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A TRACE FORMULA FOR DUAL PAIRS

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1. Introduction. Let F be a number field and $\mathbf{A}$ its adele ring. Let G' be a simple, split, simply laced group defined over F. Assume that G' is not of type A_n. There is a dual pair of reductive groups G and SL_2 in G'; i.e., $G \times SL_2$ embeds into G', with G and SL_2 satisfying the double centralizer property in G'. In this paper, we propose a study on the correspondence between the automorphic forms on $G(\mathbf{A})$ and the automorphic forms on $SL_2(\mathbf{A})$ using the trace formula.

Let Δ be the set of simple roots of G'. Let $\tilde{\alpha}$ be a highest-weight root of G'. Let α be the unique simple root that is not perpendicular to $\tilde{\alpha}$. Let P' be the maximal parabolic subgroup of G' corresponding to $\Delta \backslash \{\alpha\}$, with $P' = M'U'$ being its Levi decomposition. Then U' is a Heisenberg group with a one-dimensional center Z. The quotient space U'/Z has the structure of a symplectic vector space. The adjoint action of the group M' on U' factors to a map $j\colon M' \to GSp(U'/Z)$. Let G be the derived group of M'. Then $j(G)$ lies in $Sp(U'/Z)$, and (G, SL_2) is a dual pair in G'. If G' is of type E_i, $i = 6, 7, 8$, or D_m, then G is semisimple split of the type A_5, D_6, E_7, or $A_1 \times D_{m-2}$.

The correspondence between the automorphic forms on $G(\mathbf{A})$ and $SL_2(\mathbf{A})$ can be studied using the *theta kernel*. For the group G', one can construct an automorphic θ-module π (see [GRS2]). Here $\pi = \bigotimes \pi_v$, where each local component π_v is of class-one and of the smallest Gelfand-Kirillov dimension. At a finite place v, the representation π_v is the unique minimal representation constructed in [K], [KS], and [S]. If v is Archimedean, such a representation π_v is considered in [V]. The θ-module is an analogue of the θ-representation for the twofold cover of Sp_n constructed by A. Weil [W]. As is the case for Weil's construction, the θ-module can be realized in the residue spectrum of $L^2(G'(F)\backslash G'(\mathbf{A}))$. For any dual pair (H_1, H_2) in G', restricting the θ-module to $H_1 \times H_2$ gives a θ-correspondence between the automorphic forms on H_1 and these on H_2.

Our trace formula approach will not make use of the θ-module. However, the trace formula is directly inspired by the results on θ-correspondence. We summarize some of the works done on the θ-correspondence in the case G' is of type G_2 (not simply laced); see [GRS1]. Here the θ-module is an automorphic representation of $\tilde{G}'$, the threefold cover of G', constructed by Savin in [S]. The restriction of the θ-module to the dual pair $SL_2 \times \widetilde{SL}_2^3$ (threefold cover for the second component) produces a decomposition of the form $\bigoplus \pi \otimes \theta(\pi)$. Here π and $\theta(\pi)$ are cuspidal representations of $SL_2(\mathbf{A})$ and $\widetilde{SL}_2^3(\mathbf{A})$, respectively, with

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π being the cubic lifting of $\theta(\pi)$. More precisely, let i be the embedding of $SL_2(\mathbf{A}) \times \widetilde{SL}_2^3(\mathbf{A})$ in $\tilde{G}'(\mathbf{A})$; let π be a cuspidal representation of SL_2. Define for $\varphi \in \theta$, $f \in \pi$, and $h \in \widetilde{SL}_2^3(\mathbf{A})$,

$$\theta_{\varphi,f}(h) = \int_{SL_2(F)\backslash SL_2(\mathbf{A})} \varphi(i(g,h))f(g)\,dg. \tag{1}$$

Then $\theta(\pi)$ is the space spanned by the functions $\theta_{\varphi,f}(h)$. Now let ψ be a nontrivial additive character on $\mathbf{A}/F$, write $n(t)$ for the element $\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$ in SL_2 and $\tilde{n}(t)$ for the element $(n(t), 1)$ in $\widetilde{SL}_2^3$. The Fourier coefficient of $\theta_{\varphi,f}$ is

$$\theta^{\psi}_{\varphi,f}(h) = \int_{\mathbf{A}/F} \theta_{\varphi,f}(\tilde{n}(t)\cdot h)\psi(t)\,dt. \tag{2}$$

One has a nice formula for this Fourier coefficient. Let $s^3\colon SL_2 \to Sp_2$ be the sym^3 representation of SL_2 (which can be identified with the map j above). Let ρ_ψ be the Weil representation of $\widetilde{Sp}_2$ associated to ψ. It acts on $S(\mathbf{A}^2)$, the space of Schwartz functions on $\mathbf{A}^2$. Since $\widetilde{Sp}_2$ splits over the image of s^3, ρ_ψ restricts to a representation of $s^3(SL_2)$. For $\phi \in S(\mathbf{A}^2)$, define for $g \in SL_2(\mathbf{A})$:

$$\Theta^{\phi}_{\psi}(s^3(g)) = \sum_{X\in F^2} \rho_\psi(s^3(g))\phi(X). \tag{3}$$

One of the main results in [GRS1] is the following formula: given $\varphi \in \theta$, there exists a $\phi_\varphi \in S(\mathbf{A}^2)$ such that

$$\theta^{\psi}_{\varphi,f}(e) = \int_{SL_2(F)\backslash SL_2(\mathbf{A})} \Theta^{\phi_\varphi}_{\psi}(s^3(g))f(g)\,dg. \tag{4}$$

In particular, if the integral on the right-hand side of (4) is nonvanishing for some $f \in \pi$ and $\phi_\varphi \in S(\mathbf{A}^2)$, the space $\theta(\pi)$ is nontrivial.

We now state the trace formula. Back to the situation in the second paragraph: Let Y be a maximal isotropic subspace of U'/Z defined over F. Let ρ_ψ be the Weil representation of $\widetilde{Sp}(U'/Z)$ acting on $S(Y(\mathbf{A}))$. It is known that $\widetilde{Sp}(U'/Z)$ splits over $j(G)$. Thus ρ_ψ restricts to a representation of $j(G)$. For $\phi \in S(Y(\mathbf{A}))$, $g \in G(\mathbf{A})$, define:

$$\Theta^{\phi}_{\psi}(j(g)) = \sum_{X\in Y(F)} \rho_\psi(j(g))\phi(X). \tag{5}$$

Let Δ_1 be the set of simple roots β which are not perpendicular to α. Let $P = MU$ be the parabolic subgroup of G associated to $\Delta\backslash\Delta_1$. Then with our assumption on

G', there is a canonical bijection from the unipotent group U to a split degree-3 Jordan algebra $\mathcal{J}$ (not necessarily simple) [Ru]. Write this map as $u \to J_u$. Let Tr be the trace form on $\mathcal{J}$. Recall for $f \in C_c^\infty(G(\mathbb{A}))$, we have a kernel function

$$K_f(x,y) = \sum_{\xi \in G(F)} f(x^{-1}\xi y).$$

With the above notations, define the distribution

$$I(f,\phi) = \int_{U(F)\backslash U(\mathbb{A})} \int_{G(F)\backslash G(\mathbb{A})} K_f(g,u)\Theta_\psi^\phi(j(g))\psi(\mathrm{Tr}(J_u))\,dg\,du. \tag{6}$$

On the other hand, for $f' \in C_c^\infty(SL_2(\mathbb{A}))$, define

$$K_{f'}(x,y) = \sum_{\xi \in SL_2(F)} f'(x^{-1}\xi y).$$

We define the distribution

$$J(f') = \int_{(\mathbb{A}/F)^2} K_{f'}(n(-x),n(y))\psi(x+y)\,dx\,dy. \tag{7}$$

We expect the following trace formula to hold: for *matching* functions f, ϕ, f', one has

$$I(f,\phi) = J(f'). \tag{8}$$

Here the concept of matching is, as usual, defined in terms of conditions over local fields.

The spectral decomposition of $I(f,\phi)$ is roughly a sum over the set of automorphic representations π of

$$I_\pi(f,\phi) = \sum_{\varphi_i} P(\phi,\varphi_i)\overline{W}_{\pi(f)\varphi_i}(e), \tag{9}$$

where $\{\varphi_i\}$ is an orthonormal basis for π and

$$\begin{aligned} P(\phi,f) &= \int_{G(F)\backslash G(\mathbb{A})} \Theta_\psi^\phi(j(g))f(g)\,dg \\ W_f(g) &= \int_{U(F)\backslash U(\mathbb{A})} f(ug)\psi(\mathrm{Tr}(J_u))\,du. \end{aligned} \tag{10}$$

Meanwhile, the distribution $J(f')$ is roughly a sum over the set of cuspidal representations π' of certain distributions $J_{\pi'}(f')$. From equation (8), there will be identities between the (sums of) nontrivial distributions $I_{\pi}(f,\phi)$ and the (sums of) nontrivial distributions $J_{\pi'}(f')$. The correspondence between automorphic forms on G and SL_2 follows from these identities. For a general discussion on trace formula and its applications, we refer to the articles [A], [JLR], and [JY].

The above definition of $I(f,\phi)$ can be done in the case when G' is of type B_n, G_2, or F_4, as well. In [MR], we will prove equation (8) in the case when G' is of type G_2, with $J(f')$ being defined as a distribution on $\widetilde{SL}_2^3$ instead of SL_2. By comparing the spectral decomposition of $I(f,\phi)$ and $J(f')$, we get a correspondence between the automorphic forms on $G = SL_2$ and on $\widetilde{SL}_2^3$. This correspondence is indeed the same one obtained by θ-correspondence in [GRS1].

In [J2], a relative trace formula is used to establish the Waldspurger correspondence between cuspidal representations on PGL_2 and $\widetilde{SL}_2^2$, the double cover of SL_2. We will show that when G' is SO_7 (type B_3), the corresponding trace identity (8) is in essence the relative trace formula stated in [J2, §7].

Our main goal here is to prove the fundamental lemma for the trace formula (8). If $f = \bigotimes f_v, \phi = \bigotimes \phi_v, f' = \bigotimes f_v'$, we can unwind the integrals (6) and (7) as follows (see §3):

$$I(f,\phi) = \sum_{a\in F^\times} \prod_v I_v(a_v, f_v * \phi_v) + \prod_v I_v^+(f_v * \phi_v) + \prod_v I_v^-(f_v * \phi_v) \tag{11}$$

$$J(f') = \sum_{a\in F^\times} \prod_v J_v(a_v, f_v') + \prod_v J_v^+(f_v') + \prod_v J_v^-(f_v'). \tag{12}$$

The fundamental lemma asserts a relation between $I_v(a_v, f_v * \phi_v)$ and $J_v(a_v, f_v')$ when f_v, ϕ_v, f_v' are *unit elements* (see §3). The following equality is the key to this relation: Let n be the dimension of $\mathcal{J}$ over F, and let R_v be the ring of integers when v is finite; then for all finite places v, with $|a| < 1$, with the usual Haar measures,

$$\int_{A\in\mathcal{J}(R_v)} \psi\left(\frac{\operatorname{Tr}(A) - N(A)}{a}\right) dA = |a|^{(n-1)/2} \int_{|x|=1} \psi\left(\frac{x + x^{-1}}{a}\right) dx, \tag{13}$$

where $N(A)$ is the norm of A. The equation (13) is an equality between two exponential sums, with the right-hand side being a Kloosterman sum.

In [KS], the function

$$\phi_r(t,a,A) = |ta^{-1}|^r \psi\left(\frac{N(A)}{a}\right)$$

is considered (r some fixed integer). To construct the minimal representation of G' over a finite place v, the key step is to establish the self-duality of the function $\phi_r(t,a,A)$. Since we avoid the minimal representation in the statement of the trace formula, it shall not be surprising that our proof of equation (13) uses all the ingredients that appear in the proof of the self-duality of $\phi_r(t, a, A)$. The self-duality of $\phi_r(t, a, A)$ apparently has some relation to the theory of mirror symmetry.

In §2, we collect some preliminaries on the Jordan algebra. In §3, we unwind the integrals (6) and (7). The fundamental lemma is proved in §4. In §5, we consider the cases when G' is of the type G_2 or F_4. In §6, we consider the finite field analogue of the equation (13). The case when $G' = SO_7$ is considered in §7, where a relationship between the identity (8) and the trace formula in [J2] is explained.

2. Jordan algebra. In this section, F will be either a number field or a local field with characteristic $\neq 2, 3$. Keep the notation in the introduction. Fix the choice of simple roots for G'. Let $P = MU \subset G(F)$, as in the introduction. From the construction in [MRS], there is a canonical identification between U and $\mathcal{J}$ a split degree-3 Jordan algebra. We now collect some facts about $\mathcal{J}$. Most of the results here are included in [KS] or [Ru].

On the set $\mathcal{J}$ there is a commutative but not associative multiplication $(A, B) \to A \cdot B$. Given an element $A \in \mathcal{J}$, there exists a generic minimal polynomial

$$P_A(x) = x^3 - \mathrm{Tr}(A)x^2 + R(A)x - N(A),$$

where $\mathrm{Tr}(A)$ and $N(A)$ are defined to be the trace and norm of A; and

$$R(A) = \frac{1}{2}[\mathrm{Tr}(A)^2 - \mathrm{Tr}(A \cdot A)].$$

The norm form $N(A)$ is homogeneous of degree 3. Define

$$A \times A = A \cdot A - \mathrm{Tr}(A)A + R(A)I,$$

where I is the identity. As $P_A(A) = 0$, we have $(A \times A) \cdot A = N(A)I$.

For γ a root in G', let e_γ be the one-parameter subgroup of G' corresponding to γ. Then $Z = e_{\tilde{\alpha}}$. For $x \in F$, we will denote by $e_\gamma(x)$ the corresponding element in e_γ.

Let $\langle\, ,\rangle$ be the bilinear form on the set of roots in G'. Let Σ be the set of roots γ such that $\langle \tilde{\alpha}, \gamma\rangle = 1$. Since the Lie algebra of U' is spanned by the roots γ with $\langle \tilde{\alpha}, \gamma\rangle \neq 0$, it follows that $U' \cong Z\prod_{\gamma\in\Sigma} e_\gamma$.

The set Σ has a natural involution $\gamma^* = \tilde{\alpha} - \gamma$. Let

$$\Pi_0 = \{\alpha\} \cup \{\gamma \in \Sigma | \langle \alpha, \gamma\rangle = 1\}.$$

Then $\Sigma = \Pi_0 \cup \Pi_0^*$ a disjoint union, and $\prod_{\gamma\in\Pi_0} e_\gamma$ is isomorphic to a maximal isotropic subspace of U'/Z. Thus we can consider the Weil representation ρ_ψ of $\widetilde{Sp}(U'/Z)$ as acting on the space $S(\prod_{\gamma\in\Pi_0} e_\gamma)$.

Write $\Pi_0' = \Pi_0 \backslash \{\alpha\}$. One key fact we use is

$$U = \prod_{\gamma \in \Pi_0'} e_{\gamma - \alpha}. \tag{14}$$

Therefore as vector spaces over F

$$\prod_{\gamma \in \Pi_0} e_\gamma \cong F \oplus U \cong F \oplus \mathcal{J}.$$

We may consider the Weil representation as acting on $S(F \oplus \mathcal{J})$. With this identification, we have the following formula [Ru].

LEMMA 1. *If $u \in U$, for $\phi \in S(F \oplus \mathcal{J})$, $a \in F, A \in \mathcal{J}$,*

$$\rho_\psi(j(u))\phi(a, A) = \psi\left(\frac{N(A) - N(A + aJ_u)}{a}\right)\phi(a, A + aJ_u), \qquad a \neq 0 \tag{15}$$

$$\rho_\psi(j(u))\phi(0, A) = \psi[-\mathrm{Tr}(J_u \cdot (A \times A))]\phi(0, A). \tag{16}$$

We will need an explicit formula for $N(A)$ and $\mathrm{Tr}(A)$ for $A \in \mathcal{J}$. From (14), one may identify the set U, thus the set $\mathcal{J}$ with $F^{\Pi_0'}$ by

$$\prod_{\Pi_0'}(x_\gamma) \to u = \prod_{\Pi_0'} e_{\gamma-\alpha}(x_\gamma) \to J_u.$$

Under this identification, for $A = \prod(x_\gamma) \in \mathcal{J}$,

$$N(A) = \sum_{\{S\}} c_S x_{\eta_1} x_{\eta_2} x_{\eta_3}, \tag{17}$$

where $\{S\} = \{S = (\eta_1, \eta_2, \eta_3) \subset \Pi_0' | \eta_1 + \eta_2 + \eta_3 = \tilde{\alpha} + \alpha\}$, c_S is a number that equals either 1 or -1 [KS].

If $I = \prod(I_\gamma) \in \mathcal{J}$, then $I_\gamma \neq 0$ for three roots ζ_i, $i = 1, 2, 3$, with $\sum_i \zeta_i = \alpha + \tilde{\alpha}$. The trace of A then equals

$$\mathrm{Tr}(A) = \sum_{i=1,2,3} x_{\zeta_i}.$$

From [MRS], we see that one of the roots $\zeta_i - \alpha$ is a simple root β of G' with $\langle \beta, \alpha \rangle = -1$. We may assume $\beta = \zeta_1 - \alpha$. Let $\Pi' = \{\gamma \in \Pi_0 | \langle \beta, \gamma \rangle = 0\}$. There is a decomposition of Π_0 into a disjoint union [KS]:

$$\Pi_0 = \{\alpha, \alpha + \beta\} \cup \Pi' \cup \{\gamma_1, \ldots, \gamma_r\} \cup \{\delta_1, \ldots, \delta_r\}, \tag{18}$$

where for all i, $\langle \gamma_i, \beta \rangle = \langle \delta_i, \beta \rangle = -1$, and $\gamma_i + \delta_i + \beta = \tilde{\alpha}$. Since

$$\langle \zeta_2, \beta \rangle + \langle \zeta_3, \beta \rangle = \langle \tilde{\alpha}, \beta \rangle - \langle \beta, \beta \rangle = -2,$$

we have for $i = 2, 3$, $\langle \zeta_i, \beta \rangle = -1$. Without loss of generality, we may assume $\zeta_2 = \gamma_1$ in the above decomposition; then $\zeta_3 = \delta_1$ and

$$\operatorname{Tr}(A) = x_{\alpha+\beta} + x_{\gamma_1} + x_{\delta_1}. \tag{19}$$

We remark also that in (17), $c_S = 1$ if $S = (\alpha + \beta, \gamma_1, \delta_1)$. This follows from the identity $N(I) = 1$.

3. The distributions *I* and *J*. In this section, F is a number field and $\mathbf{A}$ its adele ring. From §2, we can identify the maximal isotropic space of U'/Z with $F \oplus \mathscr{J}$. Recall for $f \in C_c^\infty(G(\mathbf{A}))$ and $\phi \in S(\mathbf{A} \oplus \mathscr{J}(\mathbf{A}))$,

$$I(f, \phi) = \int_{U(F)\backslash U(\mathbf{A})} \int_{G(F)\backslash G(\mathbf{A})} K_f(g, u) \Theta_\psi^\phi(j(g)) \psi(\operatorname{Tr}(J_u))\, dg\, du \tag{20}$$

with

$$K_f(g, u) = \sum_{\xi \in G(F)} f(g^{-1}\xi u)$$

$$\Theta_\psi^\phi(j(g)) = \sum_{X \in F \oplus \mathscr{J}(F)} \rho_\psi(j(g))\phi(X).$$

The following proposition justifies our computations below.

Proposition 1. *The integral*

$$\int_{U(F)\backslash U(\mathbf{A})} \int_{G(F)\backslash G(\mathbf{A})} K_{|f|}(g, u) \sum_{X \in F \oplus \mathscr{J}(F)} |\rho_\psi(j(g))\phi(X)|\, dg\, du \tag{21}$$

converges.

Proof. We use the following result in [J] (Proposition 2.1, which is proved for f a K-finite function, but holds for any $f \in C_c^\infty(G(\mathbf{A}))$).

Proposition 2. *Suppose Ω_0 is a compact set of $G(\mathbf{A})$. Given a Siegel set $\mathscr{S}$ in $G(\mathbf{A})$ and an integer N, there is a c such that for $y \in \Omega_0$ and $x \in \mathscr{S}$:*

$$|K_f(x, y)| \leqslant c\|x\|^{-N}.$$

Here $\|\cdot\|$ is a height function on $G(\mathbf{A})$ defined for a finite-dimensional rational representation of G. From the proposition, over the domain for (21), $K_{|f|}(g, u)$

is rapidly decreasing in g, uniformly with respect to u. It is known that $\sum_{Y\in F\oplus\mathcal{J}(F)}|\rho_\psi(j(g))\phi(Y)|$ is moderately increasing, bounded by $c'\|j(g)\|_*^{N'}$ for some fixed c', N', $\|\cdot\|_*$ being a height function on $Sp(U'/Z)$. As j is a finite-dimensional rational representation of G, there is certainly a relationship $\|j(g)\|_* \leqslant c''\|g\|^{N''}$ for some c'', N'' [A]. Thus the integrand in (21) is a rapidly decreasing function in g, uniformly with respect to u. The integral thus equals a finite number. □

With the proposition, we can change the order of sum and integration in (20) as we wish. Since $\Theta_\psi^\phi(j(g))$ is left $G(F)$-invariant, we get

$$I(f,\phi) = \int_{U(F)\backslash U(\mathbb{A})}\int_{G(F)\backslash G(\mathbb{A})}\sum_{\xi\in G(F)} f(g^{-1}\xi u)\Theta_\psi^\phi(j(g))\psi(\mathrm{Tr}(J_u))\,dg\,du$$

$$= \int_{U(F)\backslash U(\mathbb{A})}\int_{G(\mathbb{A})} f(g^{-1}u)\Theta_\psi^\phi(j(g))\psi(\mathrm{Tr}(J_u))\,dg\,du.$$

Change $g \longrightarrow ug$:

$$I(f,\phi) = \int_{U(F)\backslash U(\mathbb{A})}\int_{G(\mathbb{A})} f(g^{-1})\Theta_\psi^\phi(j(ug))\psi(\mathrm{Tr}(J_u))\,dg\,du$$

$$= \sum_{a\in F, A\in\mathcal{J}(F)}\int_{U(F)\backslash U(\mathbb{A})}\int_{G(\mathbb{A})} f(g^{-1})\rho_\psi(j(u)j(g))\phi(a,A)\psi(\mathrm{Tr}(J_u))\,dg\,du.$$

Let

$$f*\phi(a,A) = \int_{G(\mathbb{A})} f(g^{-1})\rho_\psi(j(g))\phi(a,A)\,dg. \tag{22}$$

Then $I(f,\phi)$ becomes

$$\sum_{a\in F, A\in\mathcal{J}(F)}\int_{U(F)\backslash U(\mathbb{A})}\rho_\psi(j(u))f*\phi(a,A)\psi(\mathrm{Tr}(J_u))\,du.$$

Apply Lemma 1, and use J_u instead of u as the independent variable. The integral $I(f,\phi)$ becomes a sum

$$\sum\int_{\mathcal{J}(F)\backslash\mathcal{J}(\mathbb{A})} f*\phi(a,A+aJ)\psi\left(\frac{N(A)-N(A+aJ)}{a}\right)\psi(\mathrm{Tr}(J))\,dJ \tag{23}$$

$$+\sum_{A\in\mathcal{J}(F)}\int_{\mathcal{J}(F)\backslash\mathcal{J}(\mathbb{A})} f*\phi(0,A)\psi[-\mathrm{Tr}(J\cdot(A\times A))]\psi(\mathrm{Tr}(J))\,dJ, \tag{24}$$

where the sum in (23) is over $a \in F^\times, A \in \mathcal{J}(F)$. In (24), for the integral over J to be nonzero, the linear form $\mathrm{Tr}(J) - \mathrm{Tr}(J \cdot (A \times A))$ must be trivial over $\mathcal{J}$. Since the bilinear form $\langle A, B\rangle = \mathrm{Tr}(A \cdot B)$ is nondegenerate, this implies $A \times A = I$. Therefore $A = (A \times A) \cdot A = N(A)I$. Let $N(A) = x$; then $A = \prod(x_\gamma)$ with $x_\gamma = x$ if $\gamma = \alpha + \beta, \gamma_1$ or δ_1, and $x_\gamma = 0$ otherwise. Using the formula for $N(A)$, one gets $N(A) = x^3$. Thus x equals either 1 or -1; A is either the identity or $-I$. When $A = \pm I$, the integration in (24) over J gives a factor 1. The expression (24) equals

$$f * \phi(0, I) + f * \phi(0, -I). \tag{25}$$

In expression (23), we make a change of variable $J \to (J - A)/a$. Observe that $\psi(N(A)/a) = \psi(-\mathrm{Tr}(A)/a) = 1$ as $a \in F^\times, A \in \mathcal{J}(F)$. We get that (23) equals

$$\sum_{a \in F^\times, A \in \mathcal{J}(F)} \int_{\mathcal{J}(F)\backslash\mathcal{J}(\mathbf{A})} f * \phi(a, J)\psi\left(\frac{\mathrm{Tr}(J) - N(J)}{a}\right) dJ,$$

which is

$$\sum_{a \in F^\times} \int_{\mathcal{J}(\mathbf{A})} f * \phi(a, A)\psi\left(\frac{\mathrm{Tr}(A) - N(A)}{a}\right) dA. \tag{26}$$

We sum up our result as follows.

Proposition 3. *Let $f = \bigotimes f_v \in C_c^\infty(G(\mathbf{A}))$, $\phi = \bigotimes \phi_v \in S(\mathbf{A} \oplus \mathcal{J}(\mathbf{A}))$. Define $f * \phi = \bigotimes f_v * \phi_v$ as in (22); then*

$$I(f, \phi) = \sum_{a \in F^\times} \prod_v I_v(a_v, f_v * \phi_v) + \prod_v I_v^+(f_v * \phi_v) + \prod_v I_v^-(f_v * \phi_v), \tag{27}$$

where

$$I_v(a, f_v * \phi_v) = \int_{\mathcal{J}(F_v)} f_v * \phi_v(a, A)\psi\left(\frac{\mathrm{Tr}(A) - N(A)}{a}\right) dA \tag{28}$$

$$I_v^\pm(f_v * \phi_v) = f_v * \phi_v(0, \pm I). \tag{29}$$

The distribution $J(f')$ is known as a Kuznietsov trace. There is a standard argument in unwinding the integral $J(f')$; see [JY]. Let $f' = \bigotimes f'_v \in C_c^\infty(SL_2(\mathbf{A}))$. For $a \in F_v^\times$, define

$$J_v(a, f'_v) = \int_{F_v^2} f'_v\left(n(x)\begin{bmatrix} & -a^{-1} \\ a & \end{bmatrix} n(y)\right)\psi(x + y)\, dx\, dy$$

$$J_v^+(f'_v) = \int_{F_v} f'_v(n(x))\psi(x)\, dx, J_v^-(f'_v) = \int_{F_v} f'_v(-n(x))\psi(x)\, dx.$$

Then

$$J(f') = \sum_{a\in F^\times} \prod_v J_v(a_v, f'_v) + \prod_v J_v^+(f'_v) + \prod_v J_v^-(f'_v). \tag{30}$$

We expect the following local relations between I_v and J_v to hold: For $a_v \in F_v^\times$, there exists a *transfer factor* $\mu_v(a_v)$, such that:

(1) If $a \in F^\times$, then $\prod_v \mu_v(a_v) = 1$.

(2) Given f'_v, there exists (f_v, ϕ_v), with $I_v(a, f_v * \phi_v) = \mu_v(a) J_v(a, f'_v)$ for all $a \in F_v^\times$, and $I_v^\pm(f_v * \phi_v) = J_v^\pm(f'_v)$, and we say (f_v, ϕ_v) and f'_v match if these conditions are satisfied.

(3) Let v be a finite place and R_v be the ring of integers in F_v. Let $f_{0,v}, \phi_{0,v}, f'_{0,v}$ be the characteristic functions of $G(R_v)$, $R_v \oplus \mathscr{J}(R_v)$, and $SL_2(R_v)$, respectively. Then $(f_{0,v}, \phi_{0,v})$ and $f'_{0,v}$ match for almost all places v.

(4) The matching condition on (f_v, ϕ_v) and f'_v is consistent with *functoriality*.

We will not try to clarify condition (4) here.

If conditions (1)–(4) are satisfied, then for each pair (f, ϕ) and f' with their local components matching, we have $I(f, \phi) = J(f')$ by Proposition 3 and equation (30). One may then use the spectral decomposition for I and J to derive a correspondence between the automorphic representations of G and these of SL_2. In [MR], we verify conditions (1)–(4) in the case G' is of type G_2. There $G = SL_2$, while the distribution $J(f')$ is defined on $\widetilde{SL}_2^3$.

In next section, we will verify condition (3), which is the so-called fundamental lemma for the trace formula.

4. The fundamental lemma. Let F_v be a local non-Archimedean field. Assume that as a vector space over F_v, $\mathscr{J}$ is n-dimensional. Let $f_{0,v}$, $\phi_{0,v}$, and $f'_{0,v}$ be as in the previous section. We prove the following.

THEOREM 1. *Over almost all finite places v, if ψ is of order 0, then for $a \in F_v^\times$,*

$$I_v(a, f_{0,v} * \phi_{0,v}) = |a|^{(n+1)/2} J_v(a, f'_{0,v}) \tag{31}$$

and

$$I_v^+(f_{0,v} * \phi_{0,v}) = J_v^+(f'_{0,v}), \qquad I_v^-(f_{0,v} * \phi_{0,v}) = J_v^-(f'_{0,v}). \tag{32}$$

Proof. This is the fundamental lemma for the trace formula in the case at hand. We start the proof with the following.

LEMMA 2. *For almost all finite places v, when ψ is of order 0, $f_{0,v} * \phi_{0,v} = \phi_{0,v}$.*

Proof. The set $j(G)$ is a subgroup of Sp_{n+1} in the current case. When ψ is of order 0, for a $g \in Sp_{n+1}(R_v)$, we have $\rho_\psi(g)\phi_{0,v} = \phi_{0,v}$. Since j is a rational finite-dimensional representation of G, for almost all finite places v, the image $j(G(R_v))$

lies inside $Sp_{n+1}(R_v)$. It follows from expression (22) that over almost all finite places, $f_{0,v} * \phi_{0,v} = \phi_{0,v}$. $\square$

We may then prove the theorem with $f_{0,v} * \phi_{0,v}$ being replaced by $\phi_{0,v}$. It is clear that $I_v^{\pm}(\phi_{0,v}) = 1$. One checks easily that $J_v^{\pm}(f'_{0,v}) = 1$. For $a \in F^{\times}$, $I_v(a, \phi_{0,v})$ equals

$$\begin{aligned} &\int_{\mathscr{I}(F_v)} \phi_{0,v}(a, A)\psi\left(\frac{\mathrm{Tr}(A) - N(A)}{a}\right) dA \\ &= \int_{A \in \mathscr{I}(R_v)} \psi\left(\frac{\mathrm{Tr}(A) - N(A)}{a}\right) dA \quad \text{if } |a| \leqslant 1 \\ &= 0 \quad \text{if } |a| > 1. \end{aligned}$$

Meanwhile

$$\begin{aligned} J_v(a, f'_{0,v}) &= \int_{F_v^2} f'_{0,v}\left(n(x)\begin{bmatrix} & -a^{-1} \\ a & \end{bmatrix} n(y)\right)\psi(x+y)\,dx\,dy \\ &= \int_{|a|,|ax|,|ay|,|axy-a^{-1}| \leqslant 1} \psi(x+y)\,dx\,dy. \end{aligned} \tag{33}$$

Thus when $|a| > 1$, $J_v(a, f'_{0,v}) = I_v(a, \phi_{0,v}) = 0$. When $|a| = 1$, a simple computation shows that $J_v(a, f'_{0,v}) = I_v(a, \phi_{0,v}) = 1$. When $|a| < 1$, make changes of variables $x \to x/a, y \to y/a$ in (33); $J_v(a, f'_{0,v})$ becomes

$$\int_{|x|,|y| \leqslant 1, |xy-1| \leqslant |a| < 1} \psi\left(\frac{x+y}{a}\right)|a|^{-2}\,dx\,dy.$$

Over the domain, we have $|x| = |y| = 1$ and $\psi(y/a) = \psi(x^{-1}/a)$. Integrating over y, we get

$$J_v(a, f'_{0,v}) = \int_{|x|=1} \psi\left(\frac{x+x^{-1}}{a}\right)|a|^{-1}\,dx. \tag{34}$$

To prove Theorem 1, we only need to show the following theorem.

THEOREM 2. *When $|a| < 1$, ψ is of order* 0*:*

$$\int_{A \in \mathscr{I}(R_v)} \psi\left(\frac{\mathrm{Tr}(A) - N(A)}{a}\right) dA = |a|^{(n-1)/2} \int_{|x|=1} \psi\left(\frac{x+x^{-1}}{a}\right) dx. \tag{35}$$

We will use the formula for $N(A)$ and $\mathrm{Tr}(A)$ as stated in §2. With the notation in §2, the left-hand side of (35) equals

$$\int_{|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta}+x_{\gamma_1}+x_{\delta_1}-\sum_S c_S x_{\eta_1}x_{\eta_2}x_{\eta_3}}{a}\right)\prod_{\Pi_0'} dx_\gamma. \tag{36}$$

Recall the following lemma proved in [KS].

Lemma 3. *With the decomposition given by equation (18):*

$$\sum_S c_S x_{\eta_1}x_{\eta_2}x_{\eta_3} = \sum_{i=1}^{r} ck_i x_{\alpha+\beta}x_{\gamma_i}x_{\delta_i} + C_i x_{\gamma_i} + D_i x_{\delta_i}$$

where $C_i = \sum_{P_i} c(i,\mu_1,\mu_2)x_{\mu_1}x_{\mu_2}$, *with*

$$P_i = \{\{\mu_1,\mu_2\} \subset \Pi' | \mu_1+\mu_2+\gamma_i = \tilde{\alpha}+\alpha\}$$

and $D_i = \sum_{R_i} d(i,\chi_1,\chi_2)x_{\chi_1}x_{\chi_2}$, *with*

$$R_i = \{\{\chi_1,\chi_2\} \subset \Pi' | \chi_1+\chi_2+\delta_i = \tilde{\alpha}+\alpha\}.$$

Also

$$\sum_{i=1}^{r} k_i C_i D_i = 0.$$

Here $c, k_i, c(i,\mu_1,\mu_2), d(i,\chi,\chi_2)$ equal either 1 or -1. We only need the fact that $ck_1 = 1$, which follows from the last remark in §2.

From Lemma 3, (36) equals

$$\int_{|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta}+x_{\gamma_1}+x_{\delta_1}-\sum_i[ck_i x_{\alpha+\beta}x_{\gamma_i}x_{\delta_i}+C_i x_{\gamma_i}+D_i x_{\delta_i}]}{a}\right)\prod dx_\gamma. \tag{37}$$

We first consider the integral (37) over the subset D' with $|x_{\alpha+\beta}| < 1$. This integral has the form

$$\int_{|x_\gamma|\leqslant 1, |x_{\alpha+\beta}|<1} \psi\left(\frac{x_{\gamma_1}(1-x_{\alpha+\beta}x_{\delta_1}-C_1)}{a}\right) dx_{\gamma_1}$$

$$\times \prod_{i\neq 1}\int \psi\left(\frac{x_{\gamma_i}(-x_{\alpha+\beta}x_{\delta_i}-C_i)}{a}\right) dx_{\gamma_i} \int \psi\left(\frac{x_{\alpha+\beta}+x_{\delta_1}-\sum_i D_i x_{\delta_i}}{a}\right)\prod dx_\gamma.$$

For the integration over x_{γ_i} to be nonzero, we must have

$$|1-x_{\alpha+\beta}x_{\delta_1}-C_1| \leqslant |a|, |-x_{\alpha+\beta}x_{\delta_i}-C_i| \leqslant |a|, \qquad i\neq 1,$$

which implies the conditions $|1 - C_1| < 1$ and $|C_i| < 1$ if $i \neq 1$. We may impose these conditions on the set D' without changing its contribution to (37). A similar argument shows that we may also impose the conditions $|1 - D_1| < 1$ and $|D_i| < 1$ if $i \neq 1$. Denote by D'' the subset of D' with these restrictions. Over D'',

$$\left|\sum_{i=1}^{r} k_i C_i D_i\right| = |k_1 C_1 D_1| = 1,$$

which contradicts Lemma 3. Thus D'' is empty, and it contributes 0 to (37); so does the set D'.

We may then restrict the domain of (37) to the set with $|x_{\alpha+\beta}| = 1$. Rearranging the terms in (37), it becomes

$$\int_{|x_{\alpha+\beta}|=1,|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta} + (1 - C_1)x_{\gamma_1} + (1 - D_1)x_{\delta_1} - x_{\alpha+\beta}x_{\gamma_1}x_{\delta_1}}{a}\right)$$
$$\times \int \prod_{i\neq 1} \psi\left(\frac{-ck_i x_{\alpha+\beta}x_{\gamma_i}x_{\delta_i} - C_i x_{\gamma_i} - D_i x_{\delta_i}}{a}\right) \prod dx_\gamma dx_{\alpha+\beta}. \tag{38}$$

Note that

$$ck_i x_{\alpha+\beta}x_{\gamma_i}x_{\delta_i} + C_i x_{\gamma_i} + D_i x_{\delta_i} = ck_i x_{\alpha+\beta}\left(x_{\gamma_i} + \frac{D_i}{ck_i x_{\alpha+\beta}}\right)\left(x_{\delta_i} + \frac{C_i}{ck_i x_{\alpha+\beta}}\right) - \frac{C_i D_i}{ck_i x_{\alpha+\beta}}.$$

Make the changes of variables $x_{\gamma_i} \to x_{\gamma_i} - (D_i/ck_i x_{\alpha+\beta})$, $x_{\delta_i} \to x_{\delta_i} - (C_i/ck_i x_{\alpha+\beta})$ for $i \neq 1$; (38) becomes

$$\int_{|x_{\alpha+\beta}|=1,|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta} + (1 - C_1)x_{\gamma_1} + (1 - D_1)x_{\delta_1} - x_{\alpha+\beta}x_{\gamma_1}x_{\delta_1} + \sum_{i\neq 1}\frac{C_i D_i}{ck_i x_{\alpha+\beta}}}{a}\right)$$
$$\times \int \prod_{i\neq 1} \psi\left(\frac{ck_i x_{\alpha+\beta}x_{\gamma_i}x_{\delta_i}}{a}\right) \prod dx_\gamma \, dx_{\alpha+\beta}. \tag{39}$$

Using Lemma 3 again, we see

$$\sum_{i\neq 1} \frac{C_i D_i}{ck_i x_{\alpha+\beta}} = -\frac{C_1 D_1}{x_{\alpha+\beta}}. \tag{40}$$

We now apply a simple lemma.

LEMMA 4. *If ψ is of order* 0, $|b| \geqslant 1$, *then*

$$\int_{|x|,|y|\leqslant 1} \psi(bxy)\, dx\, dy = |b|^{-1}.$$

Thus the integration over $x_{\gamma_i}, x_{\delta_i}$, $i \neq 1$ gives a factor of $|a|^{r-1}$. From (40), the integral (39) equals

$$|a|^{r-1} \int_{|x_{\alpha+\beta}|=1,|x_\gamma|\leqslant 1} \psi\left(\frac{-x_{\alpha+\beta}x_{\gamma_1}x_{\delta_1} + (1-C_1)x_{\gamma_1} + (1-D_1)x_{\delta_1}}{a}\right)$$

$$\times \int \psi\left(\frac{x_{\alpha+\beta} - \dfrac{C_1D_1}{x_{\alpha+\beta}}}{a}\right) \prod dx_\gamma\, dx_{\alpha+\beta}.$$

Make changes of variables $x_{\gamma_1} \to x_{\gamma_1} + (1-C_1)/x_{\alpha+\beta}$ and $x_{\delta_1} \to x_{\delta_1} + (1-D_1)/x_{\alpha+\beta}$:

$$|a|^{r-1} \int_{|x_{\alpha+\beta}|=1,|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta} - x_{\alpha+\beta}x_{\gamma_1}x_{\delta_1} + \dfrac{1-C_1-D_1}{x_{\alpha+\beta}}}{a}\right) \prod dx_\gamma\, dx_{\alpha+\beta}.$$

From Lemma 4, the integration over x_{γ_1} and x_{δ_1} gives a factor $|a|$.

$$|a|^{r} \int_{|x_{\alpha+\beta}|=1,|x_\gamma|\leqslant 1} \psi\left(\frac{x_{\alpha+\beta} + \dfrac{1}{x_{\alpha+\beta}}}{a}\right) \psi\left(\frac{-C_1-D_1}{ax_{\alpha+\beta}}\right) \prod dx_\gamma\, dx_{\alpha+\beta}. \tag{41}$$

Finally we consider the integration of (41) over x_γ with $\gamma \in \Pi'$. Recall that

$$C_1 = \sum_{P_1} c(1,\mu_1,\mu_2)x_{\mu_1}x_{\mu_2}, \qquad D_1 = \sum_{R_1} d(1,\chi_1,\chi_2)x_{\chi_1}x_{\chi_2}.$$

We claim that each root $\gamma \in \Pi'$ appears exactly once in the expressions for C_1 and D_1. For $\gamma \in \Pi'$, $\langle\gamma,\gamma_1\rangle + \langle\gamma,\delta_1\rangle = \langle\gamma,\tilde\alpha\rangle - \langle\gamma,\beta\rangle = 1$. Thus either $\langle\gamma,\gamma_1\rangle = 1$ or $\langle\gamma,\delta_1\rangle = 1$. This criterion separates Π' into a disjoint union $\Pi_1' \cup \Pi_2'$. For $\mu \in \Pi_1'$ the set with $\langle\mu,\gamma_1\rangle = 1$, define $\mu' = \alpha + \beta + \gamma_1 - \mu$. For $\nu \in \Pi_2'$, define $\nu' = \alpha + \beta + \delta_1 - \nu$.

LEMMA 5. *The maps $\mu \to \mu'$, $\nu \to \nu'$ define the involutions in Π_1' and Π_2', respectively. We have $\mu \neq \mu'$, $\nu \neq \nu'$ and*

$$P_1 = \{(\nu, \nu') | \nu \in \Pi_2'\}, \qquad R_1 = \{(\mu, \mu') | \mu \in \Pi_1'\}.$$

Proof. Given $\mu \in \Pi_1'$, $\langle \mu, \gamma_1 \rangle = 1$, thus $\gamma_1 - \mu$ is a root. Since $\langle \alpha + \beta, \gamma_1 - \mu \rangle = 1$, we see μ' is a root. One checks easily that $\langle \mu', \alpha \rangle = 1, \langle \mu', \beta \rangle = 0, \langle \mu', \gamma_1 \rangle = 1$; thus $\mu' \in \Pi_1'$. Similarly, $\nu' \in \Pi_2'$. The remaining assertions are clear. □

With this lemma, we proved our claim on the expressions C_1 and D_1. The integration over $\{x_\gamma | \gamma \in \Pi'\}$ is

$$\begin{aligned}\int_{|x_\gamma| \leq 1} \psi\left(\frac{-C_1 - D_1}{a x_{\alpha+\beta}}\right) \prod dx_\gamma &= \prod_{(\mu,\mu') \in \Pi_1'} \int_{|x_\mu|, |x_{\mu'}| \leq 1} \psi\left(\frac{-d(1,\mu,\mu') x_\mu x_{\mu'}}{a x_{\alpha+\beta}}\right) dx_\mu \, dx_{\mu'} \\ &\quad \times \prod_{(\nu,\nu') \in \Pi_2'} \int_{|x_\nu|, |x_{\nu'}| \leq 1} \psi\left(\frac{-c(1,\nu,\nu') x_\nu x_{\nu'}}{a x_{\alpha+\beta}}\right) dx_\nu \, dx_{\nu'}.\end{aligned}$$

Apply Lemma 4; this integral equals $|a|^{|\Pi'|/2}$. Since $r + (|\Pi'|/2) = (n-1)/2$, we get that (41) equals

$$|a|^{(n-1)/2} \int_{|x_{\alpha+\beta}|=1} \psi\left(\frac{x_{\alpha+\beta} + x_{\alpha+\beta}^{-1}}{a}\right) dx_{\alpha+\beta}. \tag{42}$$

This is the expression on the right-hand side of (35). We proved Theorem 2, and thus the fundamental lemma. □

5. The case G_2 and F_4. In the case G' is of type G_2 or F_4, the group G is SL_2 or $\widetilde{Sp}_3$, respectively. We can define a distribution $I(f, \phi)$ on SL_2 or $\widetilde{Sp}_3$, respectively, as in the introduction; in the latter case, f is a genuine function on the metaplectic group $\widetilde{Sp}_3$. The distribution will be compared with a distribution $J(f')$ on a covering group of SL_2. Let $\widetilde{SL}_2^m$ be the m-fold covering of SL_2. Denote by $\tilde{n}(t) = (n(t), 1)$ an element in $\widetilde{SL}_2^m$. For $f' \in C_c^\infty(\widetilde{SL}_2^m(\mathbf{A}))$ an antigenuine function, define

$$J(f') = \int_{(\mathbf{A}/F)^2} K_{f'}(\tilde{n}(-x), \tilde{n}(y)) \psi(x+y) \, dx \, dy.$$

When G' is of type G_2, we let $m = 3$ and assume that F contains the cubic roots of unity; when G' is of type F_4, let $m = 2$. The correspondences that arise from the identity (8) are between the automorphic representations of SL_2 and $\widetilde{SL}_2^3$, or between the automorphic representations of $\widetilde{Sp}_3$ and $\widetilde{SL}_2^2$. Here we are interested in showing the fundamental lemma for the trace formula (8).

One can unwind the integrals $I(f,\phi)$ and $J(f')$ as in §3. We get Proposition 3 and equation (30) with $n(t)$ replaced by $\tilde{n}(t)$ and $\begin{bmatrix} & -a^{-1} \\ a & \end{bmatrix}$ replaced by $\left(\begin{bmatrix} & -a^{-1} \\ a & \end{bmatrix}, 1\right)$. The fundamental lemma over the local non-Archimedean field asserts as before a relation between $I_v(a, f_{0,v} * \phi_{0,v})$ and $J_v(a, f'_{0,v})$, with $\phi_{0,v}$ being the same as defined in §3, and $f_{0,v}$ and $f'_{0,v}$ being the unit antigenuine Hecke elements which are defined over almost all local places. (See [J2], [MR].)

Theorem 1 holds when G' is of type G_2, and with a minor modification when G' is of type F_4. We now sketch a proof. Again we may replace $f_{0,v} * \phi_{0,v}$ by $\phi_{0,v}$ in our consideration.

Assume $|m| = 1$ over the finite place v. It is easy to see that $I_v(a, \phi_{0,v})$ and $J_v(f'_{0,v})$ equals 0 when $|a| > 1$ and equals 1 when $|a| = 1$. When $|a| < 1$, from the argument in §4,

$$I_v(a, \phi_{0,v}) = \int_{A \in \mathcal{J}(R_v)} \psi\left(\frac{\operatorname{Tr}(A) - N(A)}{a}\right) dA. \tag{43}$$

In the case G' is of type G_2, $\mathcal{J}$ is the set $\{tI \mid t \in F_v\}$. Thus (43) equals

$$\int_{R_v} \psi\left(\frac{3t - t^3}{a}\right) dt. \tag{44}$$

In the case G' is of type F_4, $\mathcal{J}$ is isomorphic to the set of 3×3 symmetric matrices. The norm form is just the determinant, while the trace form is the usual trace. Thus, explicitly, (43) equals

$$\int_{R_v^6} \psi\left(\frac{x + y + z - xyz - 2efg + xe^2 + yf^2 + zg^2}{a}\right) d(xyzefg). \tag{45}$$

Meanwhile from the discussion in §4, we see

$$J_v(f'_{0,v}) = |a|^{-1} \int_{|x|=1} \psi\left(\frac{x + x^{-1}}{a}\right) (a, x)\, dx, \tag{46}$$

where $(*, *)$ is the mth-order Hilbert symbol.

Proving the fundamental lemma is equivalent to establishing the relationship between the exponential sums (44) and (46) or (45) and (46). In [MR], we prove the following proposition.

Proposition 4. *When* ψ *is of order* 0, $|a| < 1$, *if* $R_v^\times$ *contains the cubic roots of unity, then*

$$\int_{R_v} \psi\left(\frac{3t - t^3}{a}\right) dt = \int_{R_v^\times} \psi\left(\frac{x + x^{-1}}{a}\right) (a, x)\, dx.$$

In the rest of the section, we assume that F_v is a local non-Archimedean field with residue characteristic $\neq 2$. We prove the next proposition.

Proposition 5. *When ψ is of order 0, $(*,*)$ is the second-order Hilbert symbol; when $|a| < 1$, the expression (45) equals*

$$|a|^{5/2}\gamma(a,\psi)^{-1}\int_{|x|=1}\psi\left(\frac{x+x^{-1}}{a}\right)(a,x)\,dx.$$

Here $\gamma(a,\psi)$ is the Weil constant.

Proof. This is essentially the fundamental lemma in the case when G' is of type F_4. First consider the contribution from the set D with $|z| < 1$ to the integral (45). Observe that for the integration over g to be nonzero in (45), $|ef|$ must be less than 1; for the integration over x, y to be nonzero, $|e^2+1|$ and $|f^2+1|$ must be less than 1. There is no such e, f satisfying these conditions. Thus the contribution from D is 0.

Now assume $|z| = 1$ over the domain for (45). Change $x \to x + ((1+e^2)/z)$, $y \to y + ((1+f^2)/z)$; (45) becomes

$$\int_{|z|=1}\int_{R_v^5}\psi\left(\frac{z+z^{-1}-xyz+\frac{e^2}{z}+\frac{f^2}{z}+\frac{(g-ef)^2}{z}}{a}\right)d(xyzefg). \tag{47}$$

We now apply Lemma 4 and the next lemma.

Lemma 6. *If ψ is of order 0, $|b| > 1$, then*

$$\int_{R_v}\psi(bx^2)\,dx = |b|^{-1/2}\gamma(b,\psi).$$

The integral (47) becomes

$$\int_{|z|=1}\psi\left(\frac{z+z^{-1}}{a}\right)|a|^{5/2}\gamma(az,\psi)^3dz. \tag{48}$$

Since $\gamma(az,\psi)^4 = 1$, we get

$$\gamma(az,\psi)^3 = \gamma(az,\psi)^{-1} = \gamma(a,\psi)^{-1}\gamma(z,\psi)^{-1}(a,z).$$

With $|z| = 1$, we have $\gamma(z,\psi) = 1$. Thus (48) is

$$|a|^{5/2}\gamma(a,\psi)^{-1}\int_{|z|=1}\psi\left(\frac{z+z^{-1}}{a}\right)(a,z)\,dz.$$

The proposition is proved. □

6. Finite field case. There is an analogue for Theorem 2 over the finite field. Back to the situation in §4. Let F be a non-Archimedean field with residue field K. Assume $|K| = q$. For $x \in R$ the integer ring, let $x \to \bar{x}$ be the map from R to K by the reduction modulo P the maximal ideal in R. Reduce the Jordan algebra $\mathscr{J}(F_v)$ modulo P; we get a Jordan algebra $\bar{\mathscr{J}}$ over the finite field K. One may define the norm and trace form on $\bar{\mathscr{J}}$ by reduction modulo P as well. We have the following.

THEOREM 3. *Let ψ' be a nontrivial additive character on K. Then*

$$\sum_{\bar{A} \in \bar{\mathscr{J}}} \psi'(\mathrm{Tr}(\bar{A}) - N(\bar{A})) = q^{(n-1)/2} \sum_{x \in K^\times} \psi'(x + x^{-1}). \tag{49}$$

Proof. Choose $a \in P$, such that $\psi(x/a) = \psi'(\bar{x})$. Then the left-hand side of (49) equals q^n times the left-hand side of (35), while

$$\sum_{x \in K^\times} \psi'(x + x^{-1}) = q \int_{|x|=1} \psi\left(\frac{x + x^{-1}}{a}\right) dx.$$

Equation (49) follows from Theorem 2. □

We note the exponential sum on the right-hand side of (49) is a Kloosterman sum.

For the situations considered in §5, we have also the analogues over finite field. In the case of G' is of type G_2, the corresponding equation over K is (with $q = 1 \bmod 3$):

$$\sum_{t \in K} \psi'(3t - t^3) = \sum_{x \in K^\times} \psi'(x + x^{-1})\chi(x), \tag{50}$$

where χ is a nontrivial cubic character on $K^\times$. This is the equation that appears in [DI]. In the case where G' is of type F_4, we have when q is odd,

$$\sum_{\bar{A} \in \bar{\mathscr{J}}} \psi'(\mathrm{Tr}(\bar{A}) - N(\bar{A})) = q^3 \left(\sum_{v \in K} \psi'(v^2)\right)^{-1} \sum_{x \in K^\times} \psi'(x + x^{-1})\zeta(x), \tag{51}$$

where ζ is the quadratic character on $K^\times$.

7. The case B_3. Keep the notation in the introduction. When $G' = SO_7$, then the group G is $SO_3 \times SL_2$, where SO_3 is defined with the quadratic form Q on F^3:

$$Q(a, b, c) = b^2 + ac.$$

The metaplectic group $\widetilde{Sp}(U'/Z) = \widetilde{Sp}_3$ does not split over $j(G)$. Let $\tilde{G} = SO_3 \times \widetilde{SL}_2^2$. For $f \in C_c^\infty(\tilde{G}(\mathbf{A}))$ a genuine function, $\phi \in S(\mathbf{A}^3)$, we may define a distribution $I(f, \phi)$ using the equation (6), where U, G are considered as subsets in $\tilde{G}$ via the embedding $g \to (g, 1)$. This distribution will be compared with a distribution $J(f')$ on $\widetilde{SL}_2^2$, where for $f' \in C_c^\infty(\widetilde{SL}_2^2(\mathbf{A}))$ a genuine function, $J(f')$ is defined as in §5.

We briefly indicate here a relationship between the identity (8) and the relative trace formula in [J2]. A simpler form of $I(f, \phi)$ is needed for the comparison.

In [J2], a distribution $I(f, \eta, \psi')$ is defined on GL_2 for some multiplicative character η and additive character ψ':

$$I(f, \eta, \psi') = \int_{a \in \mathbf{A}^\times/F^\times} \int_{u \in \mathbf{A}/F} K_f\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}, n(u)\right) \eta(a)\, d^\times a \psi'(-u)\, du.$$

Meanwhile a distribution $J(f', \varepsilon, \psi')$ is defined on $\widetilde{SL}_2^2$, where $\varepsilon \in F^\times$. We will take $\varepsilon = 1$, and $\psi'(s) = \psi(-2s)$. Then $J(f', \varepsilon, \psi')$ is just our distribution $J(f')$. With the choice of ε, the character η is the trivial character. We now compare $I(f, 1, \psi')$ and $I(f, \phi)$.

Let T be the group of diagonal matrices in GL_2, and let

$$\Phi\left(g^{-1}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} g\right) = \int_{T(\mathbf{A})} f(hg)\, dh.$$

Then Φ is a function on the set $\mathscr{S}$ of 2×2 matrices with eigenvalues $\{1, -1\}$. The group GL_2 acts on $\mathscr{S}$ by conjugation. Let $\theta(n) = \psi(2s)$ if $n = n(s)$; let $\mathscr{S}'$ be the set of relevant elements γ in $\mathscr{S}$, i.e., $\theta(n) = 1$ if $n \in N_\gamma(\mathbf{A})$. The distribution $I(f, 1, \psi')$ can be unwinded into the expression

$$\sum_{\gamma \in \mathscr{S}'(F)/N(F)} \int_{N(\mathbf{A})/N_\gamma(\mathbf{A})} \Phi(n^{-1}\gamma n)\theta(n)\, dn. \tag{52}$$

We will write $I(f, \phi)$ into this form.

Using the argument in §3, one can unwind the integral $I(f, \phi)$. Let

$$f * \phi(X) = \int_{G(\mathbf{A})} f(g^{-1})\rho_\psi(j(g))(X)\, dg, \qquad X \in \mathbf{A}^3.$$

Then

$$I(f, \phi) = \int_{U(F)\backslash U(\mathbf{A})} \sum_{X \in F^3} \rho_\psi(j(u)) f * \phi(X)\psi(\mathrm{Tr}(J_u))\, du.$$

The group U is isomorphic to $N \times N'$ where N, N' are the one-dimensional unipotent subgroups in SL_2 and SO_3, respectively. Write an element in $u \in N \times N'$ as $n(t)n'(s)$. Then $\psi(\mathrm{Tr}(J_u)) = \psi(t+2s)$, and

$$\rho_\psi(j(u))f * \phi(X) = \psi(-Q(X)t)f * \phi(Xn'(-s)).$$

Here Q is the same quadratic form as in the definition of SO_3. Thus $I(f, \phi)$ equals

$$\int_{(\mathbf{A}/F)^2} \sum_{X \in F^3} \psi(-Q(X)t + t + 2s)f * \phi(Xn'(-s))\, ds\, dt. \tag{53}$$

For the integration over t to be nonzero, $Q(X)$ must be 1. Let $Y_0(\mathbf{A})$ be the quadric

$$\{X \in \mathbf{A}^3 | Q(X) = 1\}.$$

Integrating over t, (53) becomes

$$\int_{\mathbf{A}/F} \sum_{X \in Y_0(F)} \psi(2s)f * \phi(Xn'(-s))\, ds. \tag{54}$$

Lemma 7. *There is a bijection between the set $Y_0(\mathbf{A})$ and the set $\mathscr{S}(\mathbf{A})$.*

Proof. If $X = (a, b, c) \in F^3$, define a matrix $m(X)$ as $\begin{bmatrix} b & c \\ a & -b \end{bmatrix}$. Then $X \in Y_0$ if and only if $m(X) \in \mathscr{S}$. This proves the lemma. □

As $n'(-s) \in SO_3$, for $X \in Y_0(F)$, we have $Xn'(-s) \in Y_0(\mathbf{A})$. In fact, $m(Xn'(-s)) = n(s)m(X)n(-s)$. Thus from (54), $I(f, \phi)$ equals

$$\int_{\mathbf{A}/F} \sum_{\gamma \in \mathscr{S}(F)} \psi(2s)f * \phi[m^{-1}(n(s)\gamma n(-s))]\, ds. \tag{55}$$

Let Φ be a function on the set $\mathscr{S}(\mathbf{A})$:

$$\Phi(g) = f * \phi(m^{-1}(g^{-1})).$$

Then $I(f, \phi)$ equals

$$\int_{\mathbf{A}/F} \sum_{\gamma \in \mathscr{S}(F)} \psi(2s)\Phi(n(-s)\gamma n(s))\, ds = \sum_{\gamma \in \mathscr{S}'(F)/N(F)} \int_{N(\mathbf{A})/N_\gamma(\mathbf{A})} \Phi(n^{-1}\gamma n)\theta(n)\, dn,$$

which is exactly the expression (52). We have shown that the trace formula stated in [J2] and the identity (8) in the introduction are essentially the same.

The discussion in this section applies to the general cases where G' is of the type B_n. For any n, the set Y_0 in (54) is isomorphic to the symmetric space SO_{2n+1}/SO_{2n}. The distribution $I(f, \phi)$ in (6) is, in essence, a distribution on SO_{2n+1} of the form

$$I(\varphi) = \int_{SO_{2n}(F)\backslash SO_{2n}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} K_\varphi(h, u)\theta(u)\, dh\, du,$$

where U is a unipotent subgroup of SO_{2n+1}, θ is a certain degenerate character on U; the precise definition of U and θ can be found in the introduction. The trace identity (8) will then give a correspondence between SO_{2n}-distinguished automorphic representations of SO_{2n+1} and the automorphic representations of $\widetilde{SL}_2^2$.

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