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Airy Sums, Kloosterman Sums, and Salié Sums

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We prove an identity between Airy sums and Salié sums, extending a Duke–Iwaniec result. © 1997 Academic Press

Let $n>1$ be an odd integer. The nth-degree Airy polynomial $T_n(t)$ is defined by the equality [3, p. 320]

$$T_n(u+u^{-1})=u^n+u^{-n}, \qquad u\in \mathbf{C}^\times. \tag{1}$$

For example, $T_3(t)=t^3-3t$. As the Airy polynomials have integral coefficients, they can also be considered as polynomials over any finite field. Let F be a finite field with $q=p^r$ elements. Let $\psi(x)=e^{2\pi i Tr(x)/p}$ be an additive character on F. For n as above, we define the Airy sum to be:

$$\sum_{t\in F}\psi(T_n(t)).$$

In the case $n=3$, this sum is considered in [1]. It is shown that if $q\equiv 1 \bmod 3$, then

$$\sum_{t\in F}\psi(T_n(t))=\sum_{x\in F^\times}\chi(x)\,\psi(x+x^{-1}) \tag{2}$$

where χ is a multiplicative character of order 3. In this paper, we generalize the equation (2) to all cases $n>1$ odd.

Let χ be a multiplicative character of order n. We prove

THEOREM 1. *If $q\equiv 1 \bmod n$, then*

$$2\sum_{t\in F}\psi(T_n(t))=\sum_{i=1}^{n-1}\sum_{x\in F^\times}\chi^i(x)\,\psi(x+x^{-1}). \tag{3}$$

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Before proving the theorem, we make the following remarks:

1. The sums appearing on the right-hand side of (3) are Salié sums.
2. When $n=3$, $\chi^2(x)=\chi(x^{-1})$. A change of variable $x\to x^{-1}$ shows

$$\sum_{x\in F^\times} \chi^2(x)\,\psi(x+x^{-1}) = \sum_{x\in F^\times} \chi(x)\,\psi(x+x^{-1}).$$

Thus when $n=3$, Theorem 1 is just identity (2).

3. In [2], a trace formula for the lifting of automorphic representations from SL_2 to $\widetilde{SL}_2^3$, the threefold covering of SL_2, is proved. It is shown that the identity (2) is essentially the fundamental lemma for that trace formula. We expect that the identity (3) for $n>3$ could play a similar role in the theory of trace formulas.

4. Over the real field $\mathbf{R}$, one has the classical formula (see [3, p. 322])

$$\int_0^\infty \cos(T_n(t))\,dt = \frac{\pi}{2n\sin(\pi/2n)}\,[J_{1/n}(2)+J_{-1/n}(2)],$$

where $J_{\pm 1/n}$ are Bessel functions. The identity (3) shall be considered as a finite field analogue of this formula.

The theorem follows from an identity between Airy sums and Kloosterman sums. Let

$$\mathrm{Kl}(1) = \sum_{x\in F^\times} \psi(x+x^{-1})$$

be the classical Kloosterman sum. We define

$$\mathrm{Kl}_n(1) = \sum_{x\in F^\times} \psi(x^n+x^{-n}).$$

Theorem 2. *If* $q\equiv 1 \bmod n$, *then*

$$2\sum_{t\in F} \psi(T_n(t)) + \mathrm{Kl}(1) = \mathrm{Kl}_n(1). \tag{4}$$

We derive the identity (3) from (4). When $q\equiv 1 \bmod n$, all the nth roots of unity lie in F. The equation $y^n=x$ has either 0 or n solutions in F for a given $x\in F^\times$. The right-hand side of (3) equals

$$\sum_{x\in F^\times}\sum_{i=0}^{n-1}\chi^i(x)\,\psi(x+x^{-1})-\mathrm{Kl}(1)$$

$$=n\left[\sum_{x\in F^\times,\,x=y^n\text{ for some }y}\psi(x+x^{-1})\right]-\mathrm{Kl}(1)$$

$$=\sum_{y\in F^\times}\psi(y^n+y^{-n})-\mathrm{Kl}(1)=\mathrm{Kl}_n(1)-\mathrm{Kl}(1).$$

Thus the identity (4) implies (3).

To prove (4), we introduce the quadratic extension K of F. Let $H=K\backslash F$. Let S be a subset of F

$$\{x\in F\mid x=u+u^{-1},\,u\in H\}$$

and S' be the subset

$$\{x\in F\mid x=u^n+u^{-n},\,u\in H\}.$$

Lemma 1. *With $q\equiv 1 \bmod n$, we have $S=S'$, and the map $x\to T_n(x)$ defines a bijection from S to itself.*

Proof. From (1), T_n maps the set S into S'. As n is odd, the condition $(n,q-1)=n$ implies $(n,q+1)=1$. Thus $u\in H$ implies $u^n\in H$. Thus $S'\subset S$. To complete the proof, we only need to show that T_n is an injection from S to S'. For $x_1,x_2\in S$, with $x_1=u_1+u_1^{-1}$, $x_2=u_2+u_2^{-1}$, assume that $T_n(x_1)=T_n(x_2)$. Then

$$u_1^n+u_1^{-n}=u_2^n+u_2^{-n}.$$

We get either $u_1^n=u_2^n$ or $u_1^n=u_2^{-n}$. With the right choice of u_2, we can assume that the former is true. There is an nth root of unity ξ, necessarily in $F^\times$, satisfying $u_1=u_2\xi$. Thus

$$x_1-x_2\xi=u_1^{-1}-u_2^{-1}\xi=u_2^{-1}(\xi^{-1}-\xi)\in F$$

As $u_2\notin F$, this implies $\xi^{-1}-\xi=0$. Since -1 is not an nth root of unity, $\xi=1$ and $x_1=x_2$. ▌

Let $\hat{S}$ be the complement of S in F. Then

$$\hat{S}=\{x\in F\mid x=u+u^{-1},\,u\in F^\times\}.$$

Observe that for any function f on F,

$$\sum_{x \in F^\times} f(x+x^{-1}) = 2\sum_{t \in \hat{S}} f(t) - \sum_{t=\pm 2} f(t)$$

as $t = x + x^{-1}$ has only one solution when $t = \pm 2$. Therefore, we have

$$\mathrm{Kl}(1) = \sum_{x \in F^\times} \psi(x+x^{-1}) = 2\sum_{t \in \hat{S}} \psi(t) - \sum_{t=\pm 2} \psi(t) \tag{5}$$

and, using identity (1),

$$\mathrm{Kl}_n(1) = \sum_{x \in F^\times} \psi(T_n(x+x^{-1})) = 2\sum_{t \in \hat{S}} \psi(T_n(t)) - \sum_{t=\pm 2} \psi(T_n(t)). \tag{6}$$

Note also that

$$\sum_{t \in F} \psi(T_n(t)) = \sum_{t \in S} \psi(T_n(t)) + \sum_{t \in \hat{S}} \psi(T_n(t)). \tag{7}$$

From the lemma, we have

$$\sum_{t \in S} \psi(T_n(t)) = \sum_{t \in S} \psi(t). \tag{8}$$

From the Eqs. (5)–(8), we see

$$\begin{aligned} &2\sum_{t \in F} \psi(T_n(t)) + \mathrm{Kl}(1) - \mathrm{Kl}_n(1) \\ &\quad = 2\sum_{t \in S} \psi(t) + 2\sum_{t \in \hat{S}} \psi(t) + \sum_{t=\pm 2} \psi(T_n(t)) - \sum_{t=\pm 2} \psi(t). \end{aligned}$$

From the orthogonality, the sum of the first two terms is 0. The sum of the last two terms is 0 because $T_n(t) = t$ when $t = \pm 2$ (from identity (1)). We have proved identity (4).

We remark that Theorem 2 also holds when $(q^2-1, n) = 1$. In this case, the map $u \to u^n$ is a bijection on F, K, and H. It is then easy to see that the lemma holds in this case. The above proof carries through. However, in this case $\mathrm{Kl}_n(1) = \mathrm{Kl}(1)$, so identity (4) implies that the Airy sum vanishes when $(q^2-1, n) = 1$.

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