# Computation of central value of quadratic twists of modular $L-$functions

*Z. Mao, F. Rodriguez-Villegas and G. Tornaría*

## 1 Introduction

Let $f \in S_2(p)$ be a newform of weight two, prime level $p$. If $f(z) = \sum_{m=1}^{\infty} a(m)q^m$, where $q = e^{2\pi iz}$, and $D$ is a fundamental discriminant, we define the twisted $L$-function

$$L(f, D, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left(\frac{D}{m}\right).$$

It will be convenient to also allow $D = 1$ as a fundamental discriminant, in which case we write simply $L(f, s)$ for $L(f, 1, s)$.

In this paper we consider the question of computing the twisted central values $\{L(f, D, 1) \ : \ |D| \leqslant x\}$ for some $x$.

It is well known that the fact that $f$ is an eigenform for the Fricke involution yields a rapidly convergent series for $L(f, D, 1)$. Computing $L(f, D, 1)$ by means of this series, which we call the *standard method*, takes time very roughly proportional to $|D|$ and therefore time very roughly proportional to $x^2$ to compute $L(f, D, 1)$ for $|D| \leqslant x$. We will see that this can be improved to $x^{3/2}$ by using an explicit version of Waldspurger's theorem [W]; this theorem relates the central values $L(f, D, 1)$ to the $|D|$-th Fourier coefficient of weight $3/2$ modular forms in Shimura correspondence with $f$.

Concretely, the formulas we use have the basic form

$$L(f, D, 1) = \star\, \kappa_{\mp} \frac{|c_{\mp}(|D|)|^2}{\sqrt{|D|}}, \qquad \operatorname{sign}(D) = \mp, \tag{1.1}$$

where $\star = 1$ if $p \nmid D$, $\star = 2$ if $p \mid D$, $\kappa_-$ and $\kappa_+$ are positive constants independent of $D$, and $c_-(|D|)$ (resp. $c_+(|D|)$) is the $|D|$-th Fourier coefficient of a certain modular form $g_-$ (resp. $g_+$) of weight $3/2$.

Gross [G] proves such a formula, and gives an explicit construction of the corresponding form $g_-$, in the case that $L(f, 1) \neq 0$. The purpose of this paper is to extend Gross's work to all cases. Specifically, we give an explicit construction of both $g_-$ and $g_+$, regardless of the value of $L(f, 1)$, together with the corresponding values of $\kappa_-$ and $\kappa_+$ in (1.1). The proof of the validity of this construction will be given in a later publication and relies partly in the results of [BM].

The construction gives $g_-$ and $g_+$ as linear combinations of (generalized) theta series associated to positive definite ternary quadratic forms. Computing the Fourier coefficients of these theta series up to $x$ is tantamount to running over all lattice points in an ellipsoid of volume proportional to $x^{3/2}$. Doing this takes time roughly proportional to $x^{3/2}$ which yields our claim above.

This approach to computing $L(f, D, 1)$ has several other advantages over the standard method. First, the numbers $c(|D|)$ are algebraic integers and are computed with exact arithmetic. Once $c(|D|)$ is know it is trivial to compute $L(f, D, 1)$ to any desired precision. Second, the $c(|D|)$'s have extra information; if $f$ has coefficients in $\mathbb{Z}$, for example, (1.1) gives a specific square root of $L(f, D, 1)$ (if non-zero), whose sign remains a mystery.

Moreover, the actual running time of our method vs. the standard method is, in practice, significantly better even for small $x$.

## 2 Quaternion algebras and Brandt matrices

A quaternion algebra $B$ over a field $K$ is a central simple algebra of dimension 4 over $K$. When $2 \neq 0$ in $K$ we can give $B$ concretely by specifying a $K$-basis $\{1, i, j, k\}$ such that

$$i^2 = \alpha, \qquad j^2 = \beta, \qquad \text{and} \qquad k = ij = -ji,$$

for some non-zero $\alpha, \beta \in K$. If $K = \mathbb{Q}$ we typically rescale and assume that $\alpha, \beta \in \mathbb{Z}$. A general element of $B$ then has the form $b = b_0 + b_1 i + b_2 j + b_3 k$, with $b_i \in K$ and multiplication in $B$ is determined by the above defining relations and $K$-linearity.

The *conjugate* of $b$ is defined as

$$\bar{b} = b_0 - b_1 i - b_2 j - b_3 k.$$

We define the (reduced) norm and trace of $b$ by

$$\mathcal{N} b := b\bar{b} = b_0^2 - \alpha b_1^2 - \beta b_2^2 + \alpha\beta b_3^2, \qquad \mathsf{Tr}\, b := b + \bar{b} = 2b_0.$$

Let $B$ be a quaternion algebra over $K = \mathbb{Q}$. For $\nu$ a rational prime we let $\mathbb{Q}_\nu$ be the field of $v$-adic numbers and for $\nu = \infty$ we let $\mathbb{Q}_\nu = \mathbb{R}$. We call $\nu$, a rational prime or $\infty$, a *place* of $\mathbb{Q}$.

The localization $B_\nu := B \otimes \mathbb{Q}_\nu$ is a quaternion algebra over $\mathbb{Q}_\nu$. It is a fundamental fact of Number Theory that $B_\nu$ is either isomorphic to the algebra $M_2(\mathbb{Q}_\nu)$ of $2 \times 2$ matrices, or a division algebra, which is unique up to isomorphism. (A *division algebra* is an algebra in which every non-zero element has a multiplicative inverse.) The two options are encoded in the *Hilbert symbol* $(\alpha, \beta)_\nu$, defined as $+1$ if $B_\nu$ is a matrix algebra, $-1$ if it is a division algebra. In the first case we say that $B$ is *split* at $\nu$, in the second, that $B$ is *ramified* at $\nu$.

For example, if $\nu = \infty$ so $\mathbb{Q}_\nu = \mathbb{R}$ then $(\alpha, \beta)_\infty = -1$ if and only if $\alpha < 0, \beta < 0$ in which case $B_\infty$ is isomorphic to the usual Hamilton quaternions. A quaternion algebra $B$ is *definite* if it ramifies at $\infty$ otherwise it is *indefinite* (this notation is consistent with the nature of the quadratic form on $B_\infty$ determined by the norm $\mathcal{N}$).

A quaternion algebra $B$ is ramified at a finite number of places and the total number of ramified places must be even (e.g. the *Hilbert reciprocity law* says that $\prod_\nu (\alpha, \beta)_\nu = 1$). The set of ramified places determines $B$ up to isomorphism (the *local-global principle*). For any finite set $S$ with an even number of places there is a (unique up to isomorphism) $B$ which ramifies exactly at places in $S$.

Let $B$ be a quaternion algebra over $\mathbb{Q}$. An *order* in $B$ is a (full rank) lattice $R \subseteq B$ which is also a ring with $1 \in R$. As for number fields, an element of an order must be integral over $\mathbb{Z}$, i.e., must satisfy a monic equation with coefficients in $\mathbb{Z}$ (or even more concretely must have integral trace and norm). Unlike in the commutative case, however, the set of all integral elements of $B$ is *not* a ring. The best next thing is to consider maximal orders (which always exist), i.e., orders not properly contained in another order. But maximal orders are not unique. In fact, if $B$ is definite, a maximal order is in general not even unique up to isomorphism though there always is only a finite number of isomorphism classes of maximal orders in $B$.

As an illustration consider the classical case $\alpha = \beta = -1$ of the Hamilton quaternions. The algebra is definite and hence ramifies at $\nu = \infty$. It must ramify at a least one other prime, which turns out to be only $\nu = 2$. To see this note that

$$\mathcal{N}(b_0 + b_1 i + b_2 j + b_3 k) = b_0^2 + b_1^2 + b_2^2 + b_3^2.$$

There always is a non-trivial solution to the congruence $b_0^2 + b_1^2 + b_2^2 + b_3^2 \equiv 0 \bmod p$ for $p$ prime. If $p$ is odd we can lift this solution to a solution in $\mathbb{Z}_p$ by Hensel's lemma obtaining a non-zero quaternion in $B_p$ of zero norm. This implies that $B_p$ cannot be a divison algebra and hence $(-1,-1)_p = 1$ for $p$ odd. We must necessarily have then that $(-1,-1)_2 = -1$.

If we want to study the representation of numbers as sum of four squares it is natural to consider, as Lipschitz did, the arithmetic of the quaternions with $b_i \in \mathbb{Z}$. These quaternions form an order $R'$, but, as it turns out, it is not maximal. Indeed, as Hurwitz noted, $\rho := \frac{1}{2}(1 + i + j + k)$ is integral ($\mathcal{N}\rho = 1$ and $\mathsf{Tr}\,\rho = 1$) and $R := R' + \mathbb{Z}\rho$ is also an order of $B$ strictly containing $R'$.

Moreover, $R$ is maximal and hence its arithmetic is significantly simpler than that of $R'$. Hurwitz showed, for example, that there is a left and right division algorithm in $R$, from which it follows that every positive integer is a sum of four squares.

Fix a prime $p$ and let $B$ be the quaternion algebra over $\mathbb{Q}$ ramified precisely at $\infty$ and $p$. Let $R$ be a fixed maximal order in $B$. A right ideal $I$ of $R$ is a lattice in $B$ that is stable under right multiplication by $R$. Two right ideals $I$ and $J$ are in the same *class* if $J = bI$ with $b \in B^\times$. The set of right ideal classes is finite; let $n$ be its number. Chose a set of representatives $\{I_1, \ldots, I_n\}$ of the classes. (We should emphasize here that contrary to the commutative setting there is *no* natural group structure on the set of classes.)

Consider the vector space $V$ of formal linear combinations

$$\sum_{i=1}^{n} a_i\,[I_i], \qquad a_i \in \mathbb{C}$$

(here $[I]$ denotes the class of $I$).

For each integer $m$ there is an $n \times n$ matrix $B_m$ acting on $V$. Pizer [P] gives an efficient algorithm for computing these *Brandt matrices*: its coefficients are given by the representation numbers of the norm form for certain quaternary lattices in $B$.

The Brandt matrices commute with each other and are self-adjoint with respect to the *height pairing* on $V$ (see §1 and §2 of [G] for an account of this.) From this it follows that there is basis of $V$ consisting of simultaneous eigenvectors of all $B_m$.

It follows from Eichler's trace formula that there is a one to one correspondence between Hecke eigenforms of weight 2 and level $p$ (cf. [G, §5]) and eigenvectors in $V$ of all Brandt matrices (up to a constant multiple).

If $f$ is the Hecke eigenform we let $e_f$ be the corresponding eigenvector (well defined up to a constant). Then $B_m e_f = a_m e_f$ where $T_m f = a_m f$ and $T_m$ is the $m$-th Hecke operator.

## 3 Construction of $g_-$ and $g_+$

Let $e_f$ be the eigenvector for the Brandt matrices for $R$ corresponding to $f$ as in the last section. One can use linear algebra to find its coefficients

$$e_f = \sum_{i=1}^{n} a_i [I_i],$$

by computing the Brandt matrices, and from the knowledge of a few eigenvalues (i.e. Fourier coefficients) of $f$.

We will describe below the construction of certain generalized theta series $\Theta_{l^*}([I_i])$ corresponding to each ideal class $[I_i]$, and then define

$$\Theta_{l^*}(e_f) := \sum_{i=1}^{n} a_i\ \Theta_{l^*}([I_i]) = \sum_{n=1}^{\infty} c_{l^*}(n)\, q^n.$$

Here $l^*$ is a fundamental discriminant for which we will consider three cases: $l^* = 1$, which is Gross's construction of $g_-$; $l^* = l$ for an odd prime $l \neq p$

such that $l \equiv 1 \pmod 4$, which will generalize Gross's construction of $g_-$; and $l^* = -l$ for an odd prime $l \neq p$ such that $l \equiv 3 \pmod 4$, which will give a construction of $g_+$.

Furthermore, for any fundamental discriminant $D$ such that $Dl^* < 0$, the following formula holds

$$L(f, l^*, 1)\, L(f, D, 1) = \star\, \kappa_f \frac{|c_{l^*}(|D|)|^2}{\sqrt{|Dl^*|}}, \tag{3.1}$$

where $\star = 1$ if $p \nmid D$, $\star = 2$ if $p \mid D$, and $\kappa_f := \frac{\langle f,f\rangle}{\langle e_f,e_f\rangle}$ is a positive constant independent of $D$ or $l^*$. Here $\langle e_f, e_f\rangle$ is the height of $e_f$, and $\langle f, f\rangle$ is the Petersson norm of $f$ (cf. §4 and §7 of [G].) For $l = 1$, this formula was proved by Gross in [G, Proposition 13.5]. The proof of this formula for the case $l \neq 1$ will be given in a later publication.

Note that, as a corollary, we have $\Theta_{l^*}(e_f) \neq 0$ if and only if $L(f, l^*, 1) \neq 0$, and this happens for infinitely many $l^* > 0$ and for infinitely many $l^* < 0$, as follows from [BFH].

## 3.1 Gross's construction of $\Theta_1$

Let $R_i := \{b \in B \ :\ bI_i \subset I_i\}$ be the left order of $I_i$. The $R_i$ are maximal orders in $B$, and each conjugacy class of maximal orders has a representative $R_i$ for some $i$. For $b \in B$, we use $\mathcal{N}\, b$ and $\mathsf{Tr}\, b$ to denote the reduced norm and reduced trace of $b$, namely

$$\mathcal{N}\, b := b\bar{b}, \qquad \mathsf{Tr}\, b := b + \bar{b}.$$

We let $S_i^0 := \{b \in \mathbb{Z} + 2R_i \ :\ \mathsf{Tr}\, b = 0\}$, a ternary lattice, and define

$$\Theta_1([I_i]) := \frac{1}{2} \sum_{b \in S_i^0} q^{\mathcal{N}\, b}.$$

Then $\Theta_1([I_i])$ is a weight $3/2$ modular form of level $4p$ and trivial character.

## 3.2 Weight functions and $\Theta_l$

Fix an odd prime $l \neq p$. In order to generalize Gross's method, we need to construct certain weight functions $\omega_l(I_i, \cdot)$ on $S_i^0$ with values in $\{0, \pm 1\}$. There is a choice of sign in the construction, and some care is needed to ensure that the choice is consistent from one ideal to another. It will be the case that $\omega_l(I_i, b) = 0$ unless $l \mid \mathcal{N}\, b$, and thus we define a *generalized theta series*

$$\Theta_l([I_i]) := \frac{1}{2} \sum_{b \in S_i^0} \omega_l(I_i, b) q^{\mathcal{N}\, b/l},$$

a modular form of weight $^3\!/_2$ and level $4p$ with trivial character. In addition, $\Theta_l([I_i])$ is already a cusp form whenever $l \neq 1$, although it might be zero.

**Definition 3.1.** *Given a pair* $(L, v)$*, where* $L$ *is an integral* $\mathbb{Z}_l$*-lattice of rank* 3 *with* $l \nmid \det L$*, and* $v \in L$ *is such that* $l \mid \mathcal{N}\, v$ *but* $v \notin l\, L$*, we define its* weight function $\omega_{l,v} : L \to \{0, \pm 1\}$ *to be*

$$\omega_{l,v}(v') := \begin{cases} 0 & \text{if } l \nmid \mathcal{N}\, v', \\ \chi_l(\langle v, v' \rangle) & \text{if } l \nmid \langle v, v' \rangle, \\ \chi_l(k) & \text{if } v' - k\, v \in l\, L. \end{cases}$$

*Here* $\chi_l$ *is the quadratic character of conductor* $l$*, and* $\mathcal{N}\, v := \frac{1}{2} \langle v, v \rangle$*.*

This is well defined, because if $v, v' \in L$ are such that $\mathcal{N}\, v \equiv \mathcal{N}\, v' \equiv \langle v, v' \rangle \equiv 0 \pmod{l}$, then $v$ and $v'$ must be collinear modulo $l$, since $L$ is unimodular. This means that, assuming $v \notin l\, L$, there is indeed a well defined $k \in \mathbb{Z}/l\,\mathbb{Z}$ such that $v' - k\, v \in l\, L$.

Note that there are, for different choices of $v$, two different weight functions for each $L$, opposite to each other; the definition above singles out the one for which $\omega_{l,v}(v) = +1$.

We will apply the above definition to the ternary lattices $S_i^0(\mathbb{Z}_l) := S_i^0 \otimes \mathbb{Z}_l$. Fix a quaternion $b_0 \in S^0 := \{b \in \mathbb{Z} + 2R \;:\; \mathsf{Tr}\, b = 0\}$, and such that $l \mid \mathcal{N}\, b_0$ but $b_0 \notin l S^0$. For each $I_i$, find $x_i \in I_i$ such that $l \nmid n_i := \mathcal{N}\, x_i / \mathcal{N}\, I_i$. Then $x_i$ is a local generator of $I_i$, and $b_i := x_i\, b_0\, x_i^{-1} \in S_i^0(\mathbb{Z}_l)$. We finally set

$$\omega_l(I_i, b) := \chi_l(n_i)\, \omega_{l,b_i}(b),$$

where $\omega_{l,b_i}$ is the weight function of the pair $(S_i^0(\mathbb{Z}_l), b_i)$.

## 3.3 Odd weight functions and $\Theta_{-l}$

When $l \equiv 3 \pmod 4$ the weight functions $\omega_l(I_i, \cdot)$ are odd, since $\chi_l$ is odd. Therefore, we will have $\Theta_l = 0$. To address this problem, we will construct a different kind of weight function $\omega_p(I_i, \cdot)$, and then define

$$\Theta_{-l}([I_i]) := \frac{1}{2} \sum_{b \in S_i^0} \omega_p(I_i, b)\, \omega_l(I_i, b) q^{\mathcal{N}\, b / l},$$

which will be a modular form of weight $^3\!/_2$, this time of level $4p^2$. Again, $\Theta_{-l}([I_i])$ is a cusp form, which might be zero. Note that we could have used the product of two odd weight functions $\omega_{l_1}$ and $\omega_{l_2}$, but this construction would only lead us to the same $g_-$. By using the weight functions $\omega_p$ we get a construction of $g_+$ instead.

**Definition 3.2.** *Given a triple* $(L, v, \psi)$ *where* $L$ *is an integral* $\mathbb{Z}_p$*-lattice of rank* 3 *with level* $p$ *and determinant* $p^2$ *(i.e.* $L$ *is* $\mathbb{Z}_p$*-equivalent to* $S^0(\mathbb{Z}_p)$*,)* $v \in L$ *is such that* $p \nmid \mathcal{N}\, v$*, and* $\psi : \mathbb{Z}/p\,\mathbb{Z} \to \mathbb{C}$ *is a periodic function modulo* $p$*, the* weight function $\omega_{\psi,v} : L \to \mathbb{C}$ *is defined by*

$$\omega_{\psi,v}(v') := \psi(\langle v, v'\rangle).$$

Clearly, the weight function $\omega_{\psi,v}$ will be odd if and only if $\psi$ itself is odd. From now on we assume that $\psi$ is a fixed odd periodic function such that

$$|\psi(t)| = 1 \qquad \text{for } t \not\equiv 0 \pmod{p}. \tag{3.2}$$

Now fix $b_0 \in S^0$ such that $p \nmid \mathcal{N}\, b_0$. Find $x_i \in I_i$ such that $p \nmid \mathcal{N}\, x_i / \mathcal{N}\, I_i$; since $B$ is ramified at $p$, the maximal order at $p$ is unique, and so $b_i := x_i\, b_0\, x_i^{-1} \in S^0(\mathbb{Z}_l) = S_i^0(\mathbb{Z}_p)$. We define

$$\omega_p(I_i, b) := \omega_{\psi,b_i}(b),$$

to be the weight function for the triple $(S_i^0(\mathbb{Z}_p), b_i, \psi)$. In practice, one can use the same $b_0$ and $b_i$ for the definitions of both $\omega_l(I_i, \cdot)$ and $\omega_p(I_i, \cdot)$.

Note that different choices of $\psi$ will, in general, yield different forms $\Theta_{-l}([I_i])$, but as long as (3.2) holds their coefficients will be the same up to a constant of absolute value 1; thus formula (3.1) will not be affected. Moreover, given two such odd periodic functions it is not difficult to produce another *periodic* function $\chi$ with the property that the ratio of the $m$-th coefficients of the respective theta series will be $\chi(m)$.

The case when $\psi$ is actually a character of conductor $p$ is of particular interest, since the generalized theta series $\Theta_{-l}([I_i])$ will be a modular form of level $4p^2$ and character $\psi_1$, where $\psi_1(m) = \left(\frac{-1}{m}\right)\psi(m)$. From a computational point of view, however, it will always be preferable to choose a real $\psi$, whose values will be 0 or $\pm1$, and so that the coefficients of $\Theta_{-l}([I_i])$ will be rational integers. Only in case $p \equiv 3 \pmod{4}$ it is possible to satisfy both requirements at the same time, by taking for $\psi$ the quadratic character of conductor $p$.

# 4 Examples

## 4.1 11A

Let $f = f_{11A}$, the modular form of level 11, and consider $B = B(-1,-11)$, the quaternion algebra ramified precisely at $\infty$ and 11. A maximal order, and representatives for its right ideals classes, are given by

$$\begin{aligned} R = I_1 &= \left\langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_1 = 1, \\ I_2 &= \left\langle 2, 2i, \frac{1+2i+j}{2}, \frac{2+3i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_2 = 2. \end{aligned}$$

By computing the Brandt matrices, we find a vector

$$e_f = [I_2] - [I_1]$$

of height $\langle e_f, e_f \rangle = 5$ corresponding to $f$. Since $L(f,1) \approx 0.25384186$, Gross's method works, and it's easy to compute

$$\Theta_1(e_f) = \Theta_1([I_2]) - \Theta_1([I_1]) = q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + O(q^{20}),$$

as the difference of two regular theta series corresponding to the ternary quadratic forms (4.1) and (4.2).

### 4.1.1 Real twists in a case of rank 0

Let $l = 3$. One can compute $L(f,-3,1) \approx 1.6844963$, and thus expect $\Theta_{-3}(e_f)$ to be nonzero. We can choose $b_0 = i + k \in S^0$ with norm 12, and let $\psi = \chi_{11}$ be the quadratic character of conductor 11.

Clearly we can take $x_1 = 1$ and $x_2 = 2$, so that $n_1 = 1$, $n_2 = 2$ and $b_1 = b_2 = i + k$. Bases for $S^0_1$ and $S^0_2$ are given by

$$\begin{aligned} S^0_1 &= \langle 2i, j, i+k \rangle && \text{with } b_1 = (0,0,1), \\ S^0_2 &= \left\langle 4i, 2i+j, \frac{7i+k}{2} \right\rangle && \text{with } b_2 = (-3/2, 0, 2). \end{aligned}$$

The norm form in the given bases will be

$$\mathcal{N}_1(x_1,x_2,x_3) = 4x_1^2 + 11x_2^2 + 12x_3^2 + 4x_1x_3, \tag{4.1}$$
$$\mathcal{N}_2(x_1,x_2,x_3) = 16x_1^2 + 15x_2^2 + 15x_3^2 + 14x_2x_3 + 28x_1x_3 + 16x_1x_2. \tag{4.2}$$

This information is all that we need to compute $\Theta_{-3}$. As an example, we show how to compute $\Theta_{-3}([I_1])$. A simple calculation shows that

$$\langle (x_1,x_2,x_3), b_1 \rangle = 4x_1 + 24x_3 \equiv x_1 \pmod 3,$$

| $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.253842 | 92 | -5 | 0.661621 | 141 | -10 | 2.137734 |
| 5 | -5 | 2.838038 | 93 | 5 | 0.658054 | 152 | -10 | 2.058929 |
| 12 | -5 | 1.831946 | 97 | 5 | 0.644343 | 157 | -15 | 4.558227 |
| 37 | 5 | 1.043284 | 104 | 10 | 2.489124 | 168 | 10 | 1.958432 |
| 53 | 10 | 3.486786 | 113 | -5 | 0.596986 | 177 | 5 | 0.476998 |
| 56 | 10 | 3.392105 | 124 | -5 | 0.569892 | 181 | -15 | 4.245281 |
| 60 | -5 | 0.819271 | 133 | 10 | 2.201088 | 185 | -5 | 0.466571 |
| 69 | 15 | 6.875768 | 136 | 10 | 2.176676 | 188 | -10 | 1.851332 |
| 89 | -5 | 0.672680 | 137 | -5 | 0.542179 | | | |

Table 1: Coefficients of $\Theta_{-3}(e_f)$ and central values for $f = f_{11A}$

and thus $\omega_3(I_1, \cdot)$ can be computed by

$$\omega_3(I_1,(x_1,x_2,x_3)) = \begin{cases} 0 & \text{if } 3 \nmid \mathcal{N}_1(x_1,x_2,x_3), \\ \chi_3(x_1) & \text{if } x_1 \not\equiv 0 \pmod 3, \\ \chi_3(x_3) & \text{otherwise.} \end{cases}$$

Similarly, $\omega_{11}(I_1, \cdot)$ will be given by

$$\omega_{11}(I_1,(x_1,x_2,x_3)) = \chi_{11}(4x_1 + 2x_3).$$

Hence we compute

$$\Theta_{-3}([I_1]) = -2q^4 + 2q^5 + 2q^9 + 2q^{12} + 2q^{20} + 2q^{25} - 2q^{37} + O(q^{48}).$$

In a similar way one can easily get

$$\Theta_{-3}([I_2]) = q + q^4 - 3q^5 - 3q^{12} + 4q^{16} - 3q^{20} + 2q^{25} - 6q^{36} + 3q^{37} + O(q^{48}).$$

Table 1 shows the values of $c_{-3}(d)$ and $L(f,d,1)$, where $0 < d < 200$ is a fundamental discriminant such that $\left(\frac{d}{11}\right) = 1$. The formula

$$L(f,d,1) = k_{-3}\,\frac{c_{-3}(d)^2}{\sqrt{d}}$$

is satisfied, where

$$k_{-3} = \frac{1}{5} \cdot \frac{(f,f)}{L(f,-3,1)\sqrt{3}} = L(f,1) \approx 0.253841860855910684337758923350 9...$$

Note that when $\left(\frac{d}{11}\right) \neq 1$ it is trivial that $c_{-3}(d) = L(f,d,1) = 0$.

## 4.2 37 A

Let $f = f_{37A}$, the modular form of level 37 and rank 1, and consider $B = B(-2, -37)$, the quaternion algebra ramified precisely at $\infty$ and 37. A maximal order, and representatives for its right ideal classes, are given by

$$\begin{aligned} R = I_1 &= \left\langle 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right\rangle && \text{with } \mathcal{N}\, I_1 = 1, \\ I_2 &= \left\langle 2, 2i, \frac{1+3i+j}{2}, \frac{6+3i+k}{4} \right\rangle && \text{with } \mathcal{N}\, I_2 = 2, \\ I_3 &= \left\langle 4, 2i, \frac{3+3i+j}{2}, \frac{6+i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_3 = 4. \end{aligned}$$

By computing the Brandt matrices, we find a vector

$$e_f = \frac{[I_3] - [I_2]}{2}$$

of height $\langle e_f, e_f \rangle = 1/2$ corresponding to $f$. Since $L(f, 1) = 0$ we know that $2\,\Theta_1(e_f) = \Theta_1([I_3]) - \Theta_1([I_2]) = 0$. Indeed, one checks that $R_2$ and $R_3$ are conjugate, which explains the identity $\Theta_1([I_2]) = \Theta_1([I_3])$.

### 4.2.1 Imaginary twists in a case of rank 1

Let $l = 5$. One can compute $L(f, 5, 1) \approx 5.3548616$, and thus we expect $\Theta_5(e_f)$ to be nonzero. We note that, by the same reason that the orders are conjugate, we have $\Theta_5([I_3]) = -\,\Theta_5([I_2])$, except now there's an extra sign, ultimately coming from the fact that $\left(\frac{37}{5}\right) = -1$. Thus, $\Theta_5(e_f) = \Theta_5([I_3])$. A basis for $S_3^0$ is given by

$$S_3^0 = \left\langle 4i, 3i+j, \frac{3i+2j+k}{4} \right\rangle,$$

with the norm in this basis

$$\mathcal{N}_3\,(x_1, x_2, x_3) = 32x_1^2 + 55x_2^2 + 15x_3^2 + 46x_2x_3 + 12x_1x_3 + 48x_1x_2.$$

Choose $b_3 = (0, 0, 1)$, with norm 15. Then

$$\langle (x_1, x_2, x_3), b_3 \rangle = 12x_1 + 46x_2 + 30x_3 \equiv 2x_1 + x_2 \pmod{5}, \tag{4.3}$$

so that

$$\omega_5(I_3, (x_1, x_2, x_3)) = \begin{cases} 0 & \text{if } 5 \nmid \mathcal{N}_3\,(x_1, x_2, x_3), \\ \chi_5(2x_1 + x_2) & \text{if } 2x_1 + x_2 \not\equiv 0 \pmod{5}, \\ \chi_5(x_3) & \text{otherwise.} \end{cases}$$

| $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ |
|---|---|---|---|---|---|---|---|---|
| -3 | 1 | 2.830621 | -95 | 0 | 0.000000 | -139 | 0 | 0.000000 |
| -4 | 1 | 2.451389 | -104 | 0 | 0.000000 | -148 | -3 | 7.254107 |
| -7 | -1 | 1.853076 | -107 | 0 | 0.000000 | -151 | -2 | 1.595930 |
| -11 | 1 | 1.478243 | -111 | 1 | 0.930702 | -152 | -2 | 1.590671 |
| -40 | 2 | 3.100790 | -115 | -6 | 16.458713 | -155 | 2 | 1.575203 |
| -47 | -1 | 0.715144 | -120 | -2 | 1.790242 | -159 | 1 | 0.388816 |
| -67 | 6 | 21.562911 | -123 | 3 | 3.978618 | -164 | -1 | 0.382843 |
| -71 | 1 | 0.581853 | -127 | 1 | 0.435051 | -184 | 0 | 0.000000 |
| -83 | -1 | 0.538150 | -132 | 3 | 3.840589 | -195 | 2 | 1.404381 |
| -84 | -1 | 0.534937 | -136 | 4 | 6.726557 | | | |

Table 2: Coefficients of $\Theta_5(e_f)$ and central values for $f = f_{37A}$

Table 2 shows the values of $c_5(d)$ and $L(f,-d,1)$, where $-200 < -d < 0$ is a fundamental discriminant such that $\left(\frac{-d}{37}\right) \neq -1$. The formula

$$L(f,-d,1) = k_5 \, \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{37}\right) = +1, \\ 2 & \text{if } \left(\frac{-d}{37}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{37}\right) = -1, \end{cases}$$

is satisfied, where

$$k_5 = 2 \cdot \frac{(f,f)}{L(f,5,1)\sqrt{5}} \approx 4.902778763973580121708449663733...$$

Note that in the case $\left(\frac{-d}{37}\right) = -1$ it is trivial that $c_5(d) = L(f,-d,1) = 0$.

### 4.2.2 Real twists in a case of rank 1

Let $l = 3$, since $L(f,-3,1) \approx 2.9934586$. Keep $b_3$ as above, and let $\psi$ be the odd periodic function modulo 37 such that

$$\psi(x) = \begin{cases} +1 & \text{if } \ 1 \leqslant x \leqslant 18, \\ -1 & \text{if } 19 \leqslant x \leqslant 36. \end{cases}$$

Using again (4.3), we have that

$$\omega_3(I_1,(x_1,x_2,x_3)) = \begin{cases} 0 & \text{if } 3 \nmid \mathcal{N}_1\,(x_1,x_2,x_3), \\ \chi_3(x_2) & \text{if } x_2 \not\equiv 0 \pmod 3, \\ \chi_3(3) & \text{otherwise.} \end{cases}$$

and $\omega_{11}(I_1,\cdot)$ will be given by

$$\omega_{37}(I_1,(x_1,x_2,x_3)) = \psi(12x_1 + 9x_2 + 30x_3).$$

| $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ |
|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 5.354862 | 76 | 1 | 1.373493 | 133 | -1 | 1.038263 |
| 8 | 1 | 4.233390 | 88 | 1 | 1.276415 | 140 | -3 | 9.107764 |
| 13 | -1 | 3.320944 | 89 | -1 | 1.269224 | 156 | -1 | 0.958674 |
| 17 | 1 | 2.904081 | 92 | 2 | 4.993434 | 161 | -2 | 3.774681 |
| 24 | -1 | 2.444149 | 93 | 2 | 4.966515 | 165 | 1 | 0.932162 |
| 29 | 2 | 8.893941 | 97 | 0 | 0.000000 | 168 | -1 | 0.923801 |
| 56 | -1 | 1.600071 | 105 | 1 | 1.168527 | 172 | 1 | 0.912996 |
| 57 | 1 | 1.585973 | 109 | -1 | 1.146885 | 177 | 0 | 0.000000 |
| 60 | -1 | 1.545815 | 113 | 0 | 0.000000 | 193 | -1 | 0.861895 |
| 61 | 0 | 0.000000 | 124 | 0 | 0.000000 |  |  |  |
| 69 | 0 | 0.000000 | 129 | 1 | 1.054237 |  |  |  |

Table 3: Coefficients of $\Theta_{-3}(e_f)$ and central values for $f = f_{37A}$

Table 3 shows the values of $c_{-3}(d)$ and $L(f,d,1)$, where $0 < d < 200$ is a fundamental discriminant such that $\left(\frac{d}{37}\right) = -1$. The formula

$$L(f,d,1) = k_{-3}\,\frac{c_{-3}(d)^2}{\sqrt{d}}$$

is satisfied, where

$$k_{-3} = 2 \cdot \frac{(f,f)}{L(f,-3,1)\sqrt{3}} \approx 11.97383458492783851932803991781...$$

Note that in the case $\left(\frac{d}{37}\right) \neq -1$ it is trivial that $c_{-3}(d) = L(f,d,1) = 0$.

## 4.3 43A

Let $f = f_{43A}$, the modular form of level 43 and rank 1. Let $B = B(-1,-43)$, the quaternion algebra ramified precisely at $\infty$ and 43. A maximal order, and representatives for its right ideals classes, are given by

$$\begin{aligned}
R = I_1 &= \left\langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_1 = 1,\\
I_2 &= \left\langle 2, 2i, \frac{1+2i+j}{2}, \frac{2+3i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_2 = 2,\\
I_3 &= \left\langle 3, 3i, \frac{1+2i+j}{2}, \frac{2+5i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_3 = 3,\\
I_4 &= \left\langle 3, 3i, \frac{1+4i+j}{2}, \frac{4+5i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_4 = 3.
\end{aligned}$$

By computing the Brandt matrices, we find a vector

$$e_f = \frac{[I_4] - [I_3]}{2}$$

| $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ |
|---|---|---|---|---|---|---|---|---|
| -3 | 1 | 3.148135 | -91 | -1 | 0.571601 | -151 | -1 | 0.443737 |
| -7 | 1 | 2.060938 | -104 | 1 | 0.534684 | -155 | -1 | 0.437974 |
| -8 | -1 | 1.927831 | -115 | -3 | 4.576227 | -159 | 1 | 0.432430 |
| -19 | 2 | 5.003768 | -116 | -1 | 0.506273 | -163 | 7 | 20.927447 |
| -20 | -1 | 1.219267 | -119 | -1 | 0.499851 | -168 | -2 | 1.682749 |
| -39 | -1 | 0.873136 | -120 | 0 | 0.000000 | -179 | -1 | 0.407556 |
| -43 | 2 | 6.652268 | -123 | -5 | 12.291402 | -184 | -3 | 3.617825 |
| -51 | 1 | 0.763535 | -131 | 0 | 0.000000 | -191 | 0 | 0.000000 |
| -55 | 1 | 0.735246 | -132 | 3 | 4.271393 | -199 | 0 | 0.000000 |
| -71 | 0 | 0.000000 | -136 | -1 | 0.467568 | | | |
| -88 | 3 | 5.231366 | -148 | -4 | 7.171386 | | | |

Table 4: Coefficients of $\Theta_5(e_f)$ and central values for $f = f_{43A}$

of height $\langle e_f, e_f \rangle = 1/2$ corresponding to $f$.

### 4.3.1 Imaginary twists in a case of rank 1

We can use $l = 5$, since $L(f,5,1) \approx 4.8913446$ is nonzero; again, we find $\Theta_5(e_f) = \Theta_5([I_4])$. Table 4 shows the values of $c_5(d)$ and $L(f,-d,1)$, where $-200 < -d < 0$ is a fundamental discriminant such that $\left(\frac{-d}{43}\right) \neq -1$. The formula

$$L(f,-d,1) = k_5 \, \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{43}\right) = +1, \\ 2 & \text{if } \left(\frac{-d}{43}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{43}\right) = -1, \end{cases}$$

is satisfied, where

$$k_5 = 2 \cdot \frac{(f,f)}{L(f,5,1)\sqrt{5}} \approx 5.45272967268173438557072278528\text{3...}$$

Note that in the case $\left(\frac{-d}{43}\right) = -1$ it is trivial that $c_5(d) = L(f,-d,1) = 0$.

### 4.3.2 Real twists in a case of rank 1

We can use $l = 3$, since $L(f,-3,1) \approx 3.1481349$, and let $\psi = \chi_{43}$ be the quadratic character of conductor 43. Table 5 shows the values of $c_{-3}(d)$ and $L(f,d,1)$, where $0 < d < 200$ is a fundamental discriminant such that $\left(\frac{d}{43}\right) = -1$. The formula

$$L(f,d,1) = k_{-3} \, \frac{c_{-3}(d)^2}{\sqrt{d}}$$

is satisfied, where

$$k_{-3} = 2 \cdot \frac{(f,f)}{L(f,-3,1)\sqrt{3}} \approx 10.937379059935167648758735438779\text{...}$$

Note that in the case $\left(\frac{d}{43}\right) \neq -1$ it is trivial that $c_{-3}(d) = L(f,d,1) = 0$.

| $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ | $d$ | $c_{-3}(d)$ | $L(f,d,1)$ |
|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 4.891345 | 76 | 0 | 0.000000 | 137 | 2 | 3.737773 |
| 8 | -1 | 3.866947 | 77 | -3 | 11.217870 | 141 | -2 | 3.684374 |
| 12 | 1 | 3.157349 | 85 | 1 | 1.186325 | 149 | 0 | 0.000000 |
| 28 | -1 | 2.066970 | 88 | -1 | 1.165929 | 156 | 1 | 0.875691 |
| 29 | -1 | 2.031020 | 89 | 1 | 1.159360 | 157 | 2 | 3.491592 |
| 33 | -1 | 1.903953 | 93 | 3 | 10.207380 | 161 | -1 | 0.861986 |
| 37 | 2 | 7.192376 | 104 | 1 | 1.072498 | 168 | -2 | 3.375348 |
| 61 | 1 | 1.400388 | 105 | 0 | 0.000000 | 177 | -2 | 3.288415 |
| 65 | -1 | 1.356615 | 113 | -2 | 4.115608 | 184 | 1 | 0.806314 |
| 69 | -1 | 1.316706 | 120 | 0 | 0.000000 | | | |
| 73 | 1 | 1.280123 | 136 | 1 | 0.937873 | | | |

Table 5: Coefficients of $\Theta_{-3}(e_f)$ and central values for $f = f_{43A}$

## 4.4 389A

Let $f = f_{389A}$, the modular form of level 389 and rank 2. Let $B = B(-2,-389)$, the quaternion algebra ramified precisely at $\infty$ and 389. A maximal order, with 33 ideal classes, is given by

$$R = \left\langle 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right\rangle .$$

There is a vector $e_f$ of height $\langle e_f, e_f \rangle = {}^5\!/_2$ corresponding to $f$.

### 4.4.1 Imaginary twists in a case of rank 2

We can use $l = 5$, since $L(f,5,1) \approx 8.9092552$. We have omitted the 33 ideal classes; however, the computation of $\Theta_l(e_f)$ involves only 14 distinct theta series. In table 6 we give the value of $e_f$ and the coefficients of the norm form $\mathcal{N}_i$ and of $b_i$ on chosen bases of $S_i^0$.

Each row in the table allows one to compute an individual theta series

$$h_i(z) := \frac{1}{2} \sum_{b \in \mathbb{Z}^3} w_5(I_i, b) q^{\mathcal{N}_i(b)/5}.$$

The ternary form corresponding to a sextuple $(A_1, A_2, A_3, A_{23}, A_{13}, A_{12})$ is

$$\mathcal{N}_i(x_1,x_2,x_3) = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + A_{23} x_2 x_3 + A_{13} x_1 x_3 + A_{12} x_1 x_2,$$

and $\omega_5(I_i, \cdot)$ is the weight function of the pair $(\mathbb{Z}^3, b_i)$. As an example, we show how to compute $h_1(z)$. First, we have

$$\mathcal{N}_1(x_1,x_2,x_3) = 15x_1^2 + 107x_2^2 + 416x_3^2 - 100x_2x_3 - 8x_1x_3 - 14x_1x_2.$$

A simple calculation shows that

$$\langle (x_1,x_2,x_3), (2,4,0) \rangle \equiv 4x_1 + 3x_2 + 4x_3 \pmod 5.$$

| $i$ | $a_i$ | $\mathcal{N}_i$ | $b_i$ |
|---|---|---|---|
| 1 | 1/2 | 15, 107, 416, -100, -8, -14 | 2, 4, 0 |
| 2 | -1/2 | 15, 104, 415, 104, 2, 4 | 0, 4, 1 |
| 3 | -1/2 | 23, 136, 203, 68, 2, 8 | 2, 1, 4 |
| 4 | 1/2 | 23, 72, 407, 72, 10, 20 | 1, 1, 0 |
| 5 | -1/2 | 31, 51, 407, -46, -26, -10 | 1, 2, 0 |
| 6 | 1/2 | 31, 103, 204, 56, 20, 18 | 2, 0, 3 |
| 7 | 1/2 | 39, 128, 160, -116, -8, -36 | 1, 1, 4 |
| 8 | -1/2 | 39, 40, 399, 40, 2, 4 | 1, 0, 1 |
| 9 | 1/2 | 40, 47, 399, 18, 40, 36 | 4, 3, 0 |
| 10 | -1/2 | 47, 107, 135, 42, 22, 38 | 4, 3, 1 |
| 11 | -1/2 | 56, 84, 139, 56, 4, 12 | 3, 1, 4 |
| 12 | 1/2 | 56, 92, 151, 76, 52, 44 | 4, 2, 3 |
| 13 | 1/2 | 71, 83, 132, -16, -12, -70 | 2, 3, 4 |
| 14 | -1/2 | 71, 103, 124, -36, -64, -66 | 4, 0, 2 |

Table 6: Coefficients of the ternary forms and of $b_i$

Thus, $\omega_5$ can be computed as

$$\omega_5(I_1,(x_1,x_2,x_3)) = \begin{cases} 0 & \text{if } 5 \nmid \mathcal{N}_1(x_1,x_2,x_3), \\ \chi_5(4x_1+3x_2+4x_3) & \text{if } \not\equiv 0 \pmod 5, \\ \chi_5(x_2) & \text{otherwise,} \end{cases}$$

and we have

$$h_1(z) = q^3 - q^{12} - q^{27} + q^{39} + q^{40} + q^{48} - q^{83} - 2q^{92} + O(q^{100}).$$

Finally, we combine all of the theta series in

$$\Theta_5(e_f) = \sum_{i=1}^{14} a_i h_i(z)$$

Table 7 shows the values of $c_5(d)$ and $L(f,-d,1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{389}\right) \neq +1$. The formula

$$L(f,-d,1) = k_5\,\frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{389}\right) = -1, \\ 2 & \text{if } \left(\frac{-d}{389}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{389}\right) = +1, \end{cases}$$

is satisfied, where

$$k_5 = \frac{2}{5} \cdot \frac{(f,f)}{L(f,5,1)\sqrt{5}} \approx 7.88695080620659281768963079260 5...$$

Note that when $\left(\frac{-d}{389}\right) = +1$ it is trivial that $c_5(d) = L(f,-d,1) = 0$.

| $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ |
|---|---|---|---|---|---|---|---|---|
| -3 | 1 | 4.553533 | -83 | -1 | 0.865705 | -139 | -1 | 0.668962 |
| -8 | -1 | 2.788458 | -84 | 1 | 0.860537 | -148 | 6 | 23.338921 |
| -15 | -1 | 2.036402 | -88 | -4 | 13.452028 | -151 | 2 | 2.567324 |
| -23 | 1 | 1.644543 | -103 | 0 | 0.000000 | -152 | -1 | 0.639716 |
| -31 | 1 | 1.416538 | -104 | -1 | 0.773379 | -155 | 3 | 5.701456 |
| -39 | 1 | 1.262923 | -107 | 0 | 0.000000 | -163 | 8 | 39.536232 |
| -40 | 1 | 1.247036 | -115 | -1 | 0.735462 | -167 | -1 | 0.610311 |
| -43 | -3 | 10.824738 | -116 | -2 | 2.929140 | -191 | 1 | 0.570680 |
| -47 | 0 | 0.000000 | -123 | 3 | 6.400282 | -195 | 1 | 0.564796 |
| -51 | -2 | 4.417576 | -131 | 1 | 0.689086 | -199 | -1 | 0.559091 |
| -56 | 1 | 1.053938 | -132 | -2 | 2.745884 | | | |
| -71 | 1 | 0.936009 | -136 | -2 | 2.705202 | | | |

Table 7: Coefficients of $\Theta_5(e_f)$ and central values for $f = f_{389A}$

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Zhengyu Mao
Department of Mathematics and Computer Science
Rutgers university
Newark, NJ 07102-1811
zmao@andromeda.rutgers.edu

Fernando Rodriguez-Villegas
Gonzalo Tornaría
Department of Mathematics,
University of Texas at Austin,
Austin, TX 78712 USA
villegas@math.utexas.edu
tornaria@math.utexas.edu