Available online at www.sciencedirect.com

ScienceDirect

Journal of Number Theory 128 (2008) 71–95

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

A generalized Shimura correspondence for newforms

Zhengyu Mao [1]

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102-1811, USA

Received 19 October 2005

Available online 30 March 2007

Communicated by F. Rodriguez-Villegas

Abstract

We associate a set of half integral weight forms to an integral weight newform of odd level. We prove an explicit identity relating the central values of the twist L-functions of the newform to the Fourier coefficients of the half integral weight forms.

MSC: primary 11F37; secondary 11F70

Keywords: Shimura correspondence; Half integral weight forms; Special values of L-functions

1. Introduction

Let $e(z) = e^{2\pi i z}$. Let $S_{2k}(N)$ be the space of weight $2k$, level N cusp forms. Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{2k}(N)$ be a newform. We will assume N is odd. For D a fundamental discriminant, define the twisted L-functions

$$L(f, D, s) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) a(n) n^{-s}. \tag{1.1}$$

It is well known that the central values $L(f, D, k)$ are related to the Fourier coefficients of certain half integral weight forms that correspond to $f(z)$. It is the purpose of this paper to give an explicit statement of such a relationship.

E-mail address: zmao@andromeda.rutgers.edu.

[1] Partially supported by NSF grant DMS-0355285.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2007.03.005

The prototype of such a statement is given in [KZ], where the case $N = 1$ is treated. Associated to $f(z)$ of level 1 is a one-dimensional space of weight $k + 1/2$ and level 4 forms $g(z) = \sum_{n=1}^{\infty} c(n)e(nz)$, such that for all fundamental discriminants D,

$$\kappa \frac{|c(|D|)|^2}{\langle g, g\rangle} = \frac{L(f, D, k)}{\langle f, f\rangle} |D|^{k-1/2} \frac{(k-1)!}{\pi^k}. \tag{1.2}$$

Here $\kappa = 1$ when $(-1)^k D > 0$ and 0 otherwise.

The situation for forms of higher level is not as simple. For example when $N = p$ is an odd prime, Kohnen [K1] associates to $f(z)$ a one-dimensional space of forms $g(z)$, where Eq. (1.2) holds only for the fundamental discriminants D satisfying $(\frac{D}{p}) = \epsilon_p$ where ϵ_p is the Atkin–Lehner eigenvalue. This restriction is removed in [BM], where another one-dimensional space of weight $k + 1/2$ forms is associated to $f(z)$, whose Fourier coefficients satisfy (1.2) for the fundamental discriminants D satisfying $(\frac{D}{p}) = -\epsilon_p$. This result allows one to develop an algorithm to compute $L(f, D, k)$ for $f(z)$ of prime level [MTV].

Now let $f(z)$ be of arbitrary odd level. We will first attach to $f(z)$ a set of one-dimensional spaces $S^+(f, \chi, S)$ (Theorem 1.3), where each space consists of forms $g(z)$ that are Shimura correspondences of $f(z)$ in a generalized sense. The set of fundamental discriminants is partitioned into finitely many subsets $\Delta_{\eta,S}$. For each subset $\Delta_{\eta,S}$ we associate one $S^+(f, \chi, S)$ such that for $g(z) \in S^+(f, \chi, S)$ and $D \in \Delta_{\eta,S}$ the formula (1.2) holds with some explicit constant κ (Theorem 1.4). Some examples of $g(z)$ can be found in [PT].

The spaces $S^+(f, \chi, S)$ are dependent on a choice of character χ of $(\mathbb{Z}/N)^\times$. To remove this inconvenience, we associate a canonical set of forms $g_{f,\pm,\hat{\delta}}$ to $f(z)$, which are in the space of cusp forms for the congruence group $\Gamma_1(4N^2)$. The set of fundamental discriminants coprime to N are partitioned into finitely many subsets $\Delta_{\pm,\hat{\delta}}$. We then prove the formula (1.2) for $g = g_{f,\pm,\hat{\delta}}$ and $D \in \Delta_{\pm,\hat{\delta}}$ with an explicit constant κ (Theorem 1.5). It is an interesting question to find an algorithm for the computation of Fourier coefficients of $g_{f,\pm,\hat{\delta}}$.

1.1. Generalized Shimura correspondence

Let M be an odd positive integer and χ be a character of $(\mathbb{Z}/M)^\times$. Let χ' be the unique even character of $(\mathbb{Z}/4M)^\times$ associated to χ through the isomorphism $(\mathbb{Z}/4M)^\times \cong (\mathbb{Z}/4)^\times \times (\mathbb{Z}/M)^\times$. We denote by $S_{k+1/2}(4M, \chi)$ the space of weight $k+1/2$, level $4M$ cusp forms with character χ'. The Hecke operators for $S_{2k}(N)$ and $S_{k+1/2}(4M, \chi)$ are defined as follows:

- for $f(z) \in S_{2k}(N)$ and $(p, N) = 1$, $T_p(f)(z) = \sum_{n=1}^{\infty} b(n)e(nz)$ where
$$b(n) = a(pn) + p^{2k-1}a(n/p);$$
- for $g(z) = \sum_{n=1}^{\infty} c(n)e(nz) \in S_{k+1/2}(4M, \chi)$ and $(p, 4M) = 1$, $T_{p^2}(g)(z) = \sum_{n=1}^{\infty} d(n) \times e(nz)$ where
$$d(n) = c\bigl(p^2 n\bigr) + \chi'(p)\left(\frac{(-1)^k n}{p}\right)c(n) + \chi'\bigl(p^2\bigr)p^{2k-1}c\bigl(n/p^2\bigr).$$

Since $f(z)$ is a newform, we have $T_p(f) = a(p)f$ when $(p, N) = 1$. The usual definition of the Shimura correspondence [Sh] associates to $f(z)$ in $S_{2k}(N)$ forms $g(z)$ in $S_{k+1/2}(4M, 1)$

for some odd integer M, such that $g(z)$ is a Hecke eigenform with $T_{p^2}(g) = a(p)g$ whenever $(p, 2N) = 1$. However this definition is too restrictive to get the relation for all values of $L(f, D, k)$. Working under this restriction, one can only get a formula for a fraction of all fundamental discriminants, for a fraction of the newforms of odd level (see [K1,K2]).

Let T be the set of prime divisors of N and $N' = \prod_{l \in T} l$. Let χ be a character of $(\mathbb{Z}/N')^\times$; let $\epsilon(\chi) = \chi(-1)$. We denote the l-primary component of χ by $\chi_{(l)}$. Let M be an odd integer with the same set of prime divisors T and $\epsilon \in \{\pm 1\}$. We consider the space $S_{k+1/2}(f, 4M, \chi, \epsilon)$ of forms $g(z) \in S_{k+1/2}(4M, \chi)$ satisfying:

(C1) When $(p, 4N) = 1$, $T_{p^2} g(z) = \lambda(p) g(z)$ with $\lambda(p) = \chi'(p)(\frac{(-1)^k \epsilon}{p}) a(p)$. ($f(z)$ is the Shimura correspondence of $g(z)$ in a bit generalized sense; in the usual definition $\epsilon = (-1)^k$ and χ is trivial.)

1.2. Minimal level

We determine the lowest level M where this space $S_{k+1/2}(f, 4M, \chi, \epsilon)$ is nonzero.

To simply the situation, we only deal with the *q.t. primitive* newforms. For $l \neq 1$, we define

$$R_l(f)(z) = f_l(z) = \sum_{n=1}^{\infty} a(n) \left(\frac{n}{l}\right) e(nz). \tag{1.3}$$

From [AL], we know $f_l(z)$ is in $S_{2k}(N)$ when $l^2 | N$. We say $f(z)$ is *q.t. primitive* if $f_l(z)$ is also a newform in $S_{2k}(N)$ for all l satisfying $l^2 | N$ (*q.t.* for "quadratic twist"; these forms are called *very new forms* in [U]; we borrow here the related terminology of *q-primitive newform* from [A-Li]). Note that (Lemma 2.3) for any newform $f(z)$, there exists a q.t. primitive newform $f'(z)$ in $S_{2k}(N_1)$ with $N_1 | N$, such that $f(z) = f'_l(z)$ for some l with $l^2 | N$. Moreover when $(D, l) = 1$, we have $L(f, D, k)$ equals $L(f', (\frac{-1}{l}) Dl, k)$ (see Lemma 2.11).

In the following we will define subsets $T_1^+(\chi)$, $T_3^+(\chi)$ and $T_3^-(\chi)$ of the set T of prime divisors of N. When χ is fixed we write drop the reference to χ in the notation for these subsets. We will set $N(\chi) = N \prod_{l \in T_1^+ \cup T_3^+} l / \prod_{l \in T_3^-} l$.

Let $N = \prod_{l_i \in T} l_i^{m_i}$ with m_i being positive integers. We partition T into three subsets:

(1) $T_1 = \{l_i \in T \mid m_i = 1\}$;
(2) $T_2 = \{l_i \in T \mid m_i > 1,\ \text{odd}\}$;
(3) $T_3 = \{l_i \in T \mid m_i > 1,\ \text{even}\}$.

Let $W_{(l_i)} = W_{l_i^{m_i}}$ be the Atkin–Lehner involution at the prime l_i and $\epsilon_{l_i} = \epsilon_{l_i}(f) = \pm 1$ be the its eigenvalue. Let $\epsilon_l(f)$ and $\epsilon_l(f_l)$ be the Atkin–Lehner eigenvalues of f and f_l at a prime $l | N$. Let T_3^p be the set of primes l in T_3 such that $\epsilon_l(f_l) = (\frac{-1}{l}) \epsilon_l(f)$ and $T_3^s = T_3 - T_3^p$.

Let T' be the set of $l \in T_3^p$ with $l^2 \| N$. For $l \in T'$, we will associate a nontrivial character $\xi_{f,l}$ of $(\mathbb{Z}/l)^\times$ in Section 2.

We define T_1^+ to be the set of primes $l \in T_1$ where $\chi_{(l)}$ is nontrivial. Let T_3^+ to be the set of $l \in T_3^s$ such that $\chi_{(l)}(-1) \neq \epsilon_l(f)$. We let T_3^- to be the set of primes $l \in T'$ where $\chi_{(l)} = \xi_{f,l}$ or $\xi_{f,l}^{-1}$.

Our first claim is:

Theorem 1.1. *Let χ be a character of $(\mathbb{Z}/N')^\times$, then $S_{k+1/2}(f,4M,\chi,\epsilon)\neq\{0\}$ if and only if*

(1) *$N(\chi)$ divides M,*
(2) *$\epsilon=(-1)^k\epsilon(\chi)$, and*
(3) *$\chi_{(l)}(-1)=\epsilon_l$ when $l\in T_3^p$.*

1.3. One-dimensional subspaces of $S_{k+1/2}(f,4M,\chi,\epsilon)$

Let $e(\chi)=(-1)^k\epsilon(\chi)$. When χ is fixed we write e for $e(\chi)$. We give a more detailed description of $S(f,\chi)=S_{k+1/2}(f,4M,\chi,e)$ when $M=N(\chi)$, or rather its Kohnen subspace $S^+(f,\chi)$ consisting of $g(z)=\sum_{n=1}^\infty c(n)e(nz)$ satisfying:

(C2) $c(n)=0$ unless $en\equiv 0,1 \bmod 4$.

We will only deal with the case of characters χ where T_3^+ is empty, as these are the only cases needed for the formula on $L(f,D,k)$ and if T_3^+ is not empty, $N(\chi)$ will be larger. Assume T_3^+ is empty. If moreover the condition (3) in Theorem 1.1 is satisfied, we get $\chi_{(l)}(-1)=\epsilon_l$ for all $l\in T_3$. The Fourier coefficients of $g(z)\in S(f,\chi)$ vanish at many places. We have:

Proposition 1.2. *Let χ be a character of $(\mathbb{Z}/N')^\times$ with $\chi_{(l)}(-1)=\epsilon_l$ for $l\in T_3$. If $g(z)=\sum_{n=1}^\infty c(n)e(nz)\in S(f,\chi)$, then $c(n)=0$ if*

(1) *$l|n$ for some $l\in T_1^+$, T_2, or $T_3-T_3^-$*
or
(2) *$(\frac{en}{l})=-\epsilon_l\chi_{(l)}(-1)$ for some $l\in T_1\cup T_2$.*

We will let T_χ to be the set $(T_1-T_1^+)\cup T_3^-$. The proposition can be restated using the operator R_l. We extend the definition of R_l to half integral weight forms in the obvious way. If $g(z)\in S(f,\chi)$, $R_l^2(g)=g$ when $l\in T-T_\chi$; moreover when $l\in T_1^+\cup T_2$,

$$R_l(g)=\chi_{(l)}(-1)\epsilon_l\left(\frac{e}{l}\right)g.$$

Let S be a subset of $T_3-T_3^-$. Let $S^+(f,\chi,S)$ be the subspace of forms in $S^+(f,\chi)$ where for $l\in T_3-T_3^-$,

(C3) R_l acts with eigenvalue $-\epsilon_l(\frac{e}{l})$ when $l\in S$ and $\epsilon_l(\frac{e}{l})$ when $l\notin S$.

It is the space of $g(z)=\sum_{n=1}^\infty c(n)e(nz)\in S^+(f,\chi)$ satisfying $c(n)=0$ when for some $l\in S$ we have $(\frac{e}{l})=\epsilon_l$ or for some $l\in T_3-(S\cup T_3^-)$, $(\frac{e}{l})=-\epsilon_l$.

Theorem 1.3. *Let χ be a character of $(\mathbb{Z}/N')^\times$ and $\chi_{(l)}(-1)=\epsilon_l$ for all $l\in T_3$. The space $S^+(f,\chi)$ is the direct sum of $S^+(f,\chi,S)$ over all subsets S of $T_3-T_3^-$. Each of the spaces $S^+(f,\chi,S)$ is one-dimensional.*

Let $g_{f,\chi,S}(z)$ be a generator of $S^+(f,\chi,S)$. Thus we associate to f a set of half integral weight forms $g_{f,\chi,S}$. In [U], Ueda defined the set of newforms in the space of weight $k+1/2$

forms of level $4N$, with N odd. One can show that there is no newform in $S(f,\chi)$ when T_3^+ is not empty. When T_3^+ is empty, the generators $g_{f,\chi,S}(z)$ of $S^+(f,\chi,S)$ are newforms.

1.4. Explicit formula for $L(f,D,k)$

Let $\eta=(\eta(l))_{l\in T_1\cup T_2}\in\{\pm1\}^{|T_1\cup T_2|}$ with each component $\eta(l)\in\{\pm1\}$. Let Δ_η be the set of fundamental discriminants D satisfying $(\frac{D}{l})=-\epsilon_l$ if and only if $\eta(l)=-1$ for all $l\in T_1\cup T_2$. Let $\Xi(\eta,f)$ be the set of characters of $(\mathbb{Z}/N')^\times$ such that

(1) $\chi_{(l)}(-1)=\epsilon_l$ for $l\in T_3$;
(2) $\chi_{(l)}(-1)=\eta(l)$ for $l\in T_1\cup T_2$.

For $\eta\in\{\pm1\}^{|T_1\cup T_2|}$, fix a $\chi\in\Xi(\eta,f)$, then

$$\Delta_\eta=\bigcup_{S\subset T_3-T_3^-}\Delta_{\eta,S}$$

where $D\in\Delta_{\eta,S}$ when for $l\in T_3-T_3^-$, $(\frac{D}{l})=-\epsilon_l$ if and only if $l\in S$.

Theorem 1.4. *For $\eta\in\{\pm1\}^{|T_1\cup T_2|}$, let $\chi\in\Xi(\eta,f)$. Let $e=(-1)^k\epsilon(\chi)$ and $S\subset T_3-T_3^-$. Then for $D\in\Delta_{\eta,S}$ a fundamental discriminant satisfying $(|D|,\prod_{l\in T-T_\chi}l)=1$, we have (1.2) holds for D and $g(z)=g_{f,\chi,S}(z)$. The constant $\kappa=0$ when $\operatorname{sgn}(D)\neq e$; when $\operatorname{sgn}(D)=e$, we have $\kappa=\kappa(T,T_\chi,D)^{-1}$ where*

$$\kappa(T,T_\chi,D)=2^{|T-T_\chi|}\prod_{l\in T_3\cap T_\chi}\left(1-l^{-1}\right)\times\prod_{l\in T_1\cap T_\chi,\,(l,D)=1}2\prod_{l\in(T-T_\chi)\cap T_1}\left(1-\eta(l)l^{-1}\right)^{-1}. \tag{1.4}$$

The previous works in [KZ,K1] and [K2] considered the case when $\eta(l)\equiv1$ for all $l\in T_1\cup T_2$, with a further assumption that $\epsilon_l=1$ for $l\in T_3$. Combined assumptions allow one to choose χ to be trivial in the above theorem, then the results of Kohnen–Zagier can be recovered as the S empty case of the above theorem. As here $T_\chi=T_1$ we get $\kappa(T,T_\chi,D)=2^{|T|}$ when $(D,N)=1$. The result in [BM] is the special case of the above theorem where $T_2\cup T_3$ is empty and χ is such that $\chi_{(l)}$ is trivial whenever $\eta(l)=1$.

The theorem gives a formula for $L(f,D,k)$ as long as $(|D|,\prod_{l\in T-T_\chi}l)=1$. To get a formula for all D, we write $D=D_1D_2$ where D_1 is a fundamental discriminant that is coprime to $\prod_{l\in T-T_1}l$, and $D_2=\prod_{l\in T_0}(\frac{-1}{l})l$ with the product being taken over $T_0\subset T-T_1$. Corollary 2.6 says that $f_{T_0}=\prod_{l\in T_0}R_l(f)$ is again a q.t. primitive newform in $S_{2k}(N)$. Lemma 2.11 gives a relation between $L(f,D,k)$ and $L(f_{T_0},D_1,k)$. We apply the theorem to f_{T_0} and $D_1\in\Delta_\eta$. By choosing $\chi\in\Xi(\eta,f_{T_0})$ so that $\chi_{(l)}$ is trivial whenever $l\in T_1$ and $\eta(l)=1$, we get $D_1\in\Delta_\eta$ implies $(|D_1|,\prod_{T_1-T_\chi}l)=1$ and thus $(|D_1|,\prod_{l\in T-T_\chi}l)=1$. The theorem then gives a formula for $L(f_{T_0},D_1,k)$ and thus $L(f,D,k)$.

1.5. A canonical set of half integral weight forms

For practical purposes one may not want to fix a character χ and identify the set T_1^+, T_3^- (and thus T_χ). Let $N' = \prod_{l \in T} l$, $N'' = N\prod_{l \in T_1} l$. Let $\Gamma_1(4N'')$ be the subgroup of $SL_2(\mathbb{Z})$ consisting of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \equiv 1 \bmod 4N''$ and $c \equiv 0 \bmod 4N''$. We associate canonically to $f(z)$ a set of cusp forms for the congruence group $\Gamma_1(4N'')$.

Fix a q.t. primitive newform $f(z)$. For $\eta \in \{\pm 1\}^{|T_1 \cup T_2|}$, $\chi \in \Xi(\eta, f)$ and $S \subset T_3 - T_3^-$, for $\hat{\delta} \in (\mathbb{Z}/N')^\times$ we obtain a form $g_{f,\chi,S,\hat{\delta}}$ from $g_{f,\chi,S}$ by setting the Fourier coefficients $c(n)$ of $g_{f,\chi,S}$ to 0 when $n \not\equiv \hat{\delta} \bmod N'$. It follows from Proposition 1.2 that $g_{f,\chi,S,\hat{\delta}} = 0$ when for some $l \in T_1 \cup T_2$, $(\frac{e\hat{\delta}}{l}) \neq \epsilon_l \eta(l)$.

Let $\epsilon = \pm 1$ and $\hat{\delta} \in (\mathbb{Z}/N')^\times$. We say the pair $(\epsilon, \hat{\delta})$ is admissible if

$$\prod_{l \in T_1 \cup T_2} \left(\frac{\epsilon \hat{\delta}}{l}\right) = (-1)^k \epsilon \prod_{l \in T} \epsilon_l.$$

Theorem 1.5. *If $(\epsilon, \hat{\delta})$ is admissible, there exists a cusp form $g_{f,\epsilon,\hat{\delta}}$ of weight $k + 1/2$ for the congruence group $\Gamma_1(4N'')$ such that for any χ character of $(\mathbb{Z}/N')^\times$ with $\epsilon = e(\chi)$ and $\chi_{(l)}(-1) = \epsilon_l$ for $l \in T_3$, for any $S \subset T_3 - T_3^-(\chi)$, $g_{f,\chi,S,\hat{\delta}}$ is a scalar multiple of $g_{f,\epsilon,\hat{\delta}}$.*

Let $\Delta_{\epsilon,\hat{\delta}}$ be the set of fundamental discriminants D with $|D| = \hat{\delta} \bmod N'$ and $\operatorname{sgn}(D) = \epsilon$. For $D \in \Delta_{\epsilon,\hat{\delta}}$, when $(\epsilon, \hat{\delta})$ is not admissible, $L(f, D, k) = 0$; otherwise, let $g_{f,\epsilon,\hat{\delta}} = \sum_{n>0} c(n)e(nz)$, we have:

$$\frac{|c(|D|)|^2}{\langle g_{f,\epsilon,\hat{\delta}}, g_{f,\epsilon,\hat{\delta}}\rangle} = \frac{L(f, D, k)}{\langle f, f\rangle} |D|^{k-1/2} \frac{(k-1)!}{\pi^k} \kappa'' \tag{1.5}$$

with

$$\kappa'' = \prod_{l \in T}(l-1) \prod_{l \in T_1} \left(1 - \left(\frac{\epsilon \hat{\delta}}{l}\right) \epsilon_l l^{-1}\right)^{-1}.$$

There are total $\phi(N')$ pairs of $(\epsilon, \hat{\delta})$ satisfying the condition in the theorem (ϕ is the Euler function). Note however the map $g_{f,\epsilon,\hat{\delta}} \mapsto \hat{\delta}$ is not necessarily a bijection to $(\mathbb{Z}/N')^\times$.

The author would like to acknowledge Waldspurger and Baruch whose previous works we rely heavily upon ([W1,W2,W3,BM]). He also thanks the referee for carefully reading the manuscript and providing many helpful suggestions.

Notations

$\mathbb{Q}$ and $\mathbb{R}$ are fields of rational numbers and real numbers. $\mathbb{Z}$ is the ring of integers. We use p and l to denote prime numbers. $\mathbb{Q}_p$ is the p-adic field with $\mathbb{Z}_p$ as its ring of integers. $\mathbf{A}_\mathbb{Q}$ is the ring of adeles.

We fix an additive character ψ on $\mathbf{A}_\mathbb{Q}$ so that $\psi = \otimes \psi_v$, where $\psi_\infty(x) = e(x)$, and for $v = p$, $\psi_p(x) = e(-\hat{x})$ where $\hat{x} \in \mathbb{Q}$ such that $x - \hat{x} \in \mathbb{Z}_p$. For $a \in \mathbb{Q}_v$, $\gamma_v(a)$ is Weil factor associated to the character ψ_v and the quadratic form ax^2 [We, p. 19].

Fix N a positive odd integer. We write $p|N$ if N is divisible by p; $p^{\nu}\|N$ if $p^{\nu}|N$ and $p^{\nu+1}$ does not divide N. For any prime p, define $K_{0,p}$ as the subgroup of $GL_2(\mathbb{Z}_p)$ with elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \in N\mathbb{Z}_p$.

For $\tau \in \mathbb{Z}_p \backslash \{0\}$, we define the quadratic character $\chi_{\tau,(p)}$ to be associated to the quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$. At the infinite place $\chi_{\tau,\infty}$ is trivial if $\tau > 0$ and the sign character if $\tau < 0$. For $D \in \mathbb{Q}^{\times}$, $\chi_D = \otimes \chi_{D,(p)}$. When the local place is clear from the context, we will also use χ_D instead of $\chi_{D,(p)}$.

Let μ be a character of $\mathbb{Q}_p^{\times}$. We use $\pi(\mu)$ to denote the principal series representation of $GL_2(\mathbb{Q}_p)$ induced from the character $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \mapsto \mu(ad^{-1})$. When $\mu^2(a) = |a|$, we let $\sigma(\mu)$ be the special representation that is a component of $\pi(\mu)$.

Let $\overline{SL}_2$ be the two fold cover of SL_2. Its elements have the form $(g, \pm 1)$ with $g \in SL_2$. We will use g to denote $(g, 1)$. We use $\tilde{\pi}(\mu)$ to denote the principal series representation of $GL_2(\mathbb{Q}_p)$ induced from the character $\left(\begin{smallmatrix} a & b \\ & a^{-1} \end{smallmatrix}\right) \mapsto \mu(a)\gamma_p(a)$. When $\mu^2(a) = |a|$, we let $\tilde{\sigma}(\mu)$ be the special representation that is a component of $\tilde{\pi}(\mu)$.

The Peterson norm is defined as usual: If f is a cusp form of weight $k \in \frac{1}{2}\mathbb{Z}$ for a subgroup Γ of finite index in $\Gamma(1) = SL_2(\mathbb{Z})$, we define

$$\langle f, f \rangle = \frac{1}{[\Gamma(1) : \Gamma]} \int\limits_{\Gamma \backslash \mathcal{H}} \left| f(z) \right|^2 y^{k-2} \, dx \, dy$$

where $z = x + iy$ and $\mathcal{H}$ is the upper half plane.

When we say *unique*, we always mean unique up to a scalar multiple.

2. Q.t. primitive newforms

The concept of newforms in $S_{2k}(N)$ is introduced in [AL]. The work in [Ca] gives its interpretation in representation theory. A nice summary of the results can be found in [Sc].

Let $f \in S_{2k}(N)$ be a new form. Then f determines a vector in the space of automorphic forms on $GL_2(\mathbf{A}_{\mathbb{Q}})$ by $f \mapsto \varphi = s(f)$. The map $s(f)$ is defined as follows. For $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})$, let

$$f|_{g_\infty}(z) = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2k}.$$

Consider g_∞ as an element $(g_\infty, e, e, \ldots)$ in $GL_2(\mathbf{A}_{\mathbb{Q}})$, then $\varphi(g_\infty) = f|_{g_\infty}(i)$, and $\varphi(\gamma g k) = \varphi(g)$ whenever $\gamma \in GL_2(\mathbb{Q})Z(\mathbf{A}_{\mathbb{Q}})$, and $k \in \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p|N} K_{0,p}$. From the approximation theory, this defines φ for all $GL_2(\mathbf{A}_{\mathbb{Q}})$.

Then φ is a vector in the space of an irreducible cuspidal representation $\pi = \pi_f$ of $GL_2(\mathbf{A}_{\mathbb{Q}})$, with trivial central character (thus of $PGL_2(\mathbf{A}_{\mathbb{Q}})$). The representation $\pi = \otimes \pi_v$, and $\varphi = \otimes_v \varphi_v$ can be described as follows:

(1) When $v = \infty$, π_v is a discrete series representation with φ_∞ the minimal weight vector, with weight $2k$.
(2) When v is p-adic, then π_p has conductor $N\mathbb{Z}_p$ and φ_v is the unique new vector fixed under $K_{0,p}$ [Ca].

Lemma 2.1. *The map $f \mapsto \pi_f$ defined above gives a one–one correspondence between the newforms in $S_{2k}(N)$ and the representations $\pi = \pi_\infty \otimes_p \pi_p$ such that π_∞ has a weight $2k$ minimal weight vector, and conductors of π_p equal to $N\mathbb{Z}_p$ for all primes p.*

We can be more precise on the local components π_p.

Lemma 2.2. *Let $\pi = \pi_f$ be as above. Then:*

- (i) *When $p \nmid N$, π_p is an unramified representation $\pi(\mu)$ where μ is an unramified character such that $\mu(p) + \mu(p^{-1}) = a(p)p^{\frac{1}{2}-k}$.*
- (ii) *When $p \| N$, π_p is a special representation $\sigma(\chi \|^{\frac{1}{2}})$. When $W_{(p)}$ has eigenvalue -1, χ is the trivial character. When $W_{(p)}$ has eigenvalue 1, χ is the quadratic character χ_τ where τ is a non-square unit in $\mathbb{Z}_p$.*
- (iii) *When $p^2 \| N$, π_p could be a principal series, supercuspidal or a special representation. When it is a special representation $\sigma(\chi \|^{\frac{1}{2}})$, $\chi = \chi_\tau$ where τ is a generator of the prime ideal in $\mathbb{Z}_p$.*
- (iv) *When $p^\nu \| N$, with $\nu > 1$ and odd, π_p is a supercuspidal representation.*
- (v) *When $p^\nu \| N$, with $\nu > 2$ and even, π_p can be a supercuspidal representation or a principal series representation.*

All the above is an easy consequence of the formula for conductor of π_v, see [Sc] for example. Now we consider the twist f_l for l an odd prime. Recall the following result in [AL].

Lemma 2.3.

(1) *If $l^3 | N$, f_l is a newform in $S_{2k}(N)$.*
(2) *If l is an odd prime and $l^2 \nmid N$, then f_l is a newform in $S_{2k}(N')$, where $N' = Nl$ if $l | N$ or $N' = Nl^2$ if $l \nmid N$.*
(3) *If $l^2 \| N$ and f_l is not a newform in $S_{2k}(N)$, then there is f' in $S_{2k}(N/l)$ or $S_{2k}(N/l^2)$ such that $f = f'_l$.*

The representations corresponding to f and f_l has the following relationship.

Lemma 2.4. *When f_l is a newform, $\pi_{f_l} = \pi_f \otimes \chi_{(\frac{-1}{l})l}$.*

Proof. By the strong multiplicity one theorem for PGL_2, we only need to compare the local components of the two sides at $p \nmid 2N$. The component at left-hand side is $\pi(\mu_1)$ with

$$\mu_1(p) + \mu_1\bigl(p^{-1}\bigr) = a(p)\left(\frac{p}{l}\right)p^{\frac{1}{2}-k}.$$

The component at right-hand side is $\pi(\mu_2)$ with $\mu_2 = \mu\chi_{(\frac{-1}{l})l}$. Thus

$$\mu_2(p) + \mu_2\bigl(p^{-1}\bigr) = a(p)p^{\frac{1}{2}-k}\left(\frac{(\frac{-1}{l})l}{p}\right).$$

It follows from the quadratic reciprocity law that the right-hand sides of the above two equations are the same. As μ_1 and μ_2 are both unramified characters we get either $\mu_1 = \mu_2$ or $\mu_1 = \mu_2^{-1}$; either way we get $\pi(\mu_1) = \pi(\mu_2)$. □

Lemma 2.5. *The following are equivalent*:

(1) *$f(z)$ is a q.t. primitive newform in $S_{2k}(N)$.*
(2) *There does not exist $l|N$ and $f'(z)$ a newform in $S_{2k}(N/l)$ or $S_{2k}(N/l^2)$, such that $f(z) = f'_l(z)$.*
(3) *The local component $\pi_{f,l}$ of π_f at any places l with $l^2 \| N$ is not a special representation or a principal series representation $\pi(\mu)$ with $\mu\chi_l$ being unramified.*
(4) *The local component $\pi_{f,p}$ at any place p is not a special representation $\sigma(\chi\|^{\frac{1}{2}})$ with $\chi = \chi_\tau$ or a principal series representation $\pi(\mu)$ with $\mu\chi_\tau$ being unramified*; *here τ is a generator of the prime ideal in $\mathbb{Z}_p$.*

Proof. Lemma 2.3 gives (2) implies (1). We show (3) implies (2). If $f(z) = f'_l(z)$ for a prime l with $l^2 \| N$ and a newform f' of level N/l or N/l^2, from Lemma 2.4, we see the local component $\pi_{f,l}$ is $\pi_{f',l} \otimes \chi_{(\frac{-1}{l})l}$. From Lemma 2.2 we see $\pi_{f',l}$ is either an unramified representation or a special representation $\sigma(\chi\|^{\frac{1}{2}})$, where χ is either trivial or χ_τ with τ being a non-square unit in $\mathbb{Z}_l$. Thus $\pi_{f,l}$ is the type of representation described in (3). Thus (3) implies (2).

If f_l is a newform in $S_{2k}(N)$ and π_f has local component described as in (3) for some prime l, then its conductor is $l^2\mathbb{Z}_l$, thus $l^2 \| N$. We consider f_l. From Lemma 2.4 we get the local component of $\pi_{f_l,l}$ is either unramified or a special representation $\sigma(\chi\|^{\frac{1}{2}})$, where χ is either trivial or χ_τ with τ being a non-square unit in $\mathbb{Z}_l$. In either case the conductor of $\pi_{f_l,l}$ is not $l^2\mathbb{Z}_l$, thus f_l is not a newform in $S_{2k}(N)$. We get (1) implies (3).

The equivalence between (3) and (4) follows from Lemma 2.2. □

Corollary 2.6. *If f is a q.t. primitive newform in $S_{2k}(N)$, for any prime l with $l^2|N$, f_l is again a q.t. primitive newform in $S_{2k}(N)$.*

Proof. We only need to check the local components of π_{f_l} satisfies condition (4) in the previous lemma. This is clear from Lemma 2.4. □

The following is the well-known relation between the Atkin–Lehner eigenvalues and the ϵ factors of π_f [Sc].

Lemma 2.7. *For $l|N$, $\epsilon_l = \epsilon(\pi_{f,l}, 1/2)$. Let ν be such that $l^\nu \| N$, and φ_l be the new vector of $\pi_{f,l}$*; *we have*

$$\pi_{f,l}\begin{pmatrix} & 1 \\ l^\nu & \end{pmatrix}\varphi_l = \epsilon_l \varphi_l. \tag{2.1}$$

Recall the following definition in [W2]: for π an irreducible representation of $PGL_2(\mathbb{Q}_p)$, for $D \in \mathbb{Q}_p^\times n$,

$$\epsilon(\pi \otimes \chi_D, 1/2) = \pm\chi_D(-1)\epsilon(\pi, 1/2). \tag{2.2}$$

Let $X_+(\pi)$ be the set of D where the above equation take the plus sign, and $X_-(\pi)$ the set of D with negative sign. We have the following [W2]:

Lemma 2.8.

(0) *The square class of* 1 *is in* $X_+(\pi)$.
(1) *When* π *is a principal series representation,* $X_+(\pi) = \mathbb{Q}_p^\times$.
(2) *When* p *is odd and* π *is a supercuspidal representation,* $X_+(\pi)$ *and* $X_-(\pi)$ *consists of two square classes of* $\mathbb{Q}_p^\times$ *each.*
(3) *When* p *is odd and* π *is a special representation* $\sigma(\chi_\tau\|^{\frac{1}{2}})$, *then*:
  (i) *When* $\tau = 1$, $X_+(\pi)$ *is the square class of* 1 *and* $X_-(\pi)$ *consists of the remaining three square classes.*
  (ii) *When* τ *is a non-square,* $X_-(\pi)$ *is the square class of* τ *and* $X_+(\pi)$ *consists of the remaining three square classes.*

We will be a bit more precise about the sets $X_+(\pi)$ and $X_-(\pi)$.

Lemma 2.9. *When* p *is odd and* π *is a supercuspidal representation of* $PGL_2(\mathbb{Q}_p)$, *let* D *be a non-square unit in* $\mathbb{Z}_p$, *then* $D \in X_-(\pi)$ *if and only if the conductor of* π *is* $p^\nu\mathbb{Z}_p$ *with* ν *odd.*

Proof. Let π and D as above, then $\pi \otimes \chi_D$ is a supercuspidal representation with the same conductor. Take the Whittaker model of π and let φ_p be the new vector in π, it is a function of $GL_2(\mathbb{Q}_p)$. Then $\varphi_p\chi_D$ is the new vector of $\pi \otimes \chi_D$. From Lemma 2.7 we get $\epsilon(\pi \otimes \chi_D, 1/2) = \chi_D(-p^\nu)\epsilon(\pi, 1/2)$. Since $\chi_D(-1) = 1$ and $\chi_D(p) = -1$ over $\mathbb{Q}_l$, we get $D \in X_+(\pi_{f,l})$ if and only if ν is even. □

By the above lemma, when π has conductor $p^\nu\mathbb{Z}_l$ with $\nu > 1$ even, we have the square classes of 1 and non-square unit all lie in $X_+(\pi)$. Thus p lies in $X_+(\pi)$ if π is a principal series representation, it lies in $X_-(\pi)$ if π is a supercuspidal representation by Lemma 2.9. Assume f is a q.t. primitive newform in $S_{2k}(N)$. Let l be an odd prime such that $l^\nu \| N$ with ν an even positive integer. Thus by Lemma 2.2 $\pi_{f,l}$ is either a principal series or a supercuspidal representation. The above discussion gives:

Corollary 2.10. *Assume* f *is a q.t. primitive newform in* $S_{2k}(N)$. *Let* l *be an odd prime such that* $l^\nu \| N$ *with* ν *an even positive integer. Let* ϵ *and* ϵ' *be the Atkin–Lehner eigenvalue at* l *for* f *and* f_l, *respectively. Then* $\pi_{f,l}$ *is a principal series representation if* $\epsilon = (\frac{-1}{l})\epsilon'$; *it is a supercuspidal representation if* $\epsilon = -(\frac{-1}{l})\epsilon'$. *Or equivalently,* $\pi_{f,l}$ *is supercuspidal when* $l \in X_-(\pi_{f,l})$ *and a principal series representation when* $l \in X_+(\pi_{f,l})$.

From the corollary, we see that when $l \in T_3^p$, $\pi_{f,l}$ is a principal series representation; when $l \in T_3^s$, it is a supercuspidal representation. In particular when $l \in T'$, $\pi_{f,l} = \pi(\xi)$ for some character ξ. As the conductor of $\pi_{f,l}$ is $l^2\mathbb{Z}_l$ when $l \in T'$, we see ξ is trivial on $1 + l\mathbb{Z}_l$ (see [Sc]). Restricting ξ to $\mathbb{Z}_l^\times$ gives a character $\xi_{f,l}$ of $(\mathbb{Z}/l)^\times$ mentioned in the introduction.

The character $\xi_{f,l}$ can be introduced without resort to representation language. Given $g(z) = \sum b(n)e(nz)$ and a character $\chi_{(l)}$ of $(\mathbb{Z}/l)^\times$, we define $g_{\chi_{(l)}}(z) = \sum b(n)\chi_{(l)}(n)e(nz)$. Similar to Theorems 4.3 and 4.3 in [A-Li], $\xi_{f,l}$ is the unique character $\chi_{(l)}$ such that there is a $g(z) \in S_{2k}(N_0, \chi_{(l)}^{-2})$ of level $N_0 < N$ with $g_{\chi_{(l)}}(z) = f(z)$.

We now consider the L-values. In [BM], we observed the following relation between the values $L(\pi_f \otimes \chi_D, 1/2)$ and $L(f, D, k)$:

$$L(f, D, k) = L^{\infty\cup\{2\}}(\pi_f \otimes \chi_D, 1/2). \tag{2.3}$$

When f is a q.t. primitive newform in $S_{2k}(N)$, let $T_0 \subset T - T_1$ as in introduction and $L = \prod_{l \in T_0} l$, then $f_{T_0} = f_L$ is a q.t. primitive newform in $S_{2k}(N)$ by Corollary 2.6, with $\pi_{f_{T_0}} = \pi_f \otimes \chi_{(\frac{-1}{L})L}$ by Lemma 2.4. Thus for D a fundamental discriminant,

$$L(f_{T_0}, D, k) = L^{\infty\cup\{2\}}(\pi_f \otimes \chi_{(\frac{-1}{L})LD}, 1/2). \tag{2.4}$$

From (2.3) and (2.4) we get

Lemma 2.11. *For $T_0 \subset T - T_1$, let $D_2 = \prod_{l \in T_0} (\frac{-1}{l}) l$, we have for all fundamental discriminant D_1 with $(|D_1|, |D_2|) = 1$:*

$$L(f, D_1 D_2, k) = L(f_{T_0}, D_1, k). \tag{2.5}$$

3. Shimura correspondence by Waldspurger

Assume k is a nonnegative integer. Let M be a positive odd integer. Let $S_{k+1/2}(4M, \chi)$ be the space of holomorphic cusp forms of weight $k + 1/2$, level $4M$ and character χ'. The functions in the space satisfies [W2]:

$$g\left(\frac{az+b}{cz+d}\right) = j(\sigma, z)^{2k+1} \chi'(d) g(z), \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad 4M \mid c. \tag{3.1}$$

Here

$$j(\sigma, z) = \theta\left(\frac{az+b}{cz+d}\right) \Big/ \theta(z), \qquad \theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}. \tag{3.2}$$

Given $g(z)$ above define $\tilde{\varphi} = t(g)$, which is a function on $\overline{SL}_2(\mathbf{A}_{\mathbb{Q}})$ that is continuous and left invariant under $SL_2(\mathbb{Q})$ and satisfies:

$$t(g)\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 1\right) = y^{k/2+1/4} e^{i(k+1/2)\theta} g(x + yi), \tag{3.3}$$

where $y > 0$, $x \in \mathbb{R}$ and $-\pi < \theta \leqslant \pi$. By the strong approximation theorem for SL_2, we see $t(g)$ is unique.

Recall for an irreducible representation $\tilde{\pi}_v$ of $\overline{SL}_2$ at a local place v, the definition of the central character $\omega(\tilde{\pi}_v)$ is: $\omega(\tilde{\pi}_v) id = \tilde{\pi}_v(-1)/\gamma_v(-1)$.

Lemma 3.1. *If $S_{k+1/2}(f, 4M, \chi, \epsilon)$ is not $\{0\}$, the space $t(S_{k+1/2}(f, 4M, \chi, \epsilon))$ lies in the space of an irreducible cuspidal representation $\tilde{\pi} = \tilde{\pi}(f, \epsilon, \chi)$ of $\overline{SL}_2(\mathbf{A}_{\mathbb{Q}})$. Moreover $\omega(\tilde{\pi}_l) = \chi_{(l)}(-1)$ when $l | M$; $\omega(\tilde{\pi}_2) = e = (-1)^k \epsilon(\chi)$.*

Proof. The fact about central character follows from the description of the map t given in [W2, Chapter III(B)] (see also [BM] and the next section). Assume $\tilde{\pi} = \oplus \tilde{\pi}_i$ of irreducible representations. Then the condition (C1) says that $\tilde{\pi}_i$ all lie in the same near equivalence class. By the strong multiplicity one theorem [W3], there is a unique representation in a near equivalence class with a given central character. By the multiplicity one theorem there is one representation in the above sum and $\tilde{\pi}$ is irreducible. □

Next we describe the representation $\tilde{\pi}(f, \epsilon, \chi)$ by the theta correspondence.

Recall we have fixed a character ψ of $\mathbf{A}_{\mathbb{Q}}$. In [W1] the theta correspondences $\tilde{\pi} = \Theta(\pi, \psi)$ and $\pi = \Theta(\tilde{\pi}, \psi)$ are defined between an irreducible cuspidal representation π of PGL_2 and an irreducible cuspidal representation $\tilde{\pi}$ of $\overline{SL}_2$. The properties of these correspondences are summarized in Section 3 of [BM]. In particular associated to π is near equivalence class $\{\tilde{\pi}\}$ satisfying $\pi = S_\psi(\tilde{\pi})$ for all $\tilde{\pi}$ in the class, where $S_\psi(\tilde{\pi}) = \Theta(\tilde{\pi}, \psi^D) \otimes \chi_D$ for any D such that $\tilde{\pi}$ has ψ^D-Whittaker model. One can also define the map S_ψ locally in a similar fashion.

Proposition 4 of [W2] says:

Lemma 3.2. *We have $\pi_f \otimes \chi_\epsilon = S_\psi(\tilde{\pi}(f, \epsilon, \chi))$.*

From the description of the near equivalence class [BM], we see if $\tilde{\pi}$ is such that $S_\psi(\tilde{\pi}) = \pi_f \otimes \chi_\epsilon$, then $\tilde{\pi} = \tilde{\pi}_\epsilon^D = \Theta(\pi_f \otimes \chi_{\epsilon D}, \psi^D)$ for some $D \in \mathbb{Q}^\times$. Thus the local components of $\tilde{\pi}$ at a place v is one of possibly two representations: $\tilde{\pi}^+_{f,\epsilon,v} = \Theta(\pi_{f,v} \otimes \chi_{\epsilon D}, \psi^D)$ for all $D \in X_+(\pi_{f,v} \otimes \chi_\epsilon)$ and $\tilde{\pi}^-_{f,\epsilon,v} = \Theta(\pi_{f,v} \otimes \chi_{\epsilon D}, \psi^D)$ for all $D \in X_-(\pi_{f,v} \otimes \chi_\epsilon)$ if it is an nonempty set. Any combination of $\tilde{\pi}^\pm_{f,\epsilon,v}$ gives an automorphic representation $\tilde{\pi}$ if the global central character is trivial [W3, Lemma 40].

By [W1] assertion 10, $\omega(\tilde{\pi}^\pm_{f,\epsilon,v}) = \pm\epsilon(\pi_{f,v} \otimes \chi_\epsilon, 1/2)$. Note if $\pi = \pi(\mu)$ is a principal series representation, $\epsilon(\pi(\mu), 1/2) = \mu(-1)$. Thus from the result on $\epsilon(\pi, 1/2)$ in [Sc] and Lemma 2.8, we have:

Lemma 3.3. *If $\tilde{\pi}$ is such that $S_\psi(\tilde{\pi}) = \pi_f \otimes \chi_\epsilon$, then*:

(1) *When $p \in T_3^p$ or $p \notin T$, the local component at p of $\tilde{\pi}$ is $\tilde{\pi}^+_{f,\epsilon,p} = \tilde{\pi}^+(\pi_{f,p} \otimes \chi_{\epsilon,(p)})$. When $p \notin T \cup \{2\}$, $\omega(\tilde{\pi}^+_{f,\epsilon,p}) = 1$; at $p = 2$, $\omega(\tilde{\pi}^+_{f,\epsilon,2}) = \epsilon$; when $l \in T_3^p$, $\omega(\tilde{\pi}^+_{f,\epsilon,l}) = \epsilon_l(f)$.*
(2) *When $l \in T - T_3^p$, the local components at l of $\tilde{\pi}$ is one of the representations in the set $\{\tilde{\pi}^\pm_{f,\epsilon,l}\}$.*
(3) $\prod_{l \in T \cup \{2,\infty\}} \omega(\tilde{\pi}_l) = 1$.

Conversely any $\tilde{\pi}$ with local component given by conditions (1), (2), (3) *is an automorphic representation with $S_\psi(\tilde{\pi}) = \pi_f \otimes \chi_\epsilon$.*

Proof. In case 1, the local representation $\pi_{f,p} = \pi(\mu_p)$ is a principal series representation, thus $X_-(\pi_{f,p} \otimes \chi_{\epsilon,(p)})$ is empty. Note $\epsilon(\pi(\mu_p) \otimes \chi_{\epsilon,(p)}, 1/2) = \mu\chi_{\epsilon,(p)}(-1)$; also $\chi_{\epsilon,(p)}(-1) = 1$ when $p \neq 2$ and $\chi_{\epsilon,(2)}(-1) = e$. When $p \notin T$, μ is unramified we get $\mu\chi_{\epsilon,(p)}(-1) = 1$ or ϵ depending on whether $p = 2$; when $l \in T_3^p$, since $\mu(-1) = \epsilon(\pi(\mu)) = \epsilon_l(f)$ by Lemma 2.7, we get the claim on central characters. The assertion on case (2) is clear. The assertion (3) follows from assertion (1) and the fact that the global character is trivial. The last assertion follows from [W3, Lemma 40]. □

The lemma imposes some compatibility condition on χ, ϵ and the Atkin–Lehner eigenvalues $\epsilon_l = \epsilon_l(f)$:

Corollary 3.4. *If $S_{k+1/2}(f, 4M, \chi, \epsilon)$ is not $\{0\}$, then*

(1) $\epsilon = e = (-1)^k \epsilon(\chi)$;
(2) $\chi_{(l)}(-1) = \epsilon_l$ *for* $l \in T_3^p$.

Let $\tilde{\pi}(f, \chi) = \tilde{\pi}(f, e, \chi)$. When $l \in T - T_3^p$, let $e_l = \chi_{(l)}(-1)\epsilon(\pi_{f,l} \otimes \chi_e, 1/2)$, then $\tilde{\pi}_l(f, \chi) = \tilde{\pi}^{e_l}_{f,e,l}$.

Proof. In this case $\tilde{\pi} = \tilde{\pi}(f, \epsilon, \chi)$ has the infinite component $\tilde{\pi}_\infty = \tilde{\pi}^+_{f,\epsilon,\infty}$ being a holomorphic discrete series. The compatibility condition follows from the above lemma and the fact that $\omega(\tilde{\pi}_\infty) = (-1)^k$. The description of local components at $l \in T - T_3^p$ follows from the result on central character in Lemma 3.1. □

When $l \in T_3^p$ or T_1, $\pi_{f,l}$ is a principal series or special representation, we can be a bit more precise about $\tilde{\pi}^{\pm}_{f,e,l}$ [W3]:

Lemma 3.5. *When $\pi_{f,l} = \pi(\mu)$, $\tilde{\pi}^+_{f,e,l} = \tilde{\pi}(\mu\chi_{e,(l)})$.*

When $\pi_{f,l} = \sigma(\chi_\tau \|^{1/2})$ with τ a non-square unit in $\mathbb{Z}_l^\times$, $\tilde{\pi}^+_{f,e,l} = \tilde{\sigma}(\chi_\tau \|^{1/2})$ and $\tilde{\pi}^-_{f,e,l}$ is a supercuspidal representation $r^-_{\psi^\tau}$ (the odd Weil representation, see [BM]).

When $\pi_{f,l} = \sigma(\|^{1/2})$, $\tilde{\pi}^+_{f,e,l} = r^-_\psi$ and $\tilde{\pi}^-_{f,e,l} = \tilde{\sigma}(\|^{1/2})$.

We calculate the sign $e_l = \chi_{(l)}(-1)\epsilon(\pi_{f,l} \otimes \chi_e, 1/2)$ for $l \in T - T_3^p$. From (2.2), $\epsilon(\pi_{f,l} \otimes \chi_e, 1/2) = \pm\epsilon_l$ when $e \in X_\pm(\pi_{f,l})$. From the description of $X_\pm(\pi_{f,l})$ in Lemmas 2.8 and 2.9, we get:

Lemma 3.6. *If $l \in T_1 \cup T_2$, then $e_l = (\frac{e}{l})\chi_{(l)}(-1)\epsilon_l$. If $l \in T_3^s$, $e_l = \chi_{(l)}(-1)\epsilon_l$.*

We remark also that the representation $\tilde{\pi}(f, \chi)$ can be defined without assuming $S_{k+1/2}(f, 4M, \chi, e)$ is nonzero. From the argument for Corollary 3.4, we get:

Corollary 3.7. *Given f and a character χ of $(\mathbb{Z}/N')^\times$ such that $\epsilon_l = \chi_{(l)}(-1)$ for $l \in T_3^p$; let $e = (-1)^k\epsilon(\chi)$. Let $\tilde{\pi}(f, \chi) = \otimes\tilde{\pi}^{\pm}_{f,e,v}$ where the negative case is taken if and only if $l \in T - T_3^p$ with $\chi_{(l)}(-1) = -\epsilon(\pi_{f,l} \otimes \chi_{(e)}, 1/2)$. Then $\tilde{\pi}(f, \chi)$ is a cuspidal automorphic representation with $S_\psi(\tilde{\pi}) = \pi_f \otimes \chi_e$.*

4. Representation and level

Let $M = \prod_{l \in T} l^{\nu(l)}$. From Proposition 3 of [W2], the intersection of $t(S_{k+1/2}(4M, \chi))$ with any irreducible cuspidal representation $\tilde{\pi} = \tilde{\pi}_\infty \otimes_p \tilde{\pi}_p$ is a subspace V spanned by vectors $\otimes\tilde{\varphi}_v$ where (see [W2, Chapter III] or [BM]).

(1) At ∞, $\tilde{\varphi}_\infty$ is the minimal weight vector of $\tilde{\pi}^+_{f,e,\infty}$.

(2) At 2, $\tilde{\varphi}_2$ is a vector in a two-dimensional subspace of $\tilde{\pi}^+_{f,e,2}$ (see [BM], the Kohnen vector which lies in this space is also defined in [BM]).
(3) At $p \notin \{2\} \cup T$, $\tilde{\varphi}_p$ is the unramified vector of $\tilde{\pi}^+_{f,e,p}$.
(4) At $l \mid M$, $\tilde{\varphi}_l$ lies in the subspace $\tilde{\pi}_l(\nu, \chi_{(l)})$ of vectors v satisfying:

$$\tilde{\pi}_l(\sigma)v = \chi_{(l)}(d)v, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \in \mathbb{Z}_l^\times,\ b \in \mathbb{Z}_l,\ c \in p^\nu \mathbb{Z}_l.$$

We will describe the space $\tilde{\pi}_l(\nu, \chi_{(l)})$ for various cases of the representation $\tilde{\pi}_l$ and the integer ν. In Assertion 7 of [W2], a Kirillov space $K(\tilde{\pi}_l, \chi_{(l)})$ is associated to any irreducible representation $\tilde{\pi}_l$ of $\overline{SL}_2(\mathbb{Q}_l)$, such that $K(\tilde{\pi}_l, \chi_{(l)})$ is consisting of functions $t(x)$ with $x \in \mathbb{Q}_l^\times$ and

$$\tilde{\pi}_l \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} t(x) = \psi(ax)t(x), \tag{4.1}$$

$$\tilde{\pi}_l \begin{pmatrix} d & \\ & d^{-1} \end{pmatrix} t(x) = \gamma_l(d)|d|_l \chi_{(l)}\big(d^{-1}\big)t\big(xd^2\big). \tag{4.2}$$

Such a space is not unique. Let $\mathbb{Q}_l^\times(\tilde{\pi}_l)$ be the set of D such that $\tilde{\pi}_l$ has ψ^D-Whittaker model (which is unique), then for any such Kirillov space $K(\tilde{\pi}_l, \chi_{(l)})$, any $t(x)$ in the space, $t(x)$ is supported in $\mathbb{Q}_l^\times(\tilde{\pi}_l)$. Moreover $K(\tilde{\pi}_l, \chi_{(l)})$ contains the Schwartz space on $\mathbb{Q}_l^\times(\tilde{\pi}_l)$. From (4.1) we get

Lemma 4.1. *If $\delta \in \mathbb{Q}_l^\times(\tilde{\pi}_l)$, then the linear form $\tilde{L}^\delta : t(x) \mapsto t(\delta)$ is a nontrivial ψ^δ-Whittaker functional on the Kirillov space $K(\tilde{\pi}_l, \chi_{(l)})$.*

Given $\delta \in \mathbb{Q}_l^\times$, define a function

$$\alpha[\delta](a) = \begin{cases} 1 & a = \delta\beta^2,\ \beta \in \mathbb{Z}_l^\times, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The following is a special case of Propositions 6 and 11 in [W2].

Proposition 4.2. *Given $\tilde{\pi}$ an irreducible admissible representation of $\overline{SL}_2(\mathbb{Q}_p)$ with p an odd prime, let μ be a character with $\mu(-1) = \omega(\tilde{\pi})$ and trivial over $1 + p\mathbb{Z}_p$, let $m(\tilde{\pi})$ be such that the conductor of $S_\psi(\tilde{\pi}) \otimes \mu$ is $p^{m(\tilde{\pi})}\mathbb{Z}_p$. Let $V(n, \mu)$ be the subspace of $K(\tilde{\pi}, \mu)$ spanned by $\{\alpha[\delta] \mid \delta \in \mathbb{Q}_p^\times(\tilde{\pi}), 0 \leqslant v_p(\delta) \leqslant n - \sup(2, m(\tilde{\pi}))\}$.*

Then $\tilde{\pi}(n, \mu)$ is the direct sum of $\tilde{\pi}(1, \mu)$ and $V(n, \mu)$. Moreover when $m(\tilde{\pi}) > 1$, $\tilde{\pi}(1, \mu) = \{0\}$.

If $\mu(-1) \neq \omega(\tilde{\pi})$ then $\tilde{\pi}(n, \mu)$ is $\{0\}$.

When $m(\tilde{\pi}) \leqslant 1$, we have the following cases:

(1) $S_\psi(\tilde{\pi}) = \pi(\mu_0)$ with $\mu_0\mu$ unramified or $\mu_0\mu^{-1}$ unramified or both.
(2) $S_\psi(\tilde{\pi}) = \sigma(\mu_0\|^{1/2})$ with $\mu_0^2 = 1$ and $\mu_0\mu$ being unramified.

The space $\tilde{\pi}(1,\mu)$ is described in [W2, pp. 424–426].

We now specialize to our situation and describe the space $\tilde{\pi}_l(\nu,\chi_{(l)})$ for all $l\in T$ and positive integer ν.

Case 1. When $l\in T-T_1-T_3^p$, $\pi_{f,l}$ is a supercuspidal representation. Then $\tilde{\pi}^+_{f,e,l}$ and $\tilde{\pi}^-_{f,e,l}$ are supercuspidal. We have $m(\tilde{\pi}^{\pm}_{f,e,l})=\nu_l$ with $\nu_l>1$. Thus the proposition gives a complete description of $\tilde{\pi}_l(\nu,\chi_{(l)})$.

Case 2. When $l\in T_3^p$, $\pi_{f,l}$ is a principal series representation $\pi(\mu)$. Then $\tilde{\pi}_l=\tilde{\pi}(\mu\chi_{e,(l)})$. As $\omega(\tilde{\pi}_l)=\mu(-1)$, $\tilde{\pi}_l(\nu,\chi_{(l)})=\{0\}$ if $\mu\chi_{(l)}(-1)=-1$.

(i) If $\mu\chi_{(l)}$ and $\mu\chi_{(l)}^{-1}$ are both ramified, then $m(\tilde{\pi}_l)>1$ and the proposition gives a complete description of $\tilde{\pi}_l(\nu,\chi_{(l)})$. Note then $l\in T_3-T_3^-$.

(ii) If $\mu\chi_{(l)}$ or $\mu\chi_{(l)}^{-1}$ is unramified. As from our assumption that $f(z)$ is a q.t. primitive newform, we know these characters can not be both unramified. Since $\pi(\mu)=\pi(\mu^{-1})$, we will assume $\mu\chi_{(l)}^{-1}$ is unramified. From p. 424 of [W2] case (2)(b), we get $\tilde{\pi}(1,\chi_{(l)})$ is spanned by the characteristic function of $\mathbb{Z}_l-\{0\}$. We note in this case $\chi_{(l)}$ is nontrivial, and $l\in T_3^-$.

Case 3. When $l\in T_1$, then $\pi_{f,l}$ is the special representation $\sigma(\chi_\tau\|^{1/2})$ where τ is a unit in $\mathbb{Z}_l^\times$. From Lemma 3.5 we get $\tilde{\pi}_l$ is either $r^-_{\psi^{\tau'}}$ or $\tilde{\sigma}(\chi_{\tau'}\|^{1/2})$ with τ' a unit.

(i) In the case where $\tilde{\pi}_l=r^-_{\psi^{\tau'}}$, we get $\omega(\tilde{\pi}_l)=-1$. Thus when $\chi_{(l)}$ is even, the space $\tilde{\pi}_l(\nu,\chi_{(l)})$ is $\{0\}$. When $\chi_{(l)}$ is odd, the conductor of $\pi_{f,l}\otimes\chi_{(l)}$ is $l^2\mathbb{Z}_l$ and $\tilde{\pi}(\nu,\chi_{(l)})$ is completely described by the proposition.

(ii) In the case where $\tilde{\pi}_l=\tilde{\sigma}(\chi_{\tau'}\|^{1/2})$, we get $\omega(\tilde{\pi})=1$. When $\chi_{(l)}$ is odd, $\tilde{\pi}_l(\nu,\chi_{(l)})$ is $\{0\}$. When $\chi_{(l)}$ is even and not trivial, as in the previous case $\tilde{\pi}_l(1,\chi_{(l)})$ is $\{0\}$. When $\chi_{(l)}$ is trivial (equivalently, $l\in T_1-T_1^+$), $m(\tilde{\pi})=1$. Then $\tilde{\pi}(1,\chi_l)$ is spanned by one vector t_2 defined on p. 425 case 4 of [W2]. The same vector is also defined in [BM].

We end this section by the following description of $\mathbb{Q}_p^\times(\tilde{\pi})$ [W2]:

Lemma 4.3. *If $S_\psi(\tilde{\pi})=\pi$, then $\mathbb{Q}_p^\times(\tilde{\pi})=X_+(\pi)$ if $\tilde{\pi}=\tilde{\pi}^+(\pi)$ and $\mathbb{Q}_p^\times(\tilde{\pi})=X_-(\pi)$ if $\tilde{\pi}=\tilde{\pi}^-(\pi)$.*

5. Proof of the results

5.1. Proof of Theorem 1.1

The necessity of conditions (2) and (3) in Theorem 1.1 is established in Corollary 3.4.

On the other hand if conditions (2) and (3) are satisfied, from Corollary 3.7 there is a cuspidal representation $\tilde{\pi}=\tilde{\pi}(f,\chi,e)$ such that $t(S_{k+1/2}(f,4M,\chi,e))$ is the intersection of $t(S_{k+1/2}(4M,\chi))$ and $\tilde{\pi}$. Assume $M=\prod_{l\in T}l^{\nu_l}$. Then $S_{k+1/2}(f,4M,\chi,e)\neq\{0\}$ if and only if at $l\in T$, we have $\tilde{\pi}_l(\nu_l,\chi_{(l)})\neq\{0\}$.

Lemma 5.1. *Assume conditions* (2) *and* (3) *of Theorem* 1.1 *are satisfied. Let $\tilde{\pi}=\otimes\tilde{\pi}_v=\tilde{\pi}(f,e,\chi)$. If $l\mid N$ then $\tilde{\pi}_l(\nu,\chi_{(l)})\neq\{0\}$ if and only if $N(\chi)\in l^\nu\mathbb{Z}_l$.*

Proof. We can check this case by case. Let $N=\prod_{l\in T}l^{\nu_l(N)}$. Recall from Corollary 3.4, for $l\in T-T_3^p$, $\tilde{\pi}_l=\tilde{\pi}^{e_l}_{f,e,l}$.

(i) When $l \in T_1^+$, then from case (3) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)}) = \{0\}$. From Proposition 4.2, Lemmas 2.8 and 4.3, we get $\tilde{\pi}(2, \chi_{(l)})$ is the one-dimensional space generated by $\alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l^\times$ with $\chi_l(\delta) = e_l$.
(ii) When $l \in T_1 - T_1^+$, from case (3) of the previous section $\tilde{\pi}_l(1, \chi_{(l)})$ is one-dimensional.
(iii) When $l \in T_2$, then from case (1) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)}) = \{0\}$. From Proposition 4.2, Lemmas 2.8, 2.9 and 4.3, we get $\tilde{\pi}(\nu, \chi_{(l)}) = \{0\}$ when $\nu < \nu_l(N)$, and $\tilde{\pi}(\nu_l(N), \chi_{(l)})$ is the one-dimensional space generated by $\alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l^\times$ with $\chi_l(\delta) = e_l$.
(iv) When $l \in T_3^s - T_3^+$, then from case (1) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)}) = \{0\}$. From Proposition 4.2, Lemmas 2.8, 2.9 and 4.3, we get $\tilde{\pi}(\nu, \chi_{(l)}) = \{0\}$ when $\nu < \nu_l(N)$, and $\tilde{\pi}(\nu_l(N), \chi_{(l)})$ is the two-dimensional space generated by $\{\alpha[\delta] \mid \delta = 1, \tau\}$ where τ is a non-square unit in $\mathbb{Z}_l^\times$.
(v) When $l \in T_3^+$, then from case (1) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)}) = \{0\}$. From Proposition 4.2, Lemmas 2.8, 2.9 and 4.3, we get $\tilde{\pi}(\nu, \chi_{(l)}) = 0$ when $\nu \leqslant \nu_l(N)$, and $\tilde{\pi}(\nu_l(N) + 1, \chi_{(l)})$ is the two-dimensional space generated by $\{\alpha[\delta l] \mid \delta = 1, \tau\}$ where τ is a non-square unit in $\mathbb{Z}_l^\times$.
(vi) When $l \in T_3^p - T_3^-$, then from case (2)(i) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)}) = \{0\}$. From Proposition 4.2, Lemmas 2.8 and 4.3, we get $\tilde{\pi}(\nu, \chi_{(l)}) = \{0\}$ when $\nu < \nu_l(N)$, and $\tilde{\pi}(\nu_l(N), \chi_{(l)})$ is the two-dimensional space generated by $\{\alpha[\delta] \mid \delta = 1, \tau\}$ where τ is a non-square unit in $\mathbb{Z}_l^\times$.
(vii) When $l \in T_3^-$, then from case (2)(ii) of the previous section, we get $\tilde{\pi}(1, \chi_{(l)})$ is one-dimensional.

Our claim then follows from the definition of $N(\chi)$. □

Theorem 1.1 follows immediately from the above lemma.

We note the proof and the discussion on Kohnen vector in [BM] also gives a basis of $t(S^+(f, \chi))$.

Lemma 5.2. *Assume T_3^+ is empty. The set $\{\otimes\tilde{\varphi}_v\}$ is a basis of $t(S^+(f, \chi))$, where*

(1) *$\tilde{\varphi}_\infty$ is the minimal weight vector in $\tilde{\pi}_\infty$.*
(2) *When $p \nmid 2M$, $\tilde{\varphi}_p$ is the unramified vector in $\tilde{\pi}_p$.*
(3) *When $p = 2$, $\tilde{\varphi}_2$ is $\tilde{\varphi}_2^+$ the Kohnen vector in $\tilde{\pi}_{f,e,2}^+$ defined in* [BM].
(4) *When $l \in T_1^+ \cup T_2$, $\tilde{\varphi}_l = \alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l$ with $\chi_l(\delta) = e_l$.*
(5) *When $l \in T_3 - T_3^-$, $\tilde{\varphi}_l = \alpha[1]$ or $\alpha[\tau]$ where τ is a non-square unit in $\mathbb{Z}_l$.*
(6) *When $l \in T_\chi$, $\tilde{\varphi}_l$ is the unique vector in $\tilde{\pi}_l(1, \chi_{(l)})$.*

5.2. Proof of Proposition 1.2

The proof of Proposition 1.2 uses the following lemma:

Lemma 5.3. *Assume $g(z) = \sum_{n=1}^\infty c(n)e(nz)$ and $t(g) = \tilde{\varphi} = \otimes\tilde{\varphi}_v \in \tilde{\pi}$. If for some p and a choice of Kirillov model of $\tilde{\pi}_p$, we have $\tilde{\varphi}_p(n) = 0$, then $c(n) = 0$.*

Proof. Note $c(n)$ is a constant multiple of $\tilde{W}^n(\tilde{\varphi})$ where $\tilde{\varphi} = t(g)$ and $\tilde{W}^n$ is the global ψ^n-Whittaker functional of $\tilde{\pi}$. As the global Whittaker functional factors into a product of local Whittaker functionals, we get if $t(g) = \tilde{\varphi} = \otimes\tilde{\varphi}_v \in \otimes\tilde{\pi}_v$, then $c(n) = 0$ if $\tilde{L}^n(\tilde{\varphi}_v) = 0$ for some place v. If $n \notin \mathbb{Q}_p^\times(\tilde{\pi}_p)$, then $\tilde{L}^n(\tilde{\varphi}_v) = 0$ since there is no nontrivial ψ^n—Whittaker functional on $\tilde{\pi}_p$. If $n \in \mathbb{Q}_p^\times(\tilde{\pi}_p)$, from Lemma 4.1, we see $\tilde{L}^n(t(x)) = t(n)$ is a nontrivial local Whittaker functional. Since $\tilde{\varphi}_p(n) = 0$ we get $c(n) = 0$. □

Proof of Proposition 1.2. We show the vanishing of $\tilde{\varphi}_l(n)$ in several cases when $t(g)$ is a vector as described in Lemma 5.2.

When $l \mid n$ for some $l \in T - T_\chi$, cases (3), (4) of Lemma 5.2 shows $\tilde{\varphi}_l$ has the form $\alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l$. Thus $\tilde{\varphi}_l(n) = 0$.

When $l \in T_1^+ \cup T_2$, $\tilde{\varphi}_l = \alpha[\delta]$ with $\chi_l(\delta) = e_l$. Thus $\tilde{\varphi}_l(n) = 0$ if $(\frac{n}{l}) = -e_l$. From Lemmas 5.3 and 3.6, we get $c(n) = 0$ if $(\frac{en}{l}) = -\epsilon_l \chi_{(l)}(-1)$.

For $l \in T_1 - T_1^+$, $\tilde{\varphi}_l$ is in $\tilde{\pi}_l(1, \chi_{(l)})$. The description of $\mathbb{Q}_l^\times(\tilde{\pi}_l)$ in Lemmas 2.8 and 4.3 (or the explicit description in [W2] of $\tilde{\varphi}_l$) again implies $c(n) = 0$ if $(\frac{en}{l}) = -\epsilon_l \chi_{(l)}(-1)$.

These vanishing statements give Proposition 1.2. □

5.3. Proof of Theorem 1.3

The theorem follows from the description of the basis of $t(S^+(f, \chi))$ in Lemma 5.2 and the following lemma.

Lemma 5.4. *If $g = c_1 g_1 + c_2 g_2$ with $t(g_i) = \otimes\tilde{\varphi}_{i,v} \in \tilde{\pi}$, such that $\tilde{\varphi}_{1,p} = \alpha[1]$ and $\tilde{\varphi}_{2,p} = \alpha[\delta]$ where δ is a non-square unit in $\mathbb{Z}_p^\times$. Then $R_p(g) = c_1 g_1 - c_2 g_2$.*

Proof. From Lemma 5.3, we see the Fourier coefficients of g_1 vanishes at n unless $(\frac{n}{p}) = 1$ and Fourier coefficients of g_2 vanishes at n unless $(\frac{n}{p}) = -1$. Thus $R_p(g_1) = g_1$ and $R_p(g_2) = -g_2$. □

Thus condition (C3) determines the local component $\tilde{\varphi}_l$ for $l \in T_3 - T_3^-$. We get $S^+(f, \chi, S)$ is one-dimensional.

We supplement Lemma 5.2 with the following description of $g_{f,\chi,S}$.

Lemma 5.5. *Let $g(z) = g_{f,\chi,S}$, then $t(g) = \otimes\tilde{\varphi}_v$, where*

(1) *when $l \in S$, $\tilde{\varphi}_l = \alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l$ with $\chi_l(\delta) = -(\frac{e}{l})\epsilon_l$;*
(2) *when $l \in T_3 - (T_3^- \cup S)$, $\tilde{\varphi}_l = \alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l$ with $\chi_l(\delta) = (\frac{e}{l})\epsilon_l$.*

5.4. Proof of Theorem 1.4

It is clear with the choice of χ, T_3^+ is empty and condition (3) of Theorem 1.1 is satisfied. Thus $g_{f,\chi,S}$ is defined through Theorem 1.3. The local components of $t(g) = \tilde{\varphi} = \otimes\tilde{\varphi}_v$ are given by Lemmas 5.2 and 5.5.

We compute for $D \in \Delta_{\eta,S}$ the local ϵ factors of $\pi_{f,v} \otimes \chi_D$:

Lemma 5.6. *When $D \in \Delta_{\eta,S}$, $\epsilon(\pi_{f,v} \otimes \chi_D)$ equals*:

(1) $(-1)^k$ *when* $v=\infty$,
(2) $\chi_{D,(p)}(-1)$ *when* $p\nmid N$,
(3) $\chi_{D,(l)}(-1)\chi_{(l)}(-1)$ *when* $l|N$.

Proof. Case (1) is true because $\pi_{f,\infty}\otimes\chi_D\cong\pi_{f,\infty}$. Case (2) follows from (2.2) as $X_-(\pi_{f,p})$ is empty. In case (3), when $l\in T_3$, $D\in X_+(\pi_{f,l})$ by Lemmas 2.8 and 2.9; we get the identity since $\chi_{(l)}(-1)=\epsilon_l$. When $l\in T_1\cup T_2$, and D is a unit, we have $D\in X_+(\pi)$ if and only if $(\frac{D}{l})=1$. Thus $\epsilon(\pi_{f,v}\otimes\chi_D)=(\frac{D}{l})\epsilon_l\chi_{D,(l)}(-1)$ which equals $\chi_D(-1)\chi_{(l)}(-1)$. When $l\in T_1$ is such that $(D,l)\neq 1$, we have $\epsilon(\pi_{f,l}\otimes\chi_D)=\chi_{D,(l)}(-1)$ [Sc], and $\chi_{(l)}(-1)=1$. In all cases we get the lemma. $\square$

From the lemma,

$$\epsilon(\pi_f\otimes\chi_D)=(-1)^k\prod_{l\in T}\chi_{(l)}(-1)\prod_p\chi_{D,(p)}(-1)=\operatorname{sgn}(D)(-1)^k\epsilon(\chi).$$

Thus when $\operatorname{sgn}(D)\neq e$, the epsilon factor $\epsilon(\pi_f\otimes\chi_D)$ is -1 and $L(f,D,k)=0$.

Assume now $\operatorname{sgn}(D)=e=(-1)^k\epsilon(\chi)$. The proof of the identity (1.2) is similar to the proof of Theorem 10.1 in [BM]. We start with Theorem 4.3 in [BM] and the formula:

$$\frac{|\tilde W_{\tilde\varphi}^{|D|}|^2}{\|\tilde\varphi\|^2}=\frac{|W_\varphi|^2L(\pi_f\otimes\chi_D,1/2)}{\|\varphi\|^2}\prod_{v\in T\cup\{\infty,2\}}E_v\bigl(\varphi_v,\tilde\varphi_v,\psi,|D|\bigr)\tag{5.1}$$

where

$$E_v\bigl(\varphi_v,\tilde\varphi_v,\psi,|D|\bigr)=\frac{e(\varphi_v,\psi)}{e(\tilde\varphi_v,\psi^{|D|})L(\pi_{f,v}\otimes\chi_D,1/2)|D|_v}.\tag{5.2}$$

Here we use the notations in [BM]. φ and $\tilde\varphi$ are vectors in π_f and $\tilde\pi=\tilde\pi(f,\chi)$; W_φ and $\tilde W_{\tilde\varphi}^{|D|}$ are the ($\psi^{|D|}$) Whittaker coefficients; $\|\varphi\|$ and $\|\tilde\varphi\|$ are Petersson norms; and

$$e(\varphi_v,\psi)=\frac{\|\varphi_v\|^2}{|L_v(\varphi_v)|^2},\tag{5.3}$$

$$e\bigl(\tilde\varphi_v,\psi^D\bigr)=\frac{\|\tilde\varphi_v\|^2}{|\tilde L_v^D(\tilde\varphi_v)|^2},\tag{5.4}$$

with L_v and $\tilde L_v^D$ being fixed local Whittaker functionals and the local norms given by

$$\|\varphi_v\|^2=\int_{\mathbb{Q}_v^\times}\left|L_v\left(\pi_{f,v}\left(\begin{pmatrix}a&\\&1\end{pmatrix}\right)\varphi_v\right)\right|^2\frac{da}{|a|_v},\tag{5.5}$$

$$\|\tilde\varphi_v\|^2=\sum_{\delta_i}\frac{|2|_v}{2}\int_{\mathbb{Q}_v^\times}\left|\tilde L_v^{D\delta_i}\left(\tilde\pi\left(\begin{pmatrix}a&\\&a^{-1}\end{pmatrix}\right)\tilde\varphi_v\right)\right|^2\frac{da}{|a|_v}.\tag{5.6}$$

The sum is taken over representatives of square classes of $\mathbb{Q}_v^\times$.

From Lemma 4.1, given a Kirillov model of $\tilde{\pi}_p$, we can choose the Whittaker functional $\tilde{L}^D$ to be $\tilde{L}_p^{\delta_i}(t(x)) = c_{\delta_i} t(\delta_i)$ where c_{δ_i} is a fixed nonzero constant. Note that when $\tilde{\varphi}_l = \alpha[\delta]$, we have $\tilde{L}_l^{\delta_i}(\tilde{\pi}((\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}))\tilde{\varphi}_l) = 0$ if δ_i is not in the same square class of δ. Thus

$$\|\alpha[\delta]\|^2 = \frac{|2|_v}{2} \int\limits_{\mathbb{Q}_v^\times} \left| c_\delta \tilde{\pi}\left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right)\alpha[\delta](\delta)\right|^2 \frac{da}{|a|_v}. \tag{5.7}$$

Lemma 5.7. *When $l \in T - T_\chi$, $\tilde{\varphi}_l = \alpha[|D|]$, then $e(\tilde{\varphi}_l, \psi^{|D|}) = (1 - l^{-1})/2$.*

Proof. We can check case by case that $\tilde{\varphi}_l = \alpha[eD] = \alpha[|D|]$. Thus by Eq. (4.2),

$$\left|\tilde{\pi}_l\left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right)\tilde{\varphi}_l\big(|D|\big)\right| = 1$$

when $|a|_l = 1$ and 0 otherwise. We get $\|\alpha[|D|]\| = \frac{1}{2}(1 - l^{-1})|c_\delta|^2$. On the other hand, $|\tilde{L}_l^{|D|}(\alpha[|D|])| = |c_\delta|$. We get the lemma. □

When $l \in T_2 \cup T_3$, in the Kirillov model of $\pi_{f,l}$ the new vector φ_l is the characteristic function of $\mathbb{Z}_l^\times$ [Sc]. As the linear form L_l is just evaluation at 1, we get $L_l(\pi_{f,l}((\begin{smallmatrix} a & \\ & 1\end{smallmatrix}))\varphi_l) = 1$ if $a \in \mathbb{Z}_l^\times$ and 0 otherwise. Thus we get

Lemma 5.8. *When $l \in T_2 \cup T_3$, $e(\varphi_l, \psi) = (1 - l^{-1})$.*

We next consider the case of $\tilde{\varphi}_l$ when $l \in T_3^-$. In this case $\tilde{\pi}_l = \tilde{\pi}(\mu\chi_{e,(l)})$ is a principal series for some character μ of $\mathbb{Q}_l^\times$. From Proposition 9 of [W2], $\tilde{\varphi}_l$ corresponds to a vector $F[0,1]$ in $\tilde{\pi}(\mu\chi_{e,(l)})$, where $F[0,1]$ is determined by the fact that: $F[0,1]\big((\begin{smallmatrix} a & -1 \\ 1 & 0\end{smallmatrix})\big) = 1$ when $a \in l\mathbb{Z}_l$ and $F[0,1]\big((\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix})\big) = \chi_{(l)}(c)$ when $c \in \mathbb{Z}_l^\times$ and 0 when $c \in l\mathbb{Z}_l$.

Let $\Phi(a) = F[0,1]\big((\begin{smallmatrix} 0 & -1 \\ 1 & a\end{smallmatrix})\big)$, then we get $\Phi(a)$ equals the characteristic function of $\mathbb{Z}_l$. In Section 8 of [BM], we give an explicit formula of $\|\tilde{\varphi}_l\|$ using $\Phi(a)$ (Eq. (8.5) of [BM]):

$$\|\tilde{\varphi}_l\|^2 = |D|_l^{-1} \int \big|\Phi(a)\big|^2 \, da = |D|_l^{-1}.$$

On the other hand Eq. (8.3) in the same section gives:

$$\tilde{L}^{|D|}(\tilde{\varphi}_l) = \int \Phi(a)\psi^{|D|}(-a)\, da = 1$$

as $D \in \mathbb{Z}_l$. Thus we get

Lemma 5.9. *When $l \in T_3^-$, $e(\tilde{\varphi}_l, \psi^{|D|}) = |D|_l^{-1}$.*

All the other cases have already been treated in [BM]. From Eq. (10.4) of [BM] we get the following formulas for $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, |D|)$:

$$\frac{e(\varphi_p, \psi)}{e(\tilde{\varphi}_p, \psi^{|D|})} = \begin{cases} \frac{1}{2}e^{4\pi(1-|D|)}|D|^{1/2+k}\pi^{-k}(k-1)!, & p = \infty, \\ 2L(\pi_{f,p} \otimes \chi_D, 1/2)|D|_p, & p \in T_1 \cap T_\chi,\ p \nmid D, \\ L(\pi_{f,p} \otimes \chi_D, 1/2)|D|_p, & p \in T_1 \cap T_\chi,\ p|D, \\ 2|D|_2 L(\pi_{f,2} \otimes \chi_D, 1/2), & p = 2,\ (2, D) = 1, \\ 2|D|_2, & p = 2, 4|D. \end{cases} \tag{5.8}$$

From Section 8.3 of [BM], we see when $l \in T_1$, $e(\varphi_l, \psi) = (1 + l^{-1})^{-1}$. From the above Lemmas 5.7–5.9 we get:

$$\frac{e(\varphi_l, \psi)}{e(\tilde{\varphi}_l, \psi^{|D|})} = \begin{cases} 2(1 - l^{-2})^{-1}, & l \in T_1^+, \\ 2, & l \in (T_2 \cup T_3) - T_3^-, \\ |D|_l(1 - l^{-1}), & l \in T_3^-. \end{cases} \tag{5.9}$$

Note that when $l \in T_2 \cup T_3$, by our assumption that f is q.t. primitive, $L(\pi_{f,l} \otimes \chi_D, s) = 1$. When $l \in T_1$ [Go],

$$L(\pi_{f,l} \otimes \chi_D, 1/2) = \left(1 + \left(\frac{D}{l}\right)\epsilon_l l^{-1}\right)^{-1}. \tag{5.10}$$

Note when $l \in (T - T_\chi)$, $|D|_l = 1$. Combining the above formulas we get $\prod_{v \in T \cup \{\infty, 2\}} E_v(\varphi_v, \tilde{\varphi}_v, \psi, |D|)$ equals

$$e^{4\pi(1-|D|)}|D|^{1/2+k}\frac{(k-1)!}{\pi^k}\left(\prod_{l \in \{2\} \cup T} |D|_l\right)\frac{\kappa(T, T_\chi, D, f)}{L(\pi_{f,\infty} \otimes \chi_D, 1/2)L(\pi_{f,2} \otimes \chi_D, 1/2)}, \tag{5.11}$$

where

$$\kappa(T, T_\chi, D, f) = 2^{|T - T_\chi|}\prod_{l \in T_3 \cap T_\chi}\left(1 - l^{-1}\right) \times \prod_{l \in T_1 \cap T_\chi,\, |D|_l = 1} 2 \prod_{l \in (T - T_\chi) \cap T_1}\left(1 - \left(\frac{D}{l}\right)\epsilon_l l^{-1}\right)^{-1}. \tag{5.12}$$

Clearly $\prod_{l \in \{2\} \cup T} |D|_l = |D|^{-1}$. Also with our assumptions on D, we get $(\frac{D}{l})\epsilon_l = \chi_{(l)}(-1)$. From [BM, Section 10], we have $W_\varphi(e) = e^{-2\pi}$, and $\tilde{W}_{\tilde{\varphi}}^{|D|}(e) = e^{-2\pi|D|}c(|D|)$. Also from [BM, Section 9], we have

$$\frac{\|\varphi\|^2}{\|\tilde{\varphi}\|^2} = \frac{\langle f, f\rangle}{\langle g, g\rangle}$$

and from (2.3):

$$L(f, D, k) = L(\pi_f \otimes \chi_D, 1/2)/\bigl(L(\pi_{f,\infty} \otimes \chi_D, 1/2)L(\pi_{f,2} \otimes \chi_D, 1/2)\bigr).$$

Thus we get from (5.1) and (5.11)

$$\frac{|c(|D|)|^2}{\langle g, g\rangle} = \frac{L(f, D, k)}{\langle f, f\rangle}|D|^{k-\frac{1}{2}}\frac{(k-1)!}{\pi^k}\kappa(T, T_\chi, D, f).$$

As $(\frac{D}{l})\epsilon_l = \chi_{(l)}(-1) = \eta(l)$ for $l \in T_1$, we get the identity (1.2).

5.5. *Proof of Theorem 1.5*

Before we prove Theorem 1.5, we prove a middle step. Let η, χ be as in Theorem 1.4. Let $S \subset T_3$. We can define the subset of fundamental discriminants $\Delta_{\eta,S}$ as in the introduction. Let $S^{+\prime}(f, \chi, S)$ be the subspace of $S_{k+1/2}(4N'', \chi)$ satisfying the conditions (C1), (C2) and condition (C3) for all $l \in T_3$, and moreover for $l \in T_1$, $R_l^2(g) = g$.

Let $S_\chi = S \cap (T_3 - T_3^-)$.

Theorem 5.10. *Let η, χ be as in Theorem* 1.4 *and $S \subset T_3$. Then $S^{+\prime}(f, \chi, S)$ is one-dimensional. A generator $g'_{f,\chi,S}$ is given by* ($\epsilon'_l = \epsilon_l(\frac{e}{l})$)

$$g'_{f,\chi,S} = \prod_{l\in S}\frac{1}{2}\left(1 - \epsilon'_l R_l\right)\prod_{l\in T_3 - S}\frac{1}{2}\left(1 + \epsilon'_l R_l\right)\prod_{l\in T_1\cup T_3} R_l^2(g_{f,\chi,S_\chi}). \tag{5.13}$$

For $D \in \Delta_{\eta,S}$ and $(D, N) = 1$ Eq. (1.2) *holds with $g = g'_{f,\chi,S}$ and $\kappa = 0$ when* $\operatorname{sgn}(D) \neq e$; *otherwise $\kappa = \kappa'(\eta)^{-1}$ where*

$$\kappa'(\eta) = 2^{|T|}\prod_{l\in T_1}\left(1 - \eta(l)l^{-1}\right)^{-1}. \tag{5.14}$$

Equation (5.13) says $g'_{f,\chi,S}$ can be obtained by setting the Fourier coefficients of g_{f,χ,S_χ} to zero when

(1) $(n, N) \neq 1$, or
(2) for some $l \in S$, $(\frac{n}{l}) = \epsilon_l$, or
(3) for some $l \in T_3 - S$, $(\frac{n}{l}) = -\epsilon_l$.

Proof of Theorem 5.10. We observe a basis of

$$S_{k+1/2}\left(f, 4N\prod_{l\in T_1} l, \chi, e\right).$$

Let $N = \prod_{l\in T} l^{\nu_l(N)}$. From discussion in Section 4, we see

$$t\left(S_{k+1/2}\left(f, 4N\prod_{l\in T_1} l, \chi, e\right)\right)$$

is spanned by vectors $\otimes\tilde{\varphi}_v$ where $\tilde{\varphi}_v$ is a uniquely determined vector when $v=\infty$ or $v=p\nmid 2N$; $\tilde{\varphi}_2$ lies in a two-dimensional space and for $l\in T_2\cup T_3$, $\tilde{\varphi}_l$ lies in $\tilde{\pi}_l(\nu_l(N),\chi_{(l)})$, for $l\in T_1$, $\tilde{\varphi}_l$ lies in $\tilde{\pi}_l(2,\chi_{(l)})$.

With our assumption in the theorem, T_3^+ is empty. When $l\notin T_\chi$, $\tilde{\pi}_l(1,\chi_{(l)})$ is empty and the description of $\tilde{\pi}_l(\nu_l(N),\chi_{(l)})$ (when $l\in T_2\cup T_3$) or $\tilde{\pi}_l(2,\chi_{(l)})$ (when $l\in T_1$) is given in Lemma 5.2. When $l\in T_\chi$:

Case 1. $l\in T_1$, then $\tilde{\pi}_l(2,\chi_{(l)})$ is spanned by vectors t_2 [W2, p. 425] and $\alpha[\delta]$ where δ is a unit. An examination of t_2 gives $t_2-\alpha[\delta](x)=0$ if $x\notin l\mathbb{Z}_l$.

Case 2. $l\in T_3$, then $\tilde{\pi}_l(2,\chi_{(l)})$ is spanned by vectors t_1, $\alpha[1]$ and $\alpha[\delta]$ where δ is a non-square unit and t_1 is the characteristic function of $\mathbb{Z}_l$. Clearly $t_1-\alpha[1]-\alpha[\delta]$ is supported on $l\mathbb{Z}_l$.

Consider now the space $S^{+'}(f,\chi,S)$. Condition (C2) determines the choice of $\tilde{\varphi}_2$. The other conditions imply when $l\mid N$, $R_l^2(g)=g$ for all $g\in S^{+'}(f,\chi,S)$. We have the following:

Lemma 5.11. *If $g=c_1g_1+c_2g_2$ with $t(g_i)=\otimes\tilde{\varphi}_{i,v}\in\tilde{\pi}$, such that $\tilde{\varphi}_{1,p}$ is supported on $p\mathbb{Z}_p$ and $\tilde{\varphi}_{2,p}$ is supported on $\mathbb{Z}_p^\times$. Then $R_p^2(g)=c_2g_2$.*

Proof. From Lemma 5.3, we see the Fourier coefficients of $g_1(z)$ is 0 at n unless $p\mid n$, and the Fourier coefficients of $g_2(z)$ is 0 at n unless $p\nmid n$. Thus $R_p^2(g_1)=0$ and $R_p^2(g_2)=g_2$. □

For $l\in T_\chi$, in both above cases we have $\tilde{\pi}_l(2,\chi_{(l)})$ is spanned by a vector supported on $l\mathbb{Z}_l$ and vectors of the form $\alpha[\delta]$ where δ is a unit in $\mathbb{Z}_l$. We conclude that there is a basis of $t(S^{+'}(f,\chi,S))$ consisting of $\otimes\tilde{\varphi}_v$ where for all $l\in T_\chi$ (thus for $l\in T$ by Lemma 5.2), $\tilde{\varphi}_l=\alpha[\delta]$ with $\delta\in\mathbb{Q}^\times(\tilde{\pi}_l)$ is a unit of $\mathbb{Z}_l$.

When $l\in T_1\cup T_2$, the choice of $\tilde{\varphi}_l=\alpha[\delta]$ is unique. From Lemma 5.4, condition (C3) forces $t(S^{+'}(f,\chi,S))$ to be spanned by a unique vector. We have shown $S^{+'}(f,\chi,S)$ is one-dimensional.

Let $g'_{f,\chi,S}$ be the generator of this one-dimensional space. Let $\tilde{\varphi}'=t(g'_{f,\chi,S})=\otimes\tilde{\varphi}'_v$. Then the description of $\tilde{\varphi}_v$ is given as in Lemma 5.2 except when $v=l\in T_\chi$. In the case $l\in T_\chi$, $\tilde{\varphi}'_l=\alpha[\delta]$ where

(1) $\chi_l(\delta)=e_l$ when $l\in T_1$,
(2) $\chi_l(\delta)=-(\frac{e}{l})\epsilon_l$ when $l\in S$, and
(3) $\chi_l(\delta)=(\frac{e}{l})\epsilon_l$ when $l\in T_3-S$.

Let $g_0=g_{f,\chi,S_\chi}$, $g_1=\prod_{l\in T_1\cup T_3}R_l^2(g_0)$ and

$$g_2=\prod_{l\in S}\frac{1}{2}\bigl(1-\epsilon'_lR_l\bigr)\prod_{l\in T_3-S}\frac{1}{2}\bigl(1+\epsilon'_lR_l\bigr)g_1.$$

Let $\tilde{\varphi}_i=t(g_i)=\otimes\tilde{\varphi}_{i,v}$. We get the description of $\tilde{\varphi}_{0,v}$ from Lemma 5.2. From Lemma 5.11, we see $\tilde{\varphi}_{1,v}=\tilde{\varphi}_{0,v}$ unless $v=l\in T_\chi$; when $l\in T_\chi$, $\tilde{\varphi}_{1,l}=\tilde{\varphi}'_l$ when $l\in T_1$, and when $l\in T_3^-$, $\tilde{\varphi}_{1,l}=\alpha[1]+\alpha[\delta]$ with δ a non-square unit of $\mathbb{Z}_l$. From Lemma 5.4, we get $\tilde{\varphi}_{2,v}=\tilde{\varphi}_{1,v}$ unless $v=l\in T_3$, in which case $\tilde{\varphi}_{2,v}=\alpha[\delta]$ with $\chi_l(\delta)=-\epsilon'_l$ if $l\in S$ and ϵ'_l if $l\notin S$. Compare with the local components of $\tilde{\varphi}'$, we get $\tilde{\varphi}_2=\tilde{\varphi}'$. Thus we get Eq. (5.13).

The proof of identity (5.14) is identical to that of (1.4). We have less cases to consider. In fact for $l \in T_2 \cup T_3$, we have from Lemmas 5.7 and 5.8

$$\frac{e(\varphi_l, \psi)}{e(\tilde{\varphi}_l', \psi^{|D|})} = 2.$$

When $l \in T_1$, $\frac{e(\varphi_l, \psi)}{e(\tilde{\varphi}_l, \psi^{|D|})} = 2(1 - l^{-2})^{-1}$. Taking into account of the formulas (5.10) for $L(\pi_{f,l} \otimes \chi_D, 1/2)$ when $l \in T_1$, we get (5.14) from (5.1). □

From (5.13), we get

$$g'_{f,\chi,S} = \sum_{\hat{\delta} \in (\mathbb{Z}/N')^\times} g_{f,\chi,S_\chi,\hat{\delta}}. \tag{5.15}$$

For a fixed η we let

$$\epsilon = (-1)^k \prod_{l \in T_1 \cup T_2} \eta(l) \prod_{l \in T_3} \epsilon_l. \tag{5.16}$$

Let $S \subset T_3$, we will fix an $\hat{\delta}_{f,\eta,S} \in (\mathbb{Z}/N)^\times$ such that

(1) for $l \in T_1 \cup T_2$, $(\frac{\epsilon \hat{\delta}_{f,\eta,S}}{l}) = \epsilon_l \eta(l)$ and
(2) for $l \in T_3$, $(\frac{\epsilon \hat{\delta}_{f,\eta,S}}{l}) = \epsilon_l$ when $l \notin S$ and $-\epsilon_l$ when $l \in S$.

Note we have $\epsilon = e(\chi)$ when $\chi \in \Xi(\eta, f)$.

Lemma 5.12. *Assume $\chi \in \Xi(\eta, f)$ and $S \subset T_3$. Then $g_{f,\chi,S_\chi,\hat{\delta}} = 0$ unless $\hat{\delta} = \hat{\delta}_{f,\eta,S} d^2$ for some $d \in (\mathbb{Z}/N')^\times$.*

If $\chi_1, \chi_2 \in \Xi(\eta, f)$, there exists a constant β independent of $d \in (\mathbb{Z}/N')^\times$ such that when $\hat{\delta} = \hat{\delta}_{f,\eta,S} d^2$,

$$\beta \chi_1(d^{-1}) g_{f,\chi_1,S_{\chi_1},\hat{\delta}} = \chi_2(d^{-1}) g_{f,\chi_2,S_{\chi_2},\hat{\delta}}. \tag{5.17}$$

Proof. From Proposition 1.2 and conditions in Theorem 5.10, we see the Fourier coefficients of $g'_{f,\chi,S}$ is supported on the square class of $\hat{\delta}_{f,\eta,S}$ in $(\mathbb{Z}/N')^\times$. Thus we get the first claim.

Now let $\chi_1, \chi_2 \in \Xi(\eta, f)$. Assume $t(g'_{f,\chi_1,S})$ and $t(g'_{f,\chi_2,S})$ lie in the representations $\tilde{\pi}$ and $\tilde{\pi}'$. Then $\tilde{\pi}$ and $\tilde{\pi}'$ are in a near equivalence class by Lemma 3.2. As $\chi_{1,(p)}(-1) = \chi_{2,(p)}(-1)$ for all primes p, we get $\tilde{\pi}_p$ and $\tilde{\pi}'_p$ has the same central character. By the strong multiplicity theorem [W3], $\tilde{\pi} = \tilde{\pi}'$.

Let $t(g'_{f,\chi_1,S}) = \tilde{\varphi} = \otimes \tilde{\varphi}_v$ and $t(g'_{f,\chi_2,S}) = \tilde{\varphi}' = \otimes \tilde{\varphi}'_v$, then $\tilde{\varphi}$ and $\tilde{\varphi}'$ both lie in the space of $\tilde{\pi}$. At $l \in T$, we will use the Kirillov model $K(\tilde{\pi}_l, \chi_{(l)})$ for $\tilde{\pi}_l$. Then use the description of Kirillov models in [W2] and follow the proof of Theorem 5.10, we get for some nonzero constant γ (we need this constant since both $g'_{f,\chi_1,S}$ and $g'_{f,\chi_2,S}$ are defined up to a scalar multiple):

(1) When $v = \infty$, we have $\gamma \tilde{\varphi}_\infty = \tilde{\varphi}'_\infty$; when $v = p \nmid N$, we have $\tilde{\varphi}_v = \tilde{\varphi}'_v$.

(2) When $v = l \mid N$, $\tilde{\varphi}_l = \alpha[\delta_l]$ where δ_l and $\hat{\delta}_{f,\eta,S}$ are in the same square class of $\mathbb{Z}_l$. $\tilde{\varphi}'_l = \alpha'[\delta_l]$ where

$$\alpha'[\delta](a) = \begin{cases} \chi_{2,(l)}(\beta^{-1})\chi_{1,(l)}(\beta), & a = \delta\beta^2,\ \beta \in \mathbb{Z}_l^\times, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\alpha_0[\delta]$ be a function such that $\alpha_0[\delta](a) = 1$ when $a - \delta \in l\mathbb{Z}_l$ and 0 otherwise. Let $\tilde{\varphi}_{\hat{\delta}} = \otimes\tilde{\varphi}_{\hat{\delta},v}$ where $\tilde{\varphi}_{\hat{\delta},v} = \tilde{\varphi}_v$ unless $v = l \mid N$, in which case $\tilde{\varphi}_{\hat{\delta},l} = \alpha_0[\hat{\delta}]$. Clearly

$$\tilde{\varphi} = \frac{1}{2^{|T|}} \sum_{d \in (\mathbb{Z}/N')^\times} \tilde{\varphi}_{d^2\hat{\delta}_{f,\eta,S}}, \tag{5.18}$$

$$\tilde{\varphi}' = \beta\frac{1}{2^{|T|}} \sum_{d \in (\mathbb{Z}/N')^\times} \chi_2\big(d^{-1}\big)\chi_1(d)\tilde{\varphi}_{d^2\hat{\delta}_{f,\eta,S}}. \tag{5.19}$$

From Lemma 5.3, we see $t^{-1}(\tilde{\varphi}_{\hat{\delta}})$ has vanishing Fourier coefficient $c(n)$ unless n is in the same class as $\hat{\delta}$ in $(\mathbb{Z}/N')^\times$. Thus when $\hat{\delta} = \hat{\delta}_{f,\eta,S}d^2$,

$$\begin{aligned} g_{f,\chi,S_\chi,\hat{\delta}} &= t^{-1}(\tilde{\varphi}_{d^2\hat{\delta}_{f,\eta,S}}), \\ g_{f,\chi',S_{\chi'},\hat{\delta}} &= t^{-1}\big(\beta\chi_2\big(d^{-1}\big)\chi_1(d)\tilde{\varphi}_{d^2\hat{\delta}_{f,\eta,S}}\big). \end{aligned}$$

Equation (5.17) follows. □

Proof of Theorem 1.5. Let χ be a character as in the theorem, there is a unique η so that $\chi \in \Xi(\eta, f)$. It follows from (5.17) that for any $\chi \in \Xi(\eta, f)$ and $S \subset T_3$, $g_{f,\chi,S_\chi,\hat{\delta}}$ is either 0 or a nonzero multiple of a fixed form independent of χ. From the first claim of Lemma 5.12, that form will be 0 if $(\frac{\epsilon\hat{\delta}}{l}) \neq \epsilon_l\eta(l)$ for some $l \in T_1 \cup T_2$.

We now consider the pairs $(\eta, \hat{\delta})$ so that for all $l \in T_1 \cup T_2$ we have $(\frac{\epsilon\hat{\delta}}{l}) = \epsilon_l\eta(l)$. It is clear that $(\eta, \hat{\delta}) \mapsto (\epsilon, \hat{\delta})$ through Eq. (5.16) is a bijection between the pairs $(\eta, \hat{\delta})$ satisfying the above relation and the admissible pairs $(\epsilon, \hat{\delta})$. For an admissible pair $(\epsilon, \hat{\delta})$ and the associated pair $(\eta, \hat{\delta})$ under the bijection, we let $g_{f,\epsilon,\hat{\delta}} = t^{-1}(\tilde{\varphi}_{\hat{\delta}})$ where $\tilde{\varphi}_{\hat{\delta}}$ is the vector defined in the above lemma, with any choice of χ_1 in $\Xi(\eta, f)$. The lemma says that $g_{f,\epsilon,\hat{\delta}}$ is then uniquely determined. Moreover given any χ as in the theorem, we can associate an η thus an ϵ, so that $g_{f,\chi,S_\chi,\hat{\delta}}$ is a multiple of $g_{f,\epsilon,\hat{\delta}}$ when $(\epsilon, \hat{\delta})$ is admissible; it is 0 when $(\epsilon, \hat{\delta})$ is not admissible.

Clearly $g_{f,\epsilon,\hat{\delta}}$ is a cusp form. To see the invariance under $\Gamma_1(4N'')$, we need to check the following property: for all $l \in T$, $\alpha_0[\hat{\delta}]$ is fixed under the action of $K_1(N'')$ the group of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in 1 + N''\mathbb{Z}_l$, $b \in \mathbb{Z}_l$ and $c \in N''\mathbb{Z}_l$. Observe

$$\alpha_0[\hat{\delta}] = \sum_{\chi'(-1)=\chi(-1)} \frac{2}{l-1}\alpha_{\chi'}[\hat{\delta}].$$

Since each $\alpha_{\chi'}[\hat{\delta}]$ is equivariant under $K_0(N'')$ thus invariant under $K_1(N'')$, we get the invariance of $\alpha_0[\hat{\delta}]$.

When $D \in \Delta_{\epsilon,\hat{\delta}}$ with a pair $(\epsilon, \hat{\delta})$ that is not admissible, $L(f, D, s)$ satisfies an odd function equation thus $L(f, D, k) = 0$. When the pair $(\epsilon, \hat{\delta})$ is admissible, we can derive the formula (1.5) from (5.14). As $g_{f,\epsilon,\hat{\delta}}$ shares the same Fourier coefficient $c(|D|)$ with $g'_{f,\chi,S}$ (for any $\chi \in \Xi(\eta, f)$), we only need to note the difference between the norms of $g'_{f,\chi,S}$ and $g_{f,\epsilon,\hat{\delta}}$. This comes from the difference between the local norm of $\alpha[\delta]$ and $\alpha_0[\hat{\delta}]$ at a prime $l \in T$. Equation (5.7) applies also to $\alpha_0[\hat{\delta}]$. Note $\hat{\delta}$ and δ are in the same square class of $\mathbb{Z}_l$. We get $\|\alpha[\hat{\delta}]\| = |c_\delta|^2$. From the proof of Lemma 5.7, $\|\alpha[\delta]\| = (l-1)/2\|\alpha_0[\hat{\delta}]\|$. Thus we get the formula (1.5). □

References

[AL] A.O.L. Atkin, J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann. 185 (1970) 134–160.

[A-Li] A.O.L. Atkin, W. Li, Twists of newforms and pseudo-eigenvalues of W-operators, Invent. Math. (1978) 221–243.

[BM] E.M. Baruch, Z. Mao, Central values of automorphic L-functions, GAFA, submitted for publication.

[Ca] W. Casselman, On some results of Atkin and Lehner, Math. Ann. 201 (1973) 301–314.

[Go] R. Godement, Notes on Jacquet–Langlands Theory, The Institute for Advanced Study, 1970.

[K1] W. Kohnen, Fourier coefficients of modular forms of half-integral weight, Math. Ann. 271 (1985) 237–268.

[K2] W. Kohnen, A remark on the Shimura correspondence, Glasgow Math. J. 30 (1988) 285–291.

[KZ] W. Kohnen, D. Zagier, Values of L-series of modular forms at the center of the critical strip, Invent. Math. 64 (1981) 175–198.

[MTV] Z. Mao, G. Tornaria, F. Villegas-Rodriguez, Computation of central value of quadratic twists of modular L-functions, in: Ranks of Elliptic Curves and Random Matrix Theory, in press.

[PT] A. Pacetti, G. Tornaria, Computing central values of twisted L-series: The case of composite levels, Preprint.

[Sc] R. Shmidt, Some remarks on local newforms for $GL(2)$, J. Ramanujan Math. Soc. 17 (2002) 115–147.

[Sh] G. Shimura, On modular forms of half integral weight, Ann. of Math. (2) 97 (1973) 440–481.

[U] M. Ueda, On twisting operators and newforms of half integral weight II-complete theory of newforms for Kohnen space, Nagoya Math. J. 149 (1998) 117–171.

[W1] J.-L. Waldspurger, Correspondance de Shimura, J. Math. Pures Appl. (9) 59 (1980) 1–113.

[W2] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. Pures Appl. (9) 60 (1981) 375–484.

[W3] J.-L. Waldspurger, Correspondances de Shimura et quaternions, Forum Math. 3 (1991) 219–307.

[We] A. Weil, Sur certains groups d'opérateurs unitaires, Acta Math. 111 (1964) 143–211.