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Spherical Bessel Functions of $GL_2(F_{q^2})$

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We study the spherical Bessel functions of generic representations of a reductive group over finite fields. We show that in the case of $GL_2(F_{q^2})$, the relation between spherical Bessel functions and Bessel functions gives another characterization of the Shintani lifting.

Key Words: Bessel functions; Shintani lifting; finite fields.

1. INTRODUCTION

Let F_q be a finite field with $q=p^l$ elements and p an odd prime integer. Let $G=GL_2(F_{q^2})$. Let $F\colon (a_{ij})\mapsto (a_{ij}^q)$ be the Frobenious map. We will denote $F(g)$ by $\bar{g}$. Let U be the subgroup consisting of matrices $u(x)=\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]$. We will fix a nontrivial additive character ψ of F_q. Let $\theta(u(x))=\psi'(x)=\psi(x+\bar{x})$ for $x\in F_{q^2}$. Let $G'=G^F\cong GL_2(F_q)$. Let $U'=U\cap G'$, and $\theta'(u(x))=\psi(x)$ for $x\in F_q$. Let $w_0=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$. Let H be the unitary subgroup of G fixed by involution $g\to w_0\,{}^t\bar{g}^{-1}w_0$.

In this paper, we will look at the relation between the Bessel functions associated to the representations of G' and the H-spherical Bessel functions associated to the representations of G. Given π a representation of $GL_2(F_q)$, let χ_π be its character; then its Bessel function is defined by:

$$j_\pi(g)=|U'|^{-1}\sum_{x\in F_q}\chi_\pi(gu(x))\,\theta'(u(x)),\qquad g\in GL_2(F_q).$$

Let Π be a representation of the group G; its H-spherical Bessel function is defined by

$$k_\Pi(s)=|H/H\cap U|^{-1}\sum_{h\in H} j_\Pi(hg)=|HU|^{-1}\sum_{h\in H}\sum_{x\in F_{q^2}}\chi_\Pi(hgu(x))\,\theta(u(x))$$

whenever $s=w_0\,{}^t\bar{g}w_0\,g$. Clearly this function is well defined.

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Recall there is a bijective map ρ (called the Shintani lift) from the set of representations π of G' to the subset of representations Π of G satisfying $\Pi \cong \Pi \cdot F$. The map ρ satisfies the relation [8],

$$\chi_\pi(Ng) = \zeta_\pi \, Tr(I\rho(\pi))(g), \qquad g \in GL_2(F_{q^2}), \tag{1}$$

where Ng denotes the conjugacy class of $g\bar{g}$, ζ_π is a transfer factor, and I is an intertwining operator between $\rho(\pi)$ and $\rho(\pi) \cdot F$. Recall that a representation π is *generic* if π is a subrepresentation of $Ind_{U'}^{G'} \theta'$. Our main result is the following:

THEOREM 1. *A representation Π of G is a Shintani lift of a generic representation π of G' if and only if there exists a nonzero constant η such that for all $a, b \in F_q^\times$, we have $k_\Pi(a1_2) = \eta j_\pi(a1_2)$ and $k_\Pi(w_0\, diag[a, b]) = -\eta j_\pi(w_0\, diag[a, b])$.*

We note the values of k_Π and j_π at $a1_2, w_0\, diag[a, b]$ determines the function (see Section 3). Here 1_2 is the identity element of G.

The motivation of the theorem comes from Jacquet's relative trace formula (see [4]). It can be considered as a finite field analogue of local distribution theory for the relative trace formula. We prove the theorem by explicit computation using exponential sum identities. We note that a similar problem is considered by Leung in his thesis under the direction of Jacquet [6]. However, in his work Leung assumes a conjectured relation between spherical Green functions on G and Green functions on G'.

The theorem should be a special example of a very general phenomena. We will discuss the GL_n situation at the end of the paper. As there is no existing reference on the spherical Bessel functions, we also include some basic properties of these functions. In Section 2, we give a definition of spherical Bessel functions in a more general setting and prove some properties of the spherical Bessel functions. In Section 3, we recall the concept of a relevant double coset and give an explicit version of Theorem 1. The proof is given in Section 4. We discuss a generalization to the GL_n case in Section 5.

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2. BESSEL FUNCTIONS AND SPHERICAL BESSEL FUNCTIONS

We will adopt another set of notation in this section.

Let G be a connected reductive group over F_q. Let F be a standard Frobenious map on G. A Gelfand–Graev character of G^F is an induced character $Ind_{U^F}^{G^F}\theta$, where U is a maximal unipotent subgroup of G, and θ is a nondegenerate character on U^F, [1, Chap. 8]. For fixed U and θ, the Gelfand–Graev character is multiplicity free [9]; i.e., for π an irreducible complex representation of G^F,

$$\langle \pi, Ind_{U^F}^{G^F}\theta\rangle \leqslant 1. \tag{2}$$

When the above pairing is nonzero, we say π is θ-generic. Let $\Xi(G, \theta)$ be the set of all θ-generic representations.

2.1. *Bessel Functions*

For $\pi \in \Xi(G, \theta)$, one can associate a Bessel function on G^F [2]. It is a function J_π satisfying:

(i) $J_\pi(u_1 g u_2) = \theta(u_1 u_2)\, J_\pi(g)$, for all $u_1, u_2 \in U^F$ and $g \in G^F$.

(ii) The restriction of the right regular representation of G^F on $E(J_\pi(g))$ is equivalent to π.

Here we use $E(f)$ to denote the smallest invariant subspace containing f. Let e be the identity element of G. The following propositions can be found in [2, 3, 10].

PROPOSITION 1. *A function $J_\pi(g)$ satisfying the conditions* (i) *and* (ii) *above exists if and only if $\pi \in \Xi(G, \theta)$.*
The function

$$j_\pi(g) = |U^F|^{-1} \sum_{u \in U^F} \chi_\pi(gu)\, \theta^{-1}(u) \tag{3}$$

is nonzero if and only if $\pi \in \Xi(G, \theta)$. If $\pi \in \Xi(G, \theta)$, then $j_\pi(e) = 1$ and a Bessel function $J_\pi(g)$ is a nonzero multiple of $j_\pi(g)$.

Define the convolution of two functions on G^F as follows:

$$f_1 * f_2(g) = |G^F|^{-1} \sum_{g' \in G^F} f_1(gg'^{-1})\, f_2(g'). \tag{4}$$

Then $f_1 * f_2(g)$ lies in the space $E(f_1(g))$.

PROPOSITION 2 (Orthogonality relation). *For $\pi, \pi' \in \Xi(G, \theta)$,*

$$j_\pi * j_{\pi'}(g) = \delta(\pi, \pi')\, |U^F|\, d_\pi^{-1}\, j_\pi(g). \tag{5}$$

Here $\delta(\pi, \pi') = 1$ if $\pi = \pi'$ and 0 *otherwise; $d_\pi = \chi_\pi(e)$ is the dimension of π.*

2.2. *Spherical Bessel functions*

Let H be a subgroup of G. We will assume (G^F, H^F) is a weak Gelfand pair, namely

$$\langle \pi, Ind_{H^F}^{G^F} 1 \rangle \leqslant 1 \tag{6}$$

for all $\pi \in \Xi(G, \theta)$.

If the above pairing is nonzero, then π is said to be H-distinguished. Let $\Xi(G, H, \theta)$ be the set of θ-generic H-distinguished representations. For such π, we associate a H-Bessel function $I_\pi(g)$ on G^F satisfying:

(i′) $I_\pi(hgu) = \theta(u)\, I_\pi(g)$, for all $u \in U^F$, $h \in H^F$, $g \in G^F$.

(ii′) The restriction of the right regular representation of G^F on $E(I_\pi(g))$ is equivalent to π.

Remark. When H is the fixator of an involution σ on G, we let $K_\pi(\sigma(g)^{-1} g) = I_\pi(g)$ be a function on the symmetric space $H\backslash G$. We call K_π the H-spherical Bessel function associated to π.

In what follows, we look at the properties of the H-Bessel functions. The properties of the spherical Bessel functions follows from there.

Proposition 3. *A function $I_\pi(g)$ satisfying the conditions* (i′) *and* (ii′) *above exists if and only if $\pi \in \Xi(G, H, \theta)$.*

Given $\pi \in \Xi(G, H, \theta)$, there exists a $\alpha_\pi \in G^F$ such that the function

$$i_\pi(g) = |U^F H^F|^{-1} \sum_{u \in U^F} \sum_{h \in H^F} \chi_\pi(\alpha_\pi hgu)\, \theta^{-1}(u) \tag{7}$$

is nonzero. Such a function $i_\pi(g)$ is an H-Bessel function of π and any H-Bessel function $I_\pi(g)$ is a nonzero multiple of $i_\pi(g)$.

Proof. If $I_\pi(g)$ satisfying (i′) and (ii′) exists, then $E(I_\pi(g))$ is the space generated by $f(g) = I_\pi(gg')$. Any function in this space satisfies $f(hg) = f(g)$ for all $h \in H^F$. Thus this space is a subspace of $Ind_{H^F}^{G^F} 1$. The condition (ii′) implies π is H-distinguished. From the condition (i′), The action of π on $E(I_\pi(g))$ satisfies $\pi(u)\, I_\pi(g) = \theta(u)\, I_\pi(g)$ for $u \in G^F$. Thus π is also θ-generic.

For $\pi \in \Xi(G, H, \theta)$, let

$$s_\pi(g) = \sum_{h \in H^F} \chi_\pi(gh)$$

Then $s_\pi(g)$ lies in both $E(\chi_\pi(g))$ and $Ind_{H^F}^{G^F} 1$. From condition (6), the intersection of these two spaces is an irreducible space of π. Since $s_\pi(1) = |H^F| \neq 0$, we see $E(s_\pi(g))$ is this space for the representation π. As π is θ-generic, there exists a vector $f(g)$ in $E(s_\pi(g))$ with

$$\sum_{u \in U^F} \pi(u) f(g) \theta^{-1}(u) \neq 0. \tag{8}$$

We can choose $f(g)$ to be $s_\pi(g\alpha_\pi)$ for some $\alpha_\pi \in G^F$. For this α_π, it follows from (8) that $i_\pi(g) \neq 0$. This gives our second claim.

From (ii′), any H-Bessel function $I_\pi(g)$ lies in $E(s_\pi(g))$. As the subspace of vectors in $E(s_\pi(g))$ satisfying $\pi(u) f(g) = \theta(u) f(g)$, $u \in U^F$ is one-dimensional, as I_π and i_π both lie in this one dimensional subspace, they differ by only a scalar multiple. ∎

Remark. One cannot always set $\alpha_\pi = e$, as when θ is nontrivial on $H^F \cap U^F$,

$$\sum_{u \in U^F} \sum_{h \in H^F} \chi_\pi(hgu) \theta^{-1}(u) = \sum_{u \in U^F} \sum_{h \in H^F} \chi_\pi(guh) \theta^{-1}(u) = 0.$$

For $\pi \in \Xi(G, H, \theta)$, the choice of α_π is not unique; in the following, we will fix an arbitrary choice of α_π and consider $i_\pi(g)$ as uniquely defined.

PROPOSITION 4 (Orthogonality relation). *For $\pi \in \Xi(G, h, \theta)$, $\pi' \in \Xi(G, \theta)$,*

$$i_\pi * j_{\pi'}(g) = \delta(\pi, \pi') |U^F| d_\pi^{-1} i_\pi(g). \tag{9}$$

Proof. This follows from the identity [10]

$$|G^F|^{-1} \sum_{g \in G^F} \chi_\pi(sg^{-1}) \chi_{\pi'}(gt) = \delta(\pi, \pi') d_\pi^{-1} \chi_\pi(st). \quad ∎ \tag{10}$$

In particular, the set of functions

$$\Delta(G, H, \theta) = \{ i_\pi(g) \mid \pi \in \Xi(G, H, \theta) \}$$

is a linearly independent set. We will need this result in the proof of Theorem 1. We can show a bit more. Let $\Phi(G, H, \theta)$ be the set of functions $f(g)$ on G^F with $f(hgu) = \theta(u) f(g)$ for all $u \in U^F$, $h \in H^F$, $g \in G^F$.

PROPOSITION 5. *The set $\Delta(G, H, \theta)$ is a basis of $\Phi(G, H, \theta)$.*

Proof. We only need to show that $\Delta(G, H, \theta)$ span $\Phi(G, H, \theta)$. Let $f(g)$ be any function in $\Phi(G, H, \theta)$.

LEMMA 1. *For* $\pi \in \Xi(G, H, \theta)$, $f * j_\pi(g)$ *is a scalar multiple of* $i_\pi(g)$. *If* $\pi \notin \Xi(G, H, \theta)$, $f * j_\pi(g) = 0$.

Proof. The function $f(g)$ lies in $Ind_{H^F}^{G^F} 1 = \bigoplus \pi'$, where the sum is taken over all H-distinguished π'. Corresponding to this decomposition, one can write $f(g) = \sum f_{\pi'}(g)$, where $f_{\pi'}(g)$ is a function in the space for π'. As $f_{\pi'} * j_\pi(g)$ lies in $E(f_{\pi'}(g))$ and satisfies the condition (i′), from Proposition 3, we see

$$f_{\pi'} * j_\pi(g) = e_{\pi'} i_{\pi'}(g)$$

for some constant $e_{\pi'}$ when $\pi' \in \Xi(G, H, \theta)$ and 0 otherwise. Thus $f * j_\pi(g) = \sum e_{\pi'} i_{\pi'}(g)$ where the sum is taken over $\pi' \in \Xi(G, H, \theta)$. From Proposition 2, we then get

$$f * j_\pi(g) = |U^F|^{-1} d_\pi f * j_\pi * j_\pi(g) = \sum |U^F|^{-1} d_{\pi'} e_{\pi'} i_{\pi'} * j_\pi(g),$$

where the sum is taken over $\pi' \in \Xi(G, H, \theta)$. From Proposition 4, this expression is 0 if $\pi \notin \Xi(G, H, \theta)$, and $e_\pi i_\pi(g)$ otherwise. ∎

Now let

$$f'(g) = f(g) - \sum_{\pi \in \Xi(G, H, \theta)} d_\pi \, |U^F|^{-1} f(g) * j_\pi(g). \tag{11}$$

Then $f'(g) \in \Phi(G, H, \theta)$. We can check that $f' * j_\pi(g) = 0$ for all $\pi \in \Xi(G, H, \theta)$. From the above lemma, we see $f' * j_\pi(g) = 0$ for all irreducible representations π of G^F. Thus $f' * j_R(g) = 0$ where we use j_R to denote the function defined by (3) for the regular representation. A simple computation shows $f' * j_R$ is a scalar multiple of f'; thus $f'(g) = 0$. From (11) and Lemma 1, we see $f(g)$ is a linear combination of $i_\pi(g)$ with $\pi \in \Xi(G, H, \theta)$. ∎

Remark. Consider the case $G = G' + G'$, H is the fixator of the involution σ: $(g_1, g_2) \mapsto (g_2, g_1)$, and $\theta(u_1, u_2) = \theta(u_1 u_2^{-1})$. Then it is clear the set $\Xi(G, H, \theta)$ is given by $\{\Pi = (\pi, \pi^*) \mid \pi \in \Xi(G', \theta)\}$, where π^* is the contragradient of π. We can let $\alpha_\Pi = e$ in (7). The i_Π defined then satisfies $i_\Pi(g, 1) = j_\pi(g)$ for $g \in G'^F$. Thus the Bessel functions are a special case of the H-Bessel functions.

3. EXPLICIT VERSION OF THEOREM 1

3.1. *Relevant Cosets*

The function j_π is determined by the values on representatives of the cosets $U^F \backslash G^F / U^F$. We say a representative ϕ is *relevant* if $u_1 \phi u_2 = \phi$ $(u_1, u_2 \in U^F)$ implies $\theta(u_1 u_2) = 1$. If ϕ is not relevant, then clearly $j_\pi(\phi) = 0$

and $j_\pi(g)$ vanishes on the coset of ϕ. Thus j_π is determined by the values of $j_\pi(\phi)$ with ϕ all relevant representatives of double cosets.

Similarly, we say a representative ϕ of the double coset $H^F\backslash G^F/U^F$ is relevant if $h\phi u=\phi$ $(u\in U^F, h\in H^F)$ implies $\theta(u)=1$. The function i_π is determined by the values at ϕ with ϕ through all relevant representatives of double cosets.

We now go back to the case in the Introduction. Notations are as in the Introduction. The relevant representatives of double cosets $U'\backslash G'/U'$ and $H\backslash G/U$ have been classified; see [4]. Note that the coset of ϕ in $H\backslash G/U$ is uniquely determined by $w_0\, {}^t\bar{\phi}\, w_0\, \phi$ (denoted $p(\phi)$). One has

PROPOSITION 6. *A set R of elements in G gives a complete system of relevant representatives of $H\backslash G/U$ if and only if $\{p(\phi) \mid \phi\in R\}$ equals the set $S=\{\alpha 1_2, w_0\, diag[\alpha,\beta] \mid \alpha,\beta\in F_q^\times\}$. The set S gives a complete system of relevant representatives of $U'\backslash G'/U'$.*

We will need to have an explicit form of R. Given any $A, C\in F_q^\times$, we will choose $a, c\in F_{q^2}$ so that $a\bar{a}=A$, $c\bar{c}=C$. Define

$$o(A)=\begin{bmatrix} a & \\ & a\end{bmatrix}, \qquad o(B,C)=u(B)\begin{bmatrix} & c\\ 1 & \end{bmatrix}, \qquad A,B,C\in F_q^\times. \tag{12}$$

Then

$$p(o(A))=A1_2, \qquad p\left(o(B,C)\,u\left(-\frac{c}{2B}\right)\right)=w_0\, diag\left[2B, -\frac{C}{2B}\right] \tag{13}$$

which justifies our notation. The set $\{o(A), o(B,C) \mid A,B,C\in F_q^\times\}$ gives an example of a complete system R.

3.2. *Bessel Functions on G'*

A θ'-generic representation of G' is given by an irreducible regular character [1]. The set $\Xi(G',\theta')$ is independent of ψ and is the set of all irreducible representations π with $d_\pi>1$. These representations are given as follows: Let μ_1,μ_2 be characters of $F_q^\times$ and let ν be a nondecomposable character of $F_{q^2}^\times$; the representations are one of the following:

(1) $R_{T_0}^{G'}(\mu_1,\mu_2)$, where T_0 is the diagonal subgroup and $\mu_1\neq\mu_2$;

(2) $-R_{T_w}^{G'}\nu$, where T_w is the tori twisted from T_0 by w_0;

(3) the unique regular component of $R_{T_0}^{G'}(\mu_1,\mu_1)$. The character $R_{T_0}^{G'}(\mu_1,\mu_1)$ decomposes into the sum of this component with the one-dimensional representation $\mu_1\cdot\det$.

The Bessel functions for the representations in $\Xi(G', \theta')$ are given in [7]. We describe the values of these functions on the relevant orbits. Use the notation

$$o'(a) = \begin{bmatrix} a & \\ & a \end{bmatrix}, \qquad o'(a, b) = \begin{bmatrix} & a \\ ab & \end{bmatrix}, \qquad a, b \in F_q^\times. \tag{14}$$

Let λ_π be the central character of π. Then

$$j_\pi(o'(a)) = \lambda_\pi(a) \tag{15}$$

and if π is the component of $R_{T_0}^{G'}(\mu_1, \mu_2)$:

$$j_\pi(o'(a, b)) = q^{-1} \mu_1 \mu_2(a) \sum_{rs = -b^{-1},\, r, s \in F_q^\times} \mu_1(r^{-1})\, \mu_2(s^{-1})\, \psi(r+s). \tag{16}$$

If π is a component of $-R_{T_w}^{G'} v$:

$$j_\pi(o'(a, b)) = -q^{-1} v(a) \sum_{s\bar{s} = -b^{-1},\, s \in F_{q^2}^\times} v(s^{-1})\, \psi'(s). \tag{17}$$

These formulas of course work for Π a representation of G, with obvious modifications.

3.3. *Explicit Identity*

The H-Bessel functions i_Π for $\Pi \in \Xi(G, H, \theta)$ can be defined using (7) with $\alpha_\Pi = e$. We have

$$i_\Pi(g) = |H \cap U|^{-1} \sum_{h \in H} j_\Pi(hg), \tag{18}$$

where the functions j_Π for $\Pi \in \Xi(G, H, \theta)$ are given by the formulas (15)–(17) with q replaced by q^2 and ψ by ψ'.

Recall that the Shintani lift can be described explicitly as follows: a generic component of $R_{T_0}^{G'}(\mu_1, \mu_2)$ maps to a generic component of $R_{T_0}^{G}(\mu_1 \cdot N, \mu_2 \cdot N)$; and $-R_{T_w}^{G'} v$ lifts to $R_{T_0}^{G}(v, v \cdot F)$.

In the next section, we prove:

PROPOSITION 7. *For all $\pi \in \Xi(G', \theta')$, we have*

$$j_\pi(A 1_2) = \xi(\pi)\, i_{\rho(\pi)}(o(A)) \tag{19}$$

$$j_\pi\left(w_0 \, diag\left[2B, -\frac{C}{2B}\right]\right) = -\xi(\pi)\, i_{\rho(\pi)}(o(B, C))\, \psi'\left(\frac{c}{2B}\right), \tag{20}$$

where

(1) *if* $\rho(\pi) = R^G_{T_0}(\mu_1 \cdot N, \mu_2 \cdot N)$, $\mu_1 \neq \mu_2$, $\xi(\pi) = (q+1)^{-2}$,

(2) *if* $\rho(\pi)$ *is a component of* $R^G_{T_0}(\mu_1 \cdot N, \mu_1 \cdot N)$, $\xi(\pi) = q(q^2+1)^{-1}(q+1)^{-1}$.

(3) *if* $\rho(\pi) = R^G_{T_0}(v, v \cdot F')$, $\xi(\pi) = (q^2-1)^{-1}$.

Proof of Theorem 1. Theorem 1 follows from Proposition 7 easily. In fact from (13), we see if Π is the Shintani lift of π, the equations in the theorem hold with $\eta = \xi(\pi)^{-1}$. On the other hand, if the equations hold, then let Π' be the Shintani lift of π. It follows that $i_{\Pi} = \eta\xi(\Pi')^{-1} i_{\Pi'}$. From the orthogonality relation (or Proposition 5), we see $\Pi = \Pi'$ is the Shintani lift of π. ∎

4. PROOF OF PROPOSITION 7

The proof of the proposition, for lack of a better method, relies on the explicit computation of $i_{\rho(\pi)}$ using the identity (18).

(4.1). We need some identities of exponential sums.

LEMMA 2. *Let* ψ, ψ' *be characters as before and let* v *be a character of* $F^\times_{q^2}$.

(1) *For* $b \in F^\times_q$, $c \in F^\times_{q^2}$,

$$\sum_{y=-\bar{y},\, y \in F_{q^2}} \psi'\left(\frac{c}{b+y}\right) v\left(\frac{b+y}{b-y}\right) = \psi'\left(\frac{c}{2b}\right) \sum_{s\bar{s}=c\bar{c}} \psi'\left(\frac{s}{2b}\right) v(s^{-1}c) - v(-1).$$

(2) *For* $a \in F^\times_q$, $b \in F_q$,

$$\sum_{r \in F_q} \psi(ar^2 + br) = \gamma\zeta(a)\, \psi\left(-\frac{b^2}{4a}\right),$$

where $\zeta(a)$ *is the quadratic character on* $F^\times_q$, *and* $\gamma = \sum_{r \in F_q} \psi(r^2)$.

(3) *For* $a \in F^\times_q$, $b, c \in F^\times_{q^2}$,

$$\sum_{s \in F_{q^2}} \psi'(as\bar{s} + bs + c\bar{s}) = -q\psi'\left(-\frac{b\bar{b} + c\bar{c} + 2b\bar{c}}{4a}\right).$$

(4) $\sum_{r\in F_q} \psi'(rs)$ *equals* 0 *if* $s \neq -\bar{s}$ *or* q *if* $s = -\bar{s}$.

(5) *For* $a \in F_q^\times$,

$$\sum_{r\in F_q^\times} \psi\left(r+\frac{a}{r}\right) = -\sum_{s\bar{s}=a,\, s\in F_{q^2}^\times} \psi'(s).$$

(6) *If* v *is indecomposable*,

$$\sum_{s\in F_{q^2}^\times} \psi'(s)\, v(s\bar{s}^{-1}) = qv(-1).$$

Proof. For (1), we let $u=\frac{b+y}{b-y}$. Then $y \to u$ gives a bijection with its image being the set of u with $u\bar{u}=1$ and $u \neq -1$. As $y = b\,\frac{u-1}{u+1}$, the sum on the left can be written as

$$\sum_{u\bar{u}=1} \psi'\left(\frac{c(u+1)}{2bu}\right) v(u) - v(-1).$$

A change of variable $s = u^{-1}c$ gives the identity.

For parts (2) and (3), we make a change of variables on the left hand side, with $r \to r - \frac{b}{2a}$ or $s \to s - \frac{\bar{b}+c}{2a}$. The identities then follow from the identities:

$$\sum_{r\in F_q} \psi(ar^2) = \zeta(a)\,\gamma, \qquad \sum_{s\in F_{q^2}} \psi(as\bar{s}) = -q.$$

The identity (4) follows from the orthogonality of ψ'. For (5), the Davenport–Hasse relation shows for all χ characters of $F^\times$:

$$\begin{aligned}
\sum_{a\in F_q^\times}\sum_{r\in F_q^\times} \psi\left(r+\frac{a}{r}\right)\chi(a) &= \sum_{a\in F_q^\times}\sum_{r\in F_q^\times} \psi(r+a)\,\chi(ar) \\
&= -\sum_{s\in F_{q^2}^\times} \psi'(s)\,\chi(s\bar{s}) \\
&= -\sum_{a\in F_q^\times}\ \sum_{s\bar{s}=a,\, s\in F_{q^2}^\times} \psi(s)\,\chi(a).
\end{aligned}$$

The identity (5) then follows from Fourier inversion.

To prove (6), we change $s \to rs$ on the left hand side, for all $r \in F_q^\times$; we get:

$$(q-1)^{-1} \sum_{r\in F_q^\times,\, s\in F_{q^2}^\times} \psi'(rs)\, v(s\bar{s}^{-1}).$$

Use part (4) to compute the sum over r; the above sum equals:

$$q(q-1)^{-1}\sum_{s=-\bar{s}} v(s\bar{s}^{-1})-(q-1)^{-1}\sum_{s\in F_{q^2}^{\times}} v(s\bar{s}^{-1}).$$

Since v is indecomposable, the second sum gives 0; the first sum equals $qv(-1)$. ∎

(4.2). Let $d_a = diag[a, \bar{a}^{-1}]$. The following lemma is easily verified:

Lemma 3. *The unitary group H consists of the following elements with $x=-\bar{x}$, $y=-\bar{y}$ in F_{q^2} and $\alpha\in F_{q^2}^{\times}$:*

(1) $u(x)\, d_\alpha$.

(2) $u(x)\, w_0\, d_\alpha u(y)$.

(4.3). We prove the identity (19).

Note that for $\Pi\in\Xi(G, H, \theta)$, it is a component of $R^G_{T_0}(\mu_1', \mu_2')$ where either $\mu_1'(x)=\mu_1(x\bar{x})$, $\mu_2'(x)=\mu_2(x\bar{x})$ for a pair of characters (μ_1, μ_2) on $F_q^{\times}$, or $\mu_1'(x)=\mu_2'(\bar{x})=v(x)$ for some v on $F_{q^2}^{\times}$. The central character $\mu_1'\mu_2'(x)$ equals $\lambda(x\bar{x})$ where λ is either $\mu_1\mu_2$ or v.

From Lemma 3, and (18), $qi_\Pi(o(A))$ equals

$$\sum_{\alpha\in F_{q^2}^{\times}}\ \sum_{x\in F_{q^2},\, x=-\bar{x}} j_\Pi(u(x)\, d_\alpha)+\sum_{\alpha\in F_{q^2}^{\times}}\ \sum_{x,\, y\in F_{q^2},\, x=-\bar{x},\, y=-\bar{y}} j_\Pi(u(x)\, w_0 d_\alpha u(y)).$$

From (15) and (16), we see the above sum equals:

$$\sum_{\alpha\bar{\alpha}=1} q\mu_1'\mu_2'(a\alpha)+\sum_{\alpha\in F_{q^2}^{\times}}\mu_1'\mu_2'(a\alpha)\sum_{r,\, s\in F_{q^2}^{\times},\, rs=-\alpha\bar{\alpha}}\psi'(r+s)\,\mu_1'(r^{-1})\,\mu_2'(s^{-1}).$$

The first part of the sum equals $q(q+1)\,\lambda(A)$. Let $k=\alpha\bar{\alpha}$; then the second part of the sum is

$$(q+1)\sum_{k\in F_q^{\times},\, s\in F_{q^2}^{\times}}\lambda(Ak)\,\psi'\left(s-\frac{k}{s}\right)\mu_1'\left(-\frac{s}{k}\right)\mu_2'(s^{-1}). \tag{21}$$

In the case when $\lambda=v$, then

$$\lambda(k)\,\mu_1'\left(-\frac{s}{k}\right)\mu_2'(s^{-1})=v\left(-\frac{s}{\bar{s}}\right).$$

From Lemma 2(4), the above sum over k equals -1 if $s \neq -\bar{s}$ and $q-1$ otherwise. Thus the sum (21) is

$$q(q+1)\,\nu(A) \sum_{s=-\bar{s},\, s\neq 0} \psi'(s)\,\nu\left(-\frac{s}{\bar{s}}\right) - (q+1)\,\nu(A) \sum_{s\neq 0} \psi'(s)\,\nu\left(-\frac{s}{\bar{s}}\right)$$

which is $q^2(q+1)\,\nu(A)$ by Lemma 2(6). When $\Pi = R^G_{T_0}(\nu, \nu\cdot F')$, we get

$$i_\Pi(o(A)) = q^{-1}(q^2-q)(q+1)\,\nu(A). \tag{22}$$

In the case when $\lambda = \mu_1\mu_2$, if we make a change of variable $k \to ks\bar{s}$ in (21), it becomes

$$(q+1) \sum_{k\in F_q^\times,\, s\in F_{q^2}^\times} \mu_1\mu_2(A)\,\psi'(s-ks)\,\mu_2\mu_1^{-1}(k). \tag{23}$$

The sum over s equals q^2-1 when $k=1$ and -1 otherwise. Thus (23) equals

$$(q+1)\,\mu_1\mu_2(A)\left[q^2 - \sum_{k\in F_q^\times} \mu_2\mu_1^{-1}(k)\right]. \tag{24}$$

When $\mu_1 \neq \mu_2$, this is $q^2(q+1)\,\mu_1\mu_2(A)$ and

$$i_\Pi(o(A)) = (q+1)^2\,\mu_1\mu_2(A). \tag{25}$$

When $\mu_1 = \mu_2$, (24) equals $(q+1)(q^2-q+1)\,\mu_1^2(A)$ and

$$i_\Pi(o(A)) = q^{-1}(q^2+1)(q+1)\,\mu_1^2(A). \tag{26}$$

Compare the results so far with (15); we have shown Eq. (19) of Proposition 7.

(4.4). Now let $o(B, C)$ be as in (12). Then from Lemma 3 and (18), $qi_\Pi(o(B, C))$ is the sum $D_1 + D_2$ where

$$D_1 = \sum_{\alpha\in F_{q^2}^\times} \sum_{x\in F_{q^2},\, x=-\bar{x}} j_\Pi\left(u(x)\,d_\alpha u(B)\begin{bmatrix} & c \\ 1 & \end{bmatrix}\right) \tag{27}$$

and

$$D_2 = \sum_{\alpha\in F_{q^2}^\times} \sum_{x,\, y\in F_{q^2},\, x=-\bar{x},\, y=-\bar{y}} j_\Pi\left(u(x)\,w_0 d_\alpha u(y)\,u(B)\begin{bmatrix} & c \\ 1 & \end{bmatrix}\right). \tag{28}$$

We will simplify the sum D_2 first; the computation for D_1 is similar and easier. The sum over x in (28) gives a factor q. Using the fact that

$$wd_\alpha u(y)\, u(B) \begin{bmatrix} & c \\ 1 & \end{bmatrix}$$

$$= u([\alpha\bar{\alpha}(B+y)]^{-1}) \begin{bmatrix} & -\dfrac{c}{\bar{\alpha}(B+y)} \\ \alpha(B+y) & \end{bmatrix} u\left(\frac{c}{B+y}\right),$$

we see D_2 equals:

$$q \sum_{\alpha \in F_{q^2}^\times} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+(\alpha\bar{\alpha})^{-1}}{B+y}\right)$$
$$\times j_\Pi\left(\begin{bmatrix} & -\dfrac{c}{\bar{\alpha}(B+y)} \\ \alpha(B+y) & \end{bmatrix}\right).$$

From (16), it is:

$$q^{-1} \sum_{\alpha, r, s \in F_{q^2}^\times,\, rs = c[\alpha\bar{\alpha}(B+y)^2]^{-1}} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+(\alpha\bar{\alpha})^{-1}}{B+y} + r + s\right)$$
$$\times \mu_1'(r^{-1})\, \mu_2'(s^{-1})\, \mu_1'\mu_2'\left(-\frac{c}{\bar{\alpha}(B+y)}\right).$$

If we make a change of variable $(r, s) \to (\frac{r}{B+y}, \frac{s}{B+y})$, we get

$$q^{-1} \sum_{\alpha, r, s \in F_{q^2}^\times,\, rs = c[\alpha\bar{\alpha}]^{-1}} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+(\alpha\bar{\alpha})^{-1}+r+s}{B+y}\right)$$
$$\times \mu_1'(r^{-1})\, \mu_2'(s^{-1})\, \mu_1'\mu_2'\left(-\frac{c}{\bar{\alpha}}\right).$$

Since $\mu_1'\mu_2'(x) = \lambda(x\bar{x})$, if we let $k = \alpha\bar{\alpha}$, we get

$$q^{-1}(q+1) \sum_{k \in F_q^\times} \sum_{r, s \in F_{q^2}^\times,\, rs = ck^{-1}} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+k^{-1}+r+s}{B+y}\right)$$
$$\times \mu_1'(r^{-1})\, \mu_2'(s^{-1})\, \lambda\left(\frac{C}{k}\right).$$

If we sum over r and let $k \to k^{-1}$, D_2 equals:

$$q^{-1}(q+1) \sum_{k \in F_q^\times} \sum_{s \in F_{q^2}^\times} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+k+cks^{-1}+s}{B+y}\right) \times \mu_1'\left(\frac{s}{ck}\right) \mu_2'(s^{-1})\, \lambda(kC). \tag{29}$$

Similarly, from (16) we get D_1 equals:

$$q^{-1}(q+1) \sum_{k \in F_q^\times} \sum_{s \in F_{q^2}^\times} \psi'\left(kB - \frac{kc}{s} + s\right) \mu_1'\left(-\frac{s}{kc}\right) \mu_2'(s^{-1})\, \lambda(Ck). \tag{30}$$

(4.5). We compute the sum $D_1 + D_2$ in the case $\mu_1' = \mu_1 \cdot N$, $\mu_2' = \mu_2 \cdot N$. Then $\lambda = \mu_1 \mu_2$.

In (29) if we change $k \to ks\bar{s}$:

$$D_2 = q^{-1}(q+1) \sum_{k \in F_q^\times} \sum_{s \in F_{q^2}^\times} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c+ks\bar{s}+ck\bar{s}+s}{B+y}\right) \times \mu_2(C)\, \mu_2\, \mu_1^{-1}(k).$$

If we change $s \to s(B+y)$, we get D_2 equals $q^{-1}(q+1)\, \mu_2(C)$ times

$$\sum_{k \in F_q^\times} \sum_{s \in F_{q^2}^\times} \sum_{y \in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c}{B+y}\right) \psi\left(kBs\bar{s} + ck\,\frac{B-y}{B+y}\,\bar{s} + s\right) \mu_2 \mu_1^{-1}(k).$$

If we apply Lemma 2(3) to the sum over s and simplify, we see D_2 equals:

$$-(q+1)\, \mu_2(C) \sum_{k \in F_q^\times} \sum_{y \in F_{q^2},\, y=-\bar{y}} \left[\psi'\left(\frac{c}{2B}\right) \psi\left(-\frac{1+Ck^2}{2Bk}\right) + q^{-1}\psi'\left(\frac{c}{B+y}\right)\right] \mu_2\, \mu_1^{-1}(k).$$

Similarly, if we change $k \to ks\bar{s}$ and apply Lemma 2(3) in (30), D_1 equals:

$$-(q+1)\, \mu_2(C) \sum_{k \in FF_q^\times} \left[\psi'\left(\frac{c}{2B}\right) \psi\left(-\frac{1+Ck^2}{2kB}\right) + q^{-1}\right] \mu_2 \mu_1^{-1}(k).$$

Thus D_1+D_2 is the sum of

$$-(q+1)^2\,\psi'\left(\frac{c}{2B}\right)\mu_2(C)\sum_{k\in F_q^\times}\psi\left(-\frac{kC}{2B}-\frac{1}{2kB}\right)\mu_1^{-1}\mu_2(k) \tag{31}$$

and

$$-q^{-1}(q+1)\,\mu_2(C)\sum_{k\in F_q^\times}\mu_2\mu_1^{-1}(k)\left[1+\sum_{y\in F_{q^2},\, y=-\bar{y}}\psi'\left(\frac{c}{B+y}\right)\right]. \tag{32}$$

From Lemmas 2(1) and 2(5), Eq. (32) equals

$$q^{-1}(q+1)\,\psi'\left(\frac{c}{2B}\right)\mu_2(C)\sum_{k\in F_q^\times}\mu_2\mu_1^{-1}(k)\sum_{r\in F_q^\times}\psi\left(-\frac{rC}{2B}-\frac{1}{2rB}\right). \tag{33}$$

It is easy to check from (16) that when $\rho(\pi)=\Pi$,

$$j_\pi\left(w_0\, diag\left[2B,\, -\frac{C}{2B}\right]\right)=q^{-1}\mu_2(C)\sum_{k\in F_q^\times}\psi\left(-\frac{kC}{2B}-\frac{1}{2kB}\right)\mu_1^{-1}\,\mu_2(k). \tag{34}$$

Compare (31), (33), and (34); the identity (20) for $o(B, C)$ follows from

$$\sum_{k\in F_q^\times}\mu_2\mu_1^{-1}(k)=\delta(\mu_1,\mu_2)(q-1).$$

(4.6). We now consider the case when $\mu_1'=\nu$, $\mu_2'=\nu\cdot F'$. Here $\lambda=\nu$. From (30),

$$D_1=q^{-1}(q+1)\sum_{k\in F_q^\times}\sum_{s\in F_{q^2}^\times}\psi'\left(k\left(B-\frac{c}{s}\right)+s\right)\nu(-c^{-1}C)\,\nu\left(\frac{s}{\bar{s}}\right).$$

From Lemma 2(4),

$$D_1=(q+1)\,\nu(-\bar{c})\left[\sum_{s=c/(B+y),\, y=-\bar{y}}\psi'(s)\,\nu\left(\frac{s}{\bar{s}}\right)-q^{-1}\sum_{s\in F_{q^2}^\times}\psi'(s)\,\nu\left(\frac{s}{\bar{s}}\right)\right].$$

From Lemma 2(1), the first sum over s equals

$$\psi'\left(\frac{c}{2B}\right)\sum_{s\bar{s}=C}\psi'\left(\frac{s}{2B}\right)\nu(s^{-1}c)-\nu\left(-\frac{c}{\bar{c}}\right).$$

From Lemma 2(6), the second sum over s equals $qv(-1)$. Thus

$$D_1 = -(q+1)[v(\bar{c}) + v(c)] + (q+1)\, v(-C)\, \psi'\left(\frac{c}{2B}\right) \sum_{s\bar{s}=C} \psi'\left(\frac{s}{2B}\right) v(s^{-1}). \tag{35}$$

Similarly, we may use Lemma 2(4) to compute the sum over k in (29); D_2 equals the sum of

$$(q+1)\, v(\bar{c}) \sum_{z=(cs^{-1}+1)/(B+y),\, z=-\bar{z}} \; \sum_{y\in F_q^\times,\, y=-\bar{y}} \psi'\left(\frac{c+s}{B+y}\right) v\left(\frac{s}{\bar{s}}\right) \tag{36}$$

and

$$-(q+1)\, q^{-1} v(\bar{c}) \sum_{s\in F_{q^2}^\times} \; \sum_{y\in F_q^\times,\, y=-\bar{y}} \psi'\left(\frac{c+s}{B+y}\right) v\left(\frac{s}{\bar{s}}\right). \tag{37}$$

If we change $s \to s(B+y)$ in (37), it becomes

$$-(q+1)\, q^{-1} v(\bar{c}) \sum_{s\in F_{q^2}^\times} \psi'(s)\, v\left(\frac{s}{\bar{s}}\right) \sum_{y\in F_q^\times,\, y=-\bar{y}} \psi'\left(\frac{c}{B+y}\right) v\left(\frac{B+y}{B-y}\right).$$

From Lemmas 2(6) and 2(1), it is:

$$-(q+1)\, v(-C)\, \psi'\left(\frac{c}{2B}\right) \sum_{s\bar{s}=C} \psi'\left(\frac{s}{2B}\right) v(s^{-1}) + (q+1)\, v(\bar{c}). \tag{38}$$

The part $z=0$ in the sum (36) gives

$$v(c)\, q(q+1). \tag{39}$$

The part $z\neq 0$ in the sum (36) gives:

$$(q+1) \sum_{z=-\bar{z}\neq 0} \; \sum_{y\in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c}{B+y-z^{-1}}\right) v\left(-\frac{B-y+z^{-1}}{B+y-z^{-1}}\right) v(c)$$

$$= (q+1) \sum_{z=-\bar{z}\neq 0} \; \sum_{y\in F_{q^2},\, y=-\bar{y}} \psi'\left(\frac{c}{B+y}\right) v\left(-\frac{B-y}{B+y}\right) v(c)$$

after a change of variable $y \to y + z^{-1}$. From Lemma 2(1), this equals

$$(q^2-1)\, \nu(-C)\, \psi'\left(\frac{c}{2B}\right) \sum_{s\bar{s}=C} \psi'\left(\frac{s}{2B}\right) \nu(s^{-1}) - (q^2-1)\, \nu(c). \qquad (40)$$

If we add (35), (38), (39) and (40), we get $D_1 + D_2$ equals:

$$(q^2-1)\, \nu(-C)\, \psi'\left(\frac{c}{2B}\right) \sum_{s\bar{s}=C} \psi'\left(\frac{s}{2B}\right) \nu(s^{-1}). \qquad (41)$$

From (17), when $\Pi = \rho(\pi)$,

$$j_\pi\left(w_0\, diag\left[2B, -\frac{C}{2B}\right]\right) = -q^{-2}\nu\left(-\frac{C}{2B}\right) \sum_{s\bar{s}=C/4B^2} \psi'(s)\, \nu(s^{-1}). \qquad (42)$$

A change of variable $s \to 2Bs$ in (41) gives the identity (20) for $o = o(B, C)$. ∎

5. THE $GL(n)$ CASE

Let $G = GL_n(F_{q^2})$, U be its subgroup of upper triangular matrices with unit diagonal. Let $G' = G^F \cong GL_n(F_q)$, $U' = U \cap G'$. Let w_0 be the permutation matrix with 1's on the antidiagonal. Let H be the unitary matrix fixed under the involution of G: $\sigma(g) = w_0\, {}^t g^{-1} w_0$. We fix a nondegenerate character θ' on U' by $\theta'(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$, and let $\theta(u)$ for $u \in U$ be defined similarly by replacing ψ with ψ'.

Shintani has shown there is a bijection ρ from the set of representations π of G' to the subset of representations Π of G satisfying $\Pi \cong \Pi \cdot F$, as in the introduction. In [5], the relevant cosets of $H\backslash G/U$ and $U'\backslash G'/U$ are classified. We have a similar result as in Proposition 6. Let S be the set of $w_0 w\mathbf{a}$, where w is the longest Weyl element in some standard Levi subgroup of $GL_n(F_q)$ and $\mathbf{a}$ lies in the center of this Levi subgroup. Then the set S gives a complete set of relevant representatives of the cosets $U'\backslash G'/U'$. Meanwhile, there is a set R which gives a complete set of relevant representatives of $H\backslash G/U$ such that $p(R) = S$. (Recall $p(g) = w_0\, {}^t\bar{g}\, w_0\, g$.)

Define the Bessel functions j_π and H-spherical Bessel functions k_Π as in the introduction. We expect the following to hold: if $\pi \in \Xi(G', \theta')$, $\Pi = \rho(\pi)$, then there exists nonzero constants $\eta(\pi)$, such that for $w_0 w\mathbf{a} \in S$,

$$k_\Pi(w\mathbf{a}) = \tau(w)\, \eta(\pi)\, j_\pi(w_0 w\mathbf{a}),$$

where $\tau(w) = (-1)^l$ if the Levi subgroup corresponding to w is isomorphic to $\prod_{r=1}^{l} GL_{n_r}(F_q)$.

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