}_2(z)$, we have $T_{\Phi_0}(f_1 * ch_0^1) = f_2 * ch_0^2$.*

This equation is, in essence, the fundamental lemma for general Hecke functions for the relative trace identity considered in [MR08].

8.1 Definition of the integral in (8.1) when $F = \Psi_z^1$

The above integral is clearly well defined if F is compactly supported. For Ψ_z the spherical function described in Theorem 3.2, through identification in § 3.1, $\Psi_z^1(g) = \Psi_z(g^{-1}EgJ)$ defines a spherical function on $H_1 \backslash G_1$. The integral (8.1) can still be defined for $F = \Psi_z^1$, although the definition is more subtle.

Denote by π_z the induced representation $I(\chi_{-\nu(z)})$. Then π_z is induced from an unramified representation τ_z of GL_{2n} . Thus, a model of π_z is given by a space of functions of two variables $\phi(g, h)$ with $g \in Sp_{2n}$, $h \in GL_{2n}$ satisfying:

- (1) $\phi\left(\begin{pmatrix} h_1 & \\ & h_1^* \end{pmatrix}g, h\right) = \phi(g, hh_1)$;
- (2) $\phi(g, h)$ as a function of g is compactly supported over $P \backslash Sp_{2n}$;
- (3) for fixed g , $\phi(g, h)$ is a vector in τ_z with model in $\text{Ind}_{H'}^{GL_{2n}} 1$; here $H' \cong GL_n \times GL_n$ sitting diagonally in GL_{2n} ; we consider H' as a subgroup of Sp_{2n} through the embedding of GL_{2n} as the Levi subgroup of Sp_{2n} .

The following is a result from the theory of intertwining operators.

LEMMA 8.2. *There is an unramified vector $\phi_z(g, h)$ in the space of π_z such that*

$$\Psi_z^1(\epsilon_0 g) = \Psi_z(g^{-1}E_1 g J) = \int_{v \in V_{E_1}} \phi_z(J_{2n} \epsilon_0 v g, 1_{2n}) dv, \tag{8.2}$$

and the above integral converges absolutely.

Proof. Use $L_g(\phi)$ to denote the above integral with ϕ_z in the expression (8.2) replaced by an arbitrary vector ϕ in the space of π_z . We let $\phi'_g(g') = \phi(g'g, 1_{2n})$ where $g' \in H_1 = Sp_n \times Sp_n$. Then ϕ'_g is a left invariant under the Levi subgroup H' of H_1 . Note that $\epsilon_0 V_{E_1} \epsilon_0^{-1}$ is exactly the unipotent for the parabolic subgroup in H_1 with Levi being H' . Thus, $L_{\epsilon_0^{-1}}$ is an intertwining operator on π_z considered as a representation of H_1 ; it satisfies $L_{\epsilon_0^{-1}}(\pi_z(h)\phi) = L_{\epsilon_0^{-1}}(\phi)$. Thus, $L_{\epsilon_0^{-1}}$ is a $Sp_n \times Sp_n$ invariant linear form on π_z . Since such a form is unique up to scalar multiple, with Proposition 4.2, we obtain the lemma. \square

We use $\phi_z(g)$ to denote $\phi_z(g, 1_{2n})$; then $\phi_z(g)$ is left H' -invariant. Fix $g \in Sp_n$ and let $\phi_{z,g}(h) = \phi_z(hj(g))$, $F_g(g') = F(g'g)$ and $\Phi_g(x) = \omega_\psi((g, 1))\Phi(x)$.

Consider the integral

$$\mathcal{P}_{\phi \otimes \Phi}^c(h) = \frac{1}{\text{vol}(U_c)} \int_{U_c} \phi(hu)\theta(u)\Phi(\underline{u}) du, \tag{8.3}$$

where U_c is a compact subset of U consisting of elements whose entries have norm less than c .

LEMMA 8.3. For fixed $g \in \mathrm{Sp}_n$, the function $\mathcal{P}_{\phi_{z,g} \otimes \Phi_g}^c(h)$ is supported on the open cell $PJ_{2n}P'$ when U_c is a large enough compact set. Here P' is the upper triangular parabolic subgroup with the Levi subgroup being isomorphic to $\mathrm{GL}_1^n \times \mathrm{Sp}_n$. The choice of U_c is independent of z .

Proof. The first fact follows from the Jacquet module computation in [GRS99a]; it is a restatement of the isomorphism between Jacquet modules (1.14) in [GRS99a] for the case $m = 2k = 2n$. The second fact follows from Baier's lemma as in [CS80]. As this proof is used several times here, we recall it. We can choose a compact set $U_{c,z}$ for each z . Let $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots$ be an exhaustive system of compact sets in U . Let $X_n = \{z \mid U_{c,z} \subset U_n\}$. Then $\bigcup X_n$ is the set D^n . By Baier's lemma, there is an n such that X_n contains an open subset of D^n . As the definition of $U_{c,z}$ is holomorphic in z , X_n must also be a closed subset of D^n . Thus $X_n = D^n$ and we can let $U_c = U_n$ for this n . \square

Define, for $u \in U$,

$$\mathcal{Q}_{\phi_{z,g} \otimes \Phi_g}^0(u) = \int_{v \in V} \mathcal{P}_{\phi_{z,g} \otimes \Phi_g}^c(J_{2n} \epsilon_0 v u) \theta(vu) \Phi_g(\underline{u}) \, dv, \quad (8.4)$$

where U_c is large enough so the above lemma is satisfied. From the above lemma, the integration is actually over a compact set $V^c \subset V$, thus well defined. On the other hand, from Lemma 8.2 $\mathcal{Q}_{\phi_{z,g} \otimes \Phi_g}^0(u)$ equals (for U_c large enough)

$$\begin{aligned} & \int_{v \in V_{E_1} \setminus V} \frac{1}{\mathrm{vol}(U_c)} \int_{U_c} \int_{v' \in V_{E_1}} \phi_{z,g}(J_{2n} \epsilon_0 v' v u u') \theta(v u u') \Phi_g(\underline{u u'}) \, dv' \, du' \, dv \\ &= \int_{v \in V_{E_1} \setminus V} \frac{1}{\mathrm{vol}(U_c)} \int_{U_c} \Psi_{z,g}^1(\epsilon_0 v u u') \theta(v u u') \Phi_g(\underline{u u'}) \, du' \, dv. \end{aligned}$$

We define

$$\mathcal{Q}_{F \otimes \Phi}^0(u) = \lim_{c \rightarrow \infty} \int_{v \in V_{E_1} \setminus V} \frac{1}{\mathrm{vol}(U_c)} \int_{U_c} F(\epsilon_0 v u u') \theta(v u u') \Phi(\underline{u u'}) \, du' \, dv. \quad (8.5)$$

Thus, $\mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi_g}^0(u) = \mathcal{Q}_{\phi_{z,g} \otimes \Phi_g}^0(u)$.

We next define iteratively $\mathcal{P}_{F \otimes \Phi_g}^{c,i}(u)$ and $\mathcal{Q}_{F \otimes \Phi_g}^i(u)$ on U for $i = 1, 2, \dots, n-1$. Note that $V \setminus UV \cong U^1 U^2 \dots U^n$ where U^k consists of elements $u = (u_{i,j})$ such that for $i \leq n$, $u_{i,j} = 0$ when $i \neq j$ and k or $j > 2n$. Let $V_i = V U^1 U^2 \dots U^i$; let U_c^i be the subset of U^i whose entries have norm less than c :

$$\mathcal{P}_{F \otimes \Phi}^{c,i}(u) = \frac{1}{\mathrm{Vol}(V_{i-1} \setminus U_c^i V_{i-1})} \int_{v \in V_{i-1} \setminus U_c^i V_{i-1}} \mathcal{Q}_{F \otimes \Phi}^{i-1}(vu) \theta(v) \, dv; \quad (8.6)$$

$$\mathcal{Q}_{F \otimes \Phi}^i(u) = \lim_{c \rightarrow \infty} \int_{v \in V_{i-1} \setminus V_i} \mathcal{P}_{F \otimes \Phi}^{c,i}(vu) \theta(vu) \Phi(\underline{u}) \, dv. \quad (8.7)$$

We need the following lemma which we prove in the next section.

LEMMA 8.4. For fixed $u_0 \in U$, when $c_i = c_i(u_0)$ is large enough, $\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_i,i}(vu_0)$ is compactly supported on $V_{i-1} \setminus V_i$. Moreover, $c_i(u_0)$ is independent of z .

With the lemma, we see that $\mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi_g}^i(u)$ is well defined. Finally let

$$\mathcal{Q}_{F \otimes \Phi}^n = \int_{V_{n-1} \setminus UV} \mathcal{Q}_{F \otimes \Phi}^{n-1}(u) \theta(u) \Phi(\underline{u}) \, du. \quad (8.8)$$

Owing to the fact that function Φ is compactly supported, the above integral is supported over a compact set. Replacing the integral (8.1), we define

$$T_\Phi(\Psi_z^1)((g, 1)) = \mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi_g}^n. \quad (8.9)$$

We remark that $T_\Phi(\Psi_z^1)$ is defined through an iterated sequence of integrals over fixed compact sets. Since in (8.8) the integration over $V_{n-1} \setminus UV$ is over a compact subset $V_{n-1} \setminus U_{c_n}^n V_{n-1}$ determined by the support of Φ_g , we can then find a c_{n-1} large enough so that $c_{n-1} > c_{n-1}(u)$ for all $u \in U_{c_n}^n V$. Thus, we can drop the limit in definition (8.7) of $\mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi_g}^{n-1}(u)$ if we let $c = c_{n-1}$. Tracking backwards, we can find a sequence of numbers c_1, \dots, c_n so that

$$\Psi_{z,g}^1 \otimes \Phi_g \mapsto \mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi}^0 \mapsto \mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi}^{c_1,1} \mapsto \mathcal{Q}_{\Psi_{z,g}^1 \otimes \Phi}^1 \mapsto \dots \mapsto \mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi}^n = T_\Phi(\Psi_z^1)((g, 1)). \quad (8.10)$$

The numbers c_i are determined completely by g and Φ , as they are independent of $z \in D_n$ as well.

We remark that if we replace Ψ_z^1 by $F \in \mathcal{S}_{K_1}(H_1 \setminus G_1)$ in the sequence of maps in (8.10), it is clear that $T_\Phi(F)((g, 1))$ is the same linear map defined through the integration in (8.1).

8.2 Proof of Lemma 8.4

We only consider the case $i = 1$, the other cases can be treated similarly. Assume that $c_1 = q^l$ and denote $U_{c_1}^1$ as U_l^1 . Without loss of generality we may assume $u_0 = 1$.

An element $u_1 \in U^1$ is denoted by $u_1(x_2, x_3, \dots, x_{2n})$ if

$$u_1 \cdot e_1 = e_1 + \sum_{i=2}^{2n} x_i e_i.$$

Then $\theta(u_1) = \psi(x_2)$. From (8.6), (8.3) and (8.4) (and note that U^1 is abelian), we obtain the following expression for $\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1)$ (δ is some constant):

$$\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1) = \delta \int_{v \in V \setminus U_{c_1}^1 V} \int_{v' \in V} \int_{U_c} \phi_{z,g}(J_{2n} \epsilon_0 v' u_1 v u) \theta(v' v u) \Phi_g(\underline{u}) du dv' dv. \quad (8.11)$$

Assume now that $u_1 \in U^1 - U_l^1$. We show that when $c_1 = q^l$ is large enough, $\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1) = 0$. The idea is to pick two elements h_1 and h_2 in Sp_{2n} so that $h_1 u_1 h_2 = u_1 u_2$ for some $u_2 \in U_l^1$. We ask that $h_1 = \begin{pmatrix} g_1 & \\ & g_1^* \end{pmatrix}$, where $g_1 \in \mathrm{GL}_{2n}$ has the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$; then $(J_{2n} \epsilon_0) h_1 (J_{2n} \epsilon_0)^{-1} \in H'$. Furthermore, we ask that $\theta(h_1^{-1} v h_1) = \theta(v)$ for all $v \in V$. We require h_2 to be in a small neighborhood of 1_{2n} . Then from the expression of $\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1)$ in (8.11), the left H' invariance of ϕ_z and smoothness of ϕ_z , we obtain

$$\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1) = \mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(h_1 u_1 h_2) = \mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1 u_2).$$

On the other hand, using the equivariance condition we obtain

$$\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1) = \mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1 u_2) \theta(u_2). \quad (8.12)$$

Thus, if $\theta(u_2) \neq 1$, we obtain $\mathcal{P}_{\Psi_{z,g}^1 \otimes \Phi_g}^{c_1,1}(u_1) = 0$.

Let i_0 be the largest index such that $|x_{i_0}| q^{-li_0}$ has the largest value. Since $u_1 \notin U_l^1$, this value is larger than one.

Case 1: $i_0 > n$. Take t so that $|tx_{i_0}| = q$ and $\psi(tx_{i_0}) \neq 1$. Let $h_2 = \begin{pmatrix} 1_n & A \\ & 1_n \end{pmatrix}$ where all entries of A are zero except $A_{i_0-n,2} = t$. Let $h_1 = h_2^{-1}$. Then we can check that h_1, h_2 satisfy the above conditions with $\theta(u_2)$ in (8.12) equal to $\psi(-tx_{i_0})$.

Case 2: $2 < i_0 \leq n$. Take t so that $|tx_{i_0}| = q$ and $\psi(tx_{i_0}) \neq 1$. Let $h_2 = \begin{pmatrix} A & \\ & A \end{pmatrix}$ where all entries of A are zero except $A_{j,j} = 1$ for $1 \leq j \leq n$ and $A_{i_0,2} = t$. Let $h_1 = h_2^{-1}$. Then we can check that h_1, h_2 satisfy the above conditions with $\theta(u_2)$ in (8.12) equal to $\psi(-tx_{i_0})$.

Case 3: $i_0 = 2$. Take t so that $|tx_2| = q$ and $\psi(tx_2) \neq 1$. Let $h_2 = \begin{pmatrix} A & \\ & A \end{pmatrix}$ where A is a diagonal matrix, with the diagonal element being one except $A_{2,2} = 1 + t$. Let $h_1 = h_2^{-1}$. Then we can check that h_1, h_2 satisfy the above conditions with $\theta(u_2)$ in (8.12) equal to $\psi(-tx_2)$.

In all situations we have shown that the function vanishes for $u_1 \in U^1 - U_l^1$. The independence on z follows from the same argument as in Lemma 8.3.

8.3 The image of $T_{\Phi_0}(\Psi_z^1)$

PROPOSITION 8.5. *There is a function $c(z)$ on \mathbf{C}^n such that $T_{\Phi_0}(\Psi_z^1) = c(z)\Psi_z^2$ for $z \in \mathbf{C}^n$.*

Proof. Where we defined $T_{\Phi}(\Psi_z^1)$, we can replace ϕ_z by any vector ϕ in π_z . Let F_ϕ be the function on $H_1 \backslash G_1$ such that $F_\phi(\epsilon_0 g)$ is given by the integral in (8.2) with ϕ_z replaced by ϕ . Through the same process of iterated integration (8.10), we can define a linear map on the space of $\pi_z \otimes \omega_\psi$ to functions on $\widetilde{\text{Sp}}_n$ through

$$T(\phi \otimes \Phi)(g) = \mathcal{Q}_{F_\phi, g \otimes \Phi_g}^n.$$

We now derive some properties of the map.

It is shown in [MR08] that T is a map from $\pi_z \otimes \omega_\psi$ to $\text{Ind}_{H_2}^{G_2} \chi_2$ where χ_2 is defined in (7.1). Observe that G_2 acts on $\pi_z \otimes \omega_\psi$ through the embedding j of Sp_n in Sp_{2n} . Since $\pi_z(j(h))\phi_z(j(g)) = \phi_z(j(gh))$ for $g, h \in \text{Sp}_n$, we obtain the following.

LEMMA 8.6. *The map T is a G_2 -module homomorphism from $\pi_z \otimes \omega_\psi$ to $\text{Ind}_{H_2}^{G_2} \chi_2$.*

Note that $j(g)$ normalizes U . A change of variable shows the following.

LEMMA 8.7. *Let $u \in U$, then $T(\pi_z(u)\phi \otimes \omega_\psi(u, 1)\Phi) = T(\phi \otimes \Phi)\theta'(u)$, where for $u = (u_{i,j}) \in U$,*

$$\theta'(u) = \psi(-u_{1,2} - u_{2,3} - \cdots - u_{n-1,n}).$$

The Jacquet module $J_{U, \theta'}(\pi_z \otimes \omega_\psi)$ is considered in [GRS99b]. It is a $\widetilde{\text{Sp}}_n$ module defined in [GRS99b, (1.6)]. The above two lemmas show that T factors through to a map \bar{T} from $J_{U, \theta'}(\pi_z \otimes \omega_\psi)$ to $\text{Ind}_{H_2}^{G_2} \chi_2$. It follows from [GRS99b, Theorem B] that

$$J_{U, \theta'}(\pi_z \otimes \omega_\psi) \cong \tilde{\pi}_z,$$

where $\tilde{\pi}_z = I(\tilde{\chi})$ when $\tilde{\chi}$ is associated to $x = x_2$ defined by (7.2).

Since there is a unique Whittaker model for $\tilde{\pi}_z$, the map \bar{T} is the unique (up to a scalar multiple) map of $\tilde{\pi}_z$ into $\text{Ind}_{N_2}^{G_2} \chi_2$. The function $T_{\Phi_0}(\Psi_z^1)$ is the image of the unramified vector in $\pi_z \otimes \omega_\psi$. The image is clearly an unramified vector in $\text{Ind}_{N_2}^{G_2} \chi_2$, thus $T_{\Phi_0}(\Psi_z^1)$ corresponds to the image of the unramified vector of $\tilde{\pi}_z$ under the Whittaker map to $\text{Ind}_{N_2}^{G_2} \chi_2$. This image is just the unramified Whittaker function of $\tilde{\pi}_z$. Thus, we have proved that $T_{\Phi_0}(\Psi_z^1)(g)$ is a multiple of $\Psi_z^2(g)$. \square

8.4 Apply Lemma 1.6

LEMMA 8.8. For $F \in \mathcal{S}_{K_1}(H_1 \backslash G_1)$, and Φ_0 the characteristic function of \mathcal{O}^n ,

$$T_{\Phi_0}(F) = \int_{D_n} \hat{F}(z)c(z)\Psi_z^2 d_\mu^1 z.$$

Proof. From Corollary 1.5,

$$T_{\Phi_0}(F) = T_{\Phi_0} \left(\int_{D_n} \hat{F}(z)\Psi_z^1 d_\mu^1 z \right).$$

As T_{Φ_0} is an iterated integral (8.10) over a fixed compact set, we can interchange the integral and operator T_{Φ_0} and use Proposition 8.5 to obtain

$$\begin{aligned} T_{\Phi_0}(F) &= \int_{D_n} \hat{F}(z)T_{\Phi_0}(\Psi_z^1) d_\mu^1 z \\ &= \int_{D_n} \hat{F}(z)c(z)\Psi_z^2 d_\mu^1 z. \end{aligned} \quad \square$$

In [MR08], through direct calculation we checked the following.

LEMMA 8.9. We have $T_{\Phi_0}(ch_{\mathbf{0}}^1) = ch_{\mathbf{0}}^2$.

Since clearly $S_1(ch_{\mathbf{0}}^1) = S_2(ch_{\mathbf{0}}^2) = 1$, we checked that the two conditions in Lemma 1.6 are satisfied for the map T_{Φ_0} . From Corollary 1.5, for $f_i \in \mathcal{H}(G_i, K_i)$, $S_1(f_1 * ch_{\mathbf{0}}^1) = S_2(f_2 * ch_{\mathbf{0}}^2)$ whenever $\tilde{f}_1(z) \equiv \tilde{f}_2(z)$. The conclusion of Lemma 1.6 then gives Proposition 8.1.

ACKNOWLEDGEMENTS

We thank Omer Offen and Hervé Jacquet for generously sharing their ideas. We are very grateful to the referee who read the paper carefully and made many helpful suggestions and corrections.

REFERENCES

BFH91 D. Bump, S. Friedberg and J. Hoffstein, *p-adic Whittaker functions on the metaplectic group*, Duke Math. J. **63** (1991), 379–397.

Cas80 W. Casselman, *The unramified principal series of p-adic groups I: the spherical function*, Compositio Math. **40** (1980), 387–406.

CS80 W. Casselman and J. Shalika, *The unramified principal series of p-adic groups II: the Whittaker function*, Compositio Math. **41** (1980), 207–231.

GRS99a D. Ginzburg, S. Rallis and D. Soudry, *On a correspondence between cuspidal representations of GL_{2n} and Sp_{2n}* , J. Amer. Math. Soc. **12** (1999), 849–907.

GRS99b D. Ginzburg, S. Rallis and D. Soudry, *Lifting csup froms on GL_{2n} to \widetilde{Sp}_{2n} : the unramified correspondence*, Duke Math. J. **100** (1999), 243–266.

Hir99 Y. Hironaka, *Spherical functions and local densities on hermitian forms*, J. Math. Soc. Japan **51** (1999), 553–581.

HS88 Y. Hironaka and F. Sato, *Spherical functions and local densities of alternating forms*, Amer. J. Math. **110** (1988), 473–512.

Mac00 I. G. Macdonald, *Orthogonal polynomials associated with root systems*, Sém. Lothar. Combin. **45** (2000), Article B45a.

Mac71 I. G. Macdonald, *Spherical functions on a group of p-adic type*, Publications of Ramanujan Institute No. 2 (Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, 1971).

PLANCHEREL FORMULA

- MR08 Z. Mao and S. Rallis, *A relative trace identity for GL_{2n} and \widetilde{Sp}_n* , Preprint.
- Off04 O. Offen, *Relative spherical functions on p -adic symmetric spaces (three cases)*, Pacific J. Math. **215** (2004), 97–149.
- Off08 O. Offen, *Correction to “Relative spherical functions on p -adic symmetric spaces (three cases)”*, Pacific J. Math **236** (2008), 195–200.

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