

Global method in relative trace identities

Zhengyu Mao and Stephen Rallis

Abstract. We explain a way to establish the relative trace identity through comparison of global distributions, instead of through comparing local orbital integrals.

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1. Introduction

An important method in number theory and Langlands program is the theory of the trace formula. Let F be a number field, \mathbf{A} its adèle ring. We use v to denote a place of F . Let G be a reductive group. In studying the trace formula, one considers a distribution of the following type: for $f \in \mathcal{S}(G(\mathbf{A}))$ (the space of Schwartz functions on $G(\mathbf{A})$), let

$$\begin{aligned} & I_G(f : H_1, \chi_1, H_2, \chi_2) \\ &= \int_{H_1(F) \backslash H_1(\mathbf{A})} \int_{H_2(F) \backslash H_2(\mathbf{A})} K_f(h_1, h_2) \chi_1(h_1) \chi_2(h_2) dh_1 dh_2. \end{aligned} \quad (1)$$

Here H_1, H_2 are two closed subgroups of G , χ_i ($i = 1, 2$) is a character (or an automorphic form) on $H_i(\mathbf{A})$, and $K_f(x, y)$ is the kernel function for the representation $\rho(f)$ acting on $L^2(G(F) \backslash G(\mathbf{A}))$; more explicitly

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y). \quad (2)$$

The distribution I_G is a relative trace formula; the usual trace formula can be shown as a special case of this distribution.

Let G and G' be two reductive groups. Assume there is a homomorphism between the L -groups of G and G' . A *relative trace identity* is a relation between the two distributions (1) on G and G' . Explicitly, we say there is relative trace identity

$$I_G(f : H_1, \chi_1, H_2, \chi_2) = I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2) \quad (3)$$

if the following is true:

- (a) There exists maps ϵ_v between sufficiently large subsets of $\mathcal{S}(G_v)$ and $\mathcal{S}(G'_v)$ for all places v of F ,
- (b) There is a finite set S_0 of bad places, such that for any S a finite set of places containing S_0 , we have for any $f = \otimes_{v \in S} f_v \otimes_{v \notin S} f_v$ with $f_v \in \mathcal{S}(G_v)$ when $v \in S$ and f_v a Hecke function when $v \notin S$, the equation (3) holds for $f' = \otimes_{v \in S} \epsilon_v(f_v) \otimes_{v \notin S} \lambda_v(f_v)$. Here λ_v is the local Hecke algebra homomorphism between G_v and G'_v given by the Satake isomorphism and the homomorphism between the L -groups of G and G' .

The readers interested in more background on relative trace formula should consult Jacquet's contribution to this volume. For some applications of the relative trace identities in number theory and Langlands correspondence, see [BM],[CJ],[Gu] and [J].

In this paper we discuss a new method of proving a relative trace identity. We illustrate this method by looking at an example. We consider the case $G = SO(n + 1, n)$ the split special orthogonal group, and $G' = \tilde{S}p_n$ the double cover of the symplectic group Sp_n . In §2, we define two distributions $I_G(f : H_1, \chi_1, H_2, \chi_2)$ and $I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2)$ on the groups. We will prove the relative trace identity between these two distributions using a global method. This identity generalizes an identity considered in [J].

Let us first recall the usual way of proving a trace identity, which is by comparing orbital integrals. Assume $f = \otimes f_v$, one decomposes the distribution $I_G(f : H_1, \chi_1, H_2, \chi_2)$ into sums of orbital integrals:

$$\sum_{\phi \in \mathcal{O}} I_G(f, \phi : H_1, \chi_1, H_2, \chi_2) = \sum_{\phi \in \mathcal{O}} c(\phi) \prod_v I_{G,v}(f_v, \phi : H_1, \chi_1, H_2, \chi_2).$$

Here \mathcal{O} is the set of representatives of orbits, a subset of $G(F)$; $c(\phi)$ a positive coefficient (equals some volume), and $I_{G,v}(f_v, \phi : H_1, \chi_1, H_2, \chi_2)$ is the local orbital integral which takes the form:

$$\int_{H_{2,v} \cap \phi^{-1} H_{1,v} \phi \backslash H_{2,v}} \int_{H_{1,v}} f_v(h_1^{-1} \phi h_2) \chi_1(h_1) \chi_2(h_2) dh_1 dh_2.$$

Similar decomposition holds for $I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2)$:

$$I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2) = \sum_{\phi' \in \mathcal{O}'} c(\phi') \prod_v I_{G',v}(f'_v, \phi' : H'_1, \chi'_1, H'_2, \chi'_2).$$

To prove the identity (3), one shows:

- 1) there is a bijection ι between the set of orbits \mathcal{O} and \mathcal{O}' ;
- 2) there is a map ϵ_v such that for $f'_v = \epsilon_v(f_v)$, we have an identity of orbital integrals

$$I_{G,v}(f_v, \phi : H_1, \chi_1, H_2, \chi_2) = I_{G',v}(f'_v, \iota(\phi) : H'_1, \chi'_1, H'_2, \chi'_2) \Delta_v(\phi), \quad (4)$$

where $\Delta_v(\phi)$ is some transfer factor independent of f_v , satisfying $\prod_v \Delta_v(\phi) = c(\phi)/c(\iota(\phi))$;

3) we have the fundamental lemma, i.e. the identity (4) holds for $f'_v = \lambda(f_v)$ where $v \notin S_0$ and f_v is a Hecke function.

It is easy to see that the above three facts together with the orbital integral decompositions imply the relative trace identity (3). This method is developed in many papers, for example [FuSh],[J],[MR1] and [MR2].

The global method involves an introduction of a new distribution, on the group $G \times G'$, which we denote $I_{G \times G'}(\Phi : H_3, \chi_3)$. We establish separately the relation between $I_G(f : H_1, \chi_1, H_2, \chi_2)$ and $I_{G'}(f' : H'_1, \chi'_1, H'_2, \chi'_2)$ with $I_{G \times G'}(\Phi : H_3, \chi_3)$. We put together the two relations using a global identity for $I_{G \times G'}(\Phi : H_3, \chi_3)$.

This paper is part of our series of works on connecting the dual pair method and relative trace formula. After some early exploration of examples like in [MR2], we arrive at a general principle which predicts the relative trace identities for dual pairs G and G' . The principle is stated in [MR1] where we conjecture that there is a relative trace identity (3) if over the local places there are G' -module isomorphisms

$$\omega_\psi[H_2, \chi_2^{-1}] \cong \text{Ind}_{H'_1}^{G'} \chi'_1 \tag{5}$$

and G -module isomorphisms

$$\omega_\psi[H'_2, \chi'^{-1}_2] \cong \text{Ind}_{H_1}^G \chi_1. \tag{6}$$

Here we use $\omega_\psi[H, \chi]$ to denote the Jacquet module of ω_ψ with respect to the group H and character χ . The motivation behind the conjecture is that we can establish a vector space isomorphism

$$\text{Ind}_{H_1}^G \chi_1[H_2, \chi_2^{-1}] \cong \text{Ind}_{H'_1}^{G'} \chi'_1[H'_2, \chi'^{-1}_2] \tag{7}$$

by applying (5) and (6) to the space $\omega_\psi[H_2 \times H'_2, \chi_2^{-1} \times \chi'^{-1}_2]$. The relative trace identity can be considered as a global analogue of the isomorphism (7).

The global method presented here seems to fit the principle better than the local method used in [MR1]. The idea is the introduction of a distribution which can be considered as the global analogue of $\omega_\psi[H_2 \times H'_2, \chi_2^{-1} \times \chi'^{-1}_2]$. The key to the proof is the analogues of the isomorphisms (5) and (6): they are the easily established Propositions 4.1 and 5.1.

We note an interesting feature in the global method, namely it does not require the existence of a map ι matching the orbits $\{\phi\}$ and $\{\phi'\}$. Though in the example we are considering there is a match of orbits ([MR1]), we have found examples where the trace identity holds while there is no match of orbits, (and thus the method of comparing orbital integrals does not apply). We describe such an example.

Consider the case where $G = G' = Sp_2$ be the symplectic group. Let $H_1 = H'_1 = SL_2 \times SL_2$ and $\chi_1 = \chi'_1$ be the trivial character of $SL_2 \times SL_2$. Let H_2 be the maximal unipotent subgroup of Sp_2

$$H_2 = \{n(x, y, z, v) = \begin{pmatrix} 1 & x & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y & z \\ & 1 & v & -y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\},$$

and H'_2 be the subgroup of H_2 consisting of the $n(0, y, z, v)$. Let ψ be a nontrivial additive character of \mathbf{A}/F ; let χ'_2 be the character of H'_2 with $\chi'_2(n(0, y, z, v)) = \psi(v + z)$; let χ_2 be an automorphic form on H_2 :

$$\chi_2(n(x, y, z, v)) = \psi(v)\Theta_\Phi^\psi(n).$$

Here Φ is a Schwartz function of \mathbf{A} and

$$\Theta_\Phi^\psi(n(x, y, z, v)) = \sum_{w \in F} \psi(w^2v + wy + z)\Phi(w + x).$$

With these data, the relative trace identity (3) holds. One can check the relevant orbits do not match in this case. However following the global work of [GRS], it is not difficult to see the identity holds when

$$f(g) = \int_{\mathbf{A}} \Phi(w)f'(gn(w, 0, 0, 0))dw.$$

The paper is organized as follows. In Section 2, we define the distributions on G, G' and state the main result. In Section 3 we introduce a third distribution on $G \times G'$ and prove a global identity. In Section 4, we compare the distributions on G and $G \times G'$. In Section 5, we compare the distributions on G' and $G \times G'$. The proof of the trace identity is given in Section 6.

2. Statement of the result

Let X be a $2n + 1$ dimensional space; let e_1, \dots, e_{2n+1} be the standard basis of $X \cong F^{2n+1}$. Let $SO(n + 1, n)$ be the special orthogonal group fixing the symmetric bilinear form \langle, \rangle given by $\langle e_i, e_j \rangle = 2$ when $i + j = 2n + 2, i \neq j$, $\langle e_{n+1}, e_{n+1} \rangle = 1$, and $\langle e_i, e_j \rangle = 0$ otherwise. Let V be the subspace of F^{2n+1} spanned by $\{e_1, \dots, e_{n-1}, e_{n+1}\}$. We define some subgroups of $SO(n + 1, n)$.

Let R' be the subgroup of $SO(n + 1, n)$ fixing V :

$$R' = \{r \in SO(n + 1, n) | r(e_1, \dots, e_{n-1}, e_{n+1}) = (e_1, \dots, e_{n-1}, e_{n+1})\}. \quad (8)$$

Let N denote the subgroup of upper triangular matrices with unit diagonal in GL_n which acts on the n -dimensional space V . Define R the subgroup of

$SO(n + 1, n)$:

$$R = \{r \in SO(n + 1, n) \mid \exists n \in N, r\mathbf{v} = n\mathbf{v}, \forall \mathbf{v} \in V\}. \tag{9}$$

This is the Bessel group of $SO(n + 1, n)$. The group R' is a normal subgroup of R with an isomorphism $\rho : R/R' \cong N$ given by $r \mapsto n$ in equation (9).

Let U be the maximal unipotent subgroup of $SO(n + 1, n)$ consisting of the upper triangular matrices with unit diagonal.

Fix ψ a nontrivial additive character of \mathbf{A}/F . We define the characters θ on $N(\mathbf{A})$, χ on $R(\mathbf{A})$ and μ on $U(\mathbf{A})$ as follows:

$$\theta(n) = \psi(n_{1,2} + \dots + n_{n-1,n}), n \in N; \tag{10}$$

$$\chi(r) = \psi^{-1}(r_{1,2} + r_{2,3} + \dots + r_{n-2,n-1} + r_{n-1,n+1}), r \in R; \tag{11}$$

$$\mu(u) = \psi(u_{1,2} + u_{2,3} + \dots + u_{n-1,n} + u_{n,n+1}), u \in U. \tag{12}$$

We define the distribution on G as:

$$I_G(f : R, \chi, U, \mu) = \int_{R(F) \backslash R(\mathbf{A})} \int_{U(F) \backslash U(\mathbf{A})} K_f(r, u) \chi(r) \mu(u) dr du. \tag{13}$$

Let Y be a $2n$ dimensional space with a symplectic structure $\langle *, * \rangle'$, where explicitly with the standard basis f_1, \dots, f_{2n} of $Y \cong F^{2n}$, $\langle f_i, f_j \rangle' = 1$ when $i + j = 2n + 1$ and $i \leq n$ and 0 when $i + j \neq 2n + 1$. Let Sp_n be the corresponding symplectic group. Let N' be the group of upper triangular matrices in Sp_n with unit diagonal. Denote an element in the double cover G' of Sp_n by (g, ϵ) with $g \in Sp_n$ and $\epsilon = \pm 1$. If H is a subgroup of $Sp_n(\mathbf{A})$ where the covering splits, we will consider $h \in H$ as an element in G' through the splitting. In particular, the group N' can be considered as a subgroup of G' via the embedding $n \rightarrow (n, 1)$, $n \in N'$. Define a character θ' of N' by

$$\theta'(n) = \psi(n_{1,2} + \dots + n_{n-1,n} + n_{n,n+1}/2) \tag{14}$$

where $n = (n_{i,j}) \in N'$. We define a distribution on G' :

$$\begin{aligned} &I_{G'}(\tilde{f} : N', \theta', N', (\theta')^{-1}) \\ &= \int_{N'(F) \backslash N'(\mathbf{A})} \int_{N'(F) \backslash N'(\mathbf{A})} K_{\tilde{f}}(n_1, n_2) \theta'(n_1 n_2^{-1}) dn_1 dn_2. \end{aligned} \tag{15}$$

Our main result is:

Theorem 2.1. *For $f = \otimes f_v \in C_c^\infty(G(\mathbf{A}))$ there exist the maps $\epsilon_v : f_v \mapsto \tilde{f}_v \in \mathcal{S}(G'(F_v))$ such that there is a relative trace identity (in the sense of (3)):*

$$I_G(f : R, \chi, U, \mu) = I_{G'}(\tilde{f} : N', \theta', N', \theta'^{-1}). \tag{16}$$

3. A global identity

To prove the Theorem, we introduce another distribution. Let $Z = X \otimes Y$ be a $2n(2n + 1)$ dimensional space, then Z inherits a symplectic structure:

$$\langle x \otimes y, x' \otimes y' \rangle'' = \langle x, x' \rangle \langle y, y' \rangle', \quad x, x' \in X, y, y' \in Y.$$

Let $Sp_{n(2n+1)}$ be the corresponding symplectic group. Recall that G and G' consist of a dual pair inside the metaplectic group $\tilde{G} = \tilde{Sp}_{n(2n+1)}$. We will denote by (g, g') the image of $g \in G$ and $g' \in G'$.

Associated to ψ is a Weil representation ω_ψ of \tilde{G} , $[\mathbf{W}]$. Choose a maximal isotropic subspace Z^+ of Z , the Weil representation acts on the space $\mathcal{S}(Z^+)$ of Schwartz functions on Z^+ . For $\Phi \in \mathcal{S}(Z^+(\mathbf{A}))$, define the Theta function:

$$\Theta_\Phi^\psi(g) = \sum_{A \in Z^+(F)} \omega_\psi(g)\Phi(A), \quad g \in \tilde{G}. \tag{17}$$

We now define the distribution

$$\begin{aligned} & I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}) \\ &= \int_{U(F) \backslash U(\mathbf{A})} \int_{N'(F) \backslash N'(\mathbf{A})} \Theta_\Phi^\psi(u, n) \mu(u) \theta'(n^{-1}) du dn. \end{aligned} \tag{18}$$

This definition depends on our choice of the maximal isotropic subspace Z^+ . Given any two choices Z_1^+ and Z_2^+ of the maximal isotropic subspaces, there is an element $w_0 \in Sp_{n(2n+1)}(F)$ such that $Z_2^+ = w_0 Z_1^+$. We use ω'_ψ to denote the action of the Weil representation on $\mathcal{S}(Z_2^+)$, then for $g \in \tilde{G}(\mathbf{A})$:

$$\omega'_\psi(g)\Phi(A) = \omega_\psi(w_0 g w_0^{-1})\Phi'(w_0^{-1}A), \quad A \in Z_2^+(\mathbf{A}), \tag{19}$$

where $\Phi \in \mathcal{S}(Z_2^+)$, and $\Phi'(w_0^{-1}A) = \Phi(A)$ is a Schwartz function on Z_1^+ ; (we use also the fact that the covering splits over $Sp_{n(2n+1)}(F)$ to consider w_0 as an element in $\tilde{G}(\mathbf{A})$).

The following simple global identity is the key in our proof of Theorem 2.1:

Proposition 3.1. *Let Z_1^+ and Z_2^+ be two maximal isotropic subspaces of Z with $Z_2^+ = w_0 Z_1^+$. Let $\Phi \in \mathcal{S}(Z_2^+)$ and $\Phi' \in \mathcal{S}(Z_1^+)$ be related by $\omega_\psi(w_0)\Phi'(w_0^{-1}A) = \Phi(A)$ with $A \in Z_2^+(\mathbf{A})$. Then*

$$I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}) = I_{\tilde{G}}(\Phi' : U \times N', \mu \times (\theta')^{-1}). \tag{20}$$

Proof. We only need to show $\Theta_\Phi^\psi(g) = \Theta_{\Phi'}^\psi(g)$. Using the fact that the Theta function is automorphic, (which follows from the Poisson summation formula),

we get:

$$\begin{aligned}
 \Theta_{\Phi'}^\psi(g) &= \Theta_{\Phi'}^\psi(w_0g) \\
 &= \sum_{A \in Z_1^+(F)} \omega_\psi(w_0gw_0^{-1})\omega_\psi(w_0)\Phi'(A) \\
 &= \sum_{A \in Z_2^+(F)} \omega_\psi(w_0gw_0^{-1})\omega_\psi(w_0)\Phi'(w_0^{-1}A) \\
 &= \sum_{A \in Z_2^+(F)} \omega'_\psi(g)\Phi(A) = \Theta_\Phi^\psi(g).
 \end{aligned}$$

□

Two obvious choices of Z^+ are: $Z_1^+ = X \otimes Y^+$ where Y^+ is the maximal isotropic subspace of Y spanned by $\{f_1, \dots, f_n\}$; $Z_2^+ = X^+ \otimes Y \oplus X^0 \otimes Y^+$ where X^+ is the maximal isotropic subspace of X spanned by $\{e_1, \dots, e_n\}$ and X^0 is the one dimensional subspace generated by e_{n+1} . We will use the choice Z_1^+ when comparing $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1})$ with the distribution on G , and use the choice Z_2^+ when comparing it with the distribution on G' .

We remark that $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1})$ can be considered as a relative trace formula on $G \times G'$. Consider the relative trace formula on $G \times G'$:

$$\int \int K_{f \otimes \tilde{f}}((g, h), (u, n)) \Theta_\Phi^\psi(g, h) \mu(u) \theta'(n^{-1}) d(u, n) d(g, h) \tag{21}$$

where the first integral is over $G \times G'(F) \backslash G \times G'(\mathbf{A})$, and the second over $U \times N'(F) \backslash U \times N'(\mathbf{A})$. If we define for $A \in Z^+(\mathbf{A})$,

$$\Phi_{f \otimes \tilde{f}}(A) = \int_{G \times G'(\mathbf{A})} f(g^{-1}) \tilde{f}(h^{-1}) \omega_\psi(g, h) \Phi(A) d(g, h),$$

then the distribution (21) equals $I_{\tilde{G}}(\Phi_{f \otimes \tilde{f}} : U \times N', \mu \times (\theta')^{-1})$.

4. Comparison with $I_G(f : R, \chi, U, \mu)$

We identify $Z_1^+ = X \otimes Y^+$ with $M_{2n+1, n}$ the space of $(2n + 1) \times n$ matrices. The action of ω_ψ on $\mathcal{S}(M_{2n+1, n})$ restricted to $G \times G'$ can be described as follows:

$$\omega_\psi(g, \delta(h)) \Phi(A) = |\det(h)|^{n+1/2} \frac{\gamma(1, \psi)}{\gamma(\det(h)^{2n+1}, \psi)} \Phi(g^{-1}Ah), \tag{22}$$

$$\omega_\psi(1, \sigma'_i) \Phi(A) = \gamma(1, \psi)^{-(2n+1)l/2} \Phi^{\wedge l}(A), \tag{23}$$

$$\omega_\psi(1, \left(\begin{pmatrix} 1_n & V \\ & 1_n \end{pmatrix}, 1 \right))\Phi(A) = \psi(\text{tr}({}^t A \sigma_{2n+1} A V \sigma_n)/2)\Phi(A). \tag{24}$$

Here $g \in G, h \in GL_n, \sigma_l$ is the matrix in GL_l with 1's on anti-diagonal and 0's elsewhere; $h^* = \sigma_n^t h^{-1} \sigma_n$;

$$\delta(h) = \left(\begin{pmatrix} h & \\ & h^* \end{pmatrix}, 1 \right), \sigma'_l = \left(\begin{pmatrix} & & -\sigma_l \\ & 1_{2n-2l} & \\ \sigma_l & & \end{pmatrix}, 1 \right); \tag{25}$$

$\gamma(a, \psi)$ denotes the Weil constant, and for $A = [A_1, A_2] \in M_{2n+1, n}$, where $A_1 \in M_{2n+1, l}$,

$$\Phi^{\wedge l}(A) = \int_{A' \in M_{2n+1, l}} \psi(\text{tr}({}^t A' \sigma_{2n+1} A_1))\Phi([A', A_2])dA'. \tag{26}$$

The distribution $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1})$ equals:

$$\sum_{A \in M_{2n+1, n}(F)} \int_{U(F) \backslash U(\mathbf{A})} \int_{N'(F) \backslash N'(\mathbf{A})} \omega_\psi(u, n)\Phi(A)\mu(u)\theta'(n^{-1})dudn. \tag{27}$$

Let $E_1 \in M_{2n+1, n}$ be the matrix $[e_1, \dots, e_{n-1}, e_{n+1}]$.

Proposition 4.1. *With suitable choice of the measure, when for $g \in G(\mathbf{A})$*

$$\int_{R'(F) \backslash R(\mathbf{A})} \omega_\psi(r^{-1}g, 1_{2n})\Phi(E_1)\chi(r)dr = \int_{R(\mathbf{A})} f(r^{-1}g)\chi(r)dr, \tag{28}$$

we have

$$\begin{aligned} & \sum_{A \in M_{2n+1, n}(F)} \int_{N'(F) \backslash N'(\mathbf{A})} \omega_\psi(g, n)\Phi(A)\theta'(n^{-1})dn \\ &= \int_{R(F) \backslash R(\mathbf{A})} K_f(r, g)\chi(r)dr. \end{aligned} \tag{29}$$

Proof. Let S be the variety of $A = [A_1, \dots, A_n] \in M_{2n+1, n}(F)$ with $\langle A_i, A_j \rangle = 0$ for all $1 \leq i, j \leq n$ except $\langle A_n, A_n \rangle = 1$, and with A being of rank n .

Lemma 4.2. *If $A \notin S$, then*

$$\int_{N'(F) \backslash N'(\mathbf{A})} \omega_\psi(g, n)\Phi(A)\theta'(n^{-1})dn = 0.$$

Proof. If the value of $\langle A_i, A_j \rangle$ is not as prescribed above, then the integration over the unipotent radical of the Siegel parabolic N'_U of Sp_n (which is a normal subgroup of N') would vanish by formula (24). If A is not of the

maximal rank, then there is a $n = \begin{pmatrix} n_1 & \\ & n_1^* \end{pmatrix} \in N'(\mathbf{A})$ with $\theta'(n) \neq 1$ and $An_1 = A$; from formula (22), the integral vanishes. \square

From the Witt's Theorem, the map $g \mapsto A_g = g^{-1}E_1$ gives a bijection between $R'(F)\backslash G(F)$ and the variety S . From the above Lemma, we get the left hand side of (29) equals:

$$\sum_{h \in R'(F)\backslash G(F)} \int_{N'(F)\backslash N'(\mathbf{A})} \omega_\psi(g, n)\Phi(A_h)\theta'(n^{-1})dn. \tag{30}$$

Lemma 4.3. *With suitable choice of the measure, the expression (30) equals*

$$\sum_{h \in R(F)\backslash G(F)} \int_{R'(\mathbf{A})\backslash R(\mathbf{A})} \omega_\psi(r^{-1}hg, 1_{2n})\Phi(E_1)\chi(r)dr. \tag{31}$$

Proof. We first integrate out the unipotent radical of the Siegel parabolic subgroup N'_U of Sp_n in (30). From (24), the integration yields a factor which equals the volume of $N'_U(F)\backslash N'_U(\mathbf{A})$; with suitable measure, this volume equals 1. As $N'_U\backslash N' \cong N$, from (22) the expression (30) equals:

$$\sum_{h \in R'(F)\backslash G(F)} \int_{N(F)\backslash N(\mathbf{A})} \omega_\psi(g, 1_{2n})\Phi(A_hn)\theta(n^{-1})dn. \tag{32}$$

Recall there is an isomorphism $\rho : R'\backslash R \cong N$. It is easy to verify that $rE_1 = E_1n$ when $\rho(r) = n$, thus $A_hn = h^{-1}rE_1$. Also clear is that $\chi(r) = \theta^{-1}(\rho(r))$. From (22) we get (32) equals:

$$\sum_{h \in R'(F)\backslash G(F)} \int_{(R'\backslash R(F))\backslash (R'\backslash R(\mathbf{A}))} \omega_\psi(r^{-1}hg, 1_{2n})\Phi(E_1)\chi(r)dr. \tag{33}$$

A standard unwinding of the summation and integration gives the Lemma. \square

On the other hand from (2), the right hand side of (29) equals

$$\sum_{h \in R(F)\backslash G(F)} \int_{R(\mathbf{A})} f(r^{-1}hg)\chi(r)dr. \tag{34}$$

Thus the Proposition 4.1 follows from (31) and (34). \square

It follows immediately from Proposition 4.1, (13) and (27) that:

Proposition 4.4. *When Φ and f satisfy (28), we have $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}) = I_G(f : R, \chi, U, \mu)$.*

5. Comparison with $I_{G'}(\tilde{f} : N', \theta', N', (\theta')^{-1})$

We use the maximal isotropic subspace $Z_2^+ \cong X^+ \otimes Y \oplus X^0 \otimes Y^+$. We will write an element in this space as (A, B) where $A \in M_{2n,n}$ and $B \in F^n$. Use ω'_ψ to denote the Weil representation of \tilde{G} acting on $\mathcal{S}(Z_2^+)$. We will use the following formulas on this action:

For a function $\Phi \otimes \Phi_0(A, B) = \Phi(A)\Phi_0(B)$ in $\mathcal{S}(Z_2^+)$, for $g' \in Sp_n$:

$$\omega'_\psi(1, (g', 1))\Phi \otimes \Phi_0(A, B) = \Phi(g'^{-1}A)\omega_\psi(g', 1)\Phi_0(B) \tag{35}$$

the second ω_ψ being the Weil representation of G' acting on $\mathcal{S}(Y^+)$. Let U^1, U^2, U^3 be subgroups of U consisting respectively of

$$u_1(n) = \begin{pmatrix} n & & \\ & 1 & \\ & & n^* \end{pmatrix}, u_2(v) = \begin{pmatrix} 1_n & v & -vv^*/2 \\ & 1 & v^* \\ & & 1_n \end{pmatrix}, u_3(V) = \begin{pmatrix} 1_n & 0 & V \\ & 1 & 0 \\ & & 1_n \end{pmatrix}$$

Here in the definition of $u_2(v)$, we understand v as a vector in F^n . Then U^3 and U^2U^3 are normal subgroups of U . The action of U on $\mathcal{S}(Z_2^+)$ can be described as follows:

$$\omega_\psi(u_1(n), 1)\Phi \otimes \Phi_0(A, B) = \Phi \otimes \Phi_0(An, B), \tag{36}$$

$$\omega_\psi(u_3(V), 1)\Phi \otimes \Phi_0(A, B) = \psi(\text{tr}({}^tAV\sigma_nA)/2)\Phi \otimes \Phi_0(A, B). \tag{37}$$

We only need to know the value of $\omega_\psi(u_2(v), 1)\Phi \otimes \Phi_0(A, B)$ for some special choice of A :

$$\omega_\psi(u_2(v), 1)\Phi \otimes \Phi_0(A_0, B) = \psi({}^tBv)\Phi \otimes \Phi_0(A_0, B) \tag{38}$$

where $A_0 = [f_1, \dots, f_n]$ and $v \in F^n$.

For $\Phi \in \mathcal{S}(Z_2^+(\mathbf{A}))$, the distribution $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1})$ equals:

$$\sum_{A \in M_{2n,n}(F), B \in F^n} \int_{U(F) \setminus U(\mathbf{A})} \int_{N'(F) \setminus N'(\mathbf{A})} \omega'_\psi(u, n)\Phi(A, B)\mu(u)\theta'(n^{-1})dudn. \tag{39}$$

Let $E_2 = (A_0, B_0)$ with $B_0 = e_n \in F^n$. Recall N'_U is the unipotent radical of the Siegel parabolic subgroup of Sp_n .

Proposition 5.1. *With suitable choice of the measure, when for $g \in G'(\mathbf{A})$*

$$\int_{N'_U(\mathbf{A}) \setminus N'(\mathbf{A})} \omega'_\psi(1_{2n+1}, ng)\Phi(E_2)\theta'(n^{-1})dn = \int_{N'(\mathbf{A})} \tilde{f}(n^{-1}g)\theta'(n)dn, \tag{40}$$

we have

$$\begin{aligned} & \sum_{A \in M_{2n,n}(F), B \in F^n} \int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, g) \Phi(A, B) \mu(u) du \\ &= \int_{N'(F) \backslash N'(\mathbf{A})} K_{\bar{f}}(n, g) \theta'(n) dn. \end{aligned} \tag{41}$$

Proof. Let S' be the variety of $A = [A_1, \dots, A_n] \in M_{2n,n}(F)$ with $\langle A_i, A_j \rangle' = 0$ for all $1 \leq i, j \leq n$ and with A being of rank n . Similar to Lemma 4.2, using (37) and (36) we get

Lemma 5.2. *If $A \notin S'$, then*

$$\int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, g) \Phi(A, B) \mu(u) du = 0.$$

Thus from Witt's Theorem, there is a bijection from $N'_U(F) \backslash Sp_n(F)$ to S' given by $h \mapsto h^{-1}A_0$; the left hand side of (41) becomes:

$$\sum_{h \in N'_U \backslash Sp_n(F), B \in F^n} \int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, g) \Phi(h^{-1}A_0, B) \mu(u) du. \tag{42}$$

From Lemma 2 of [Fu], this expression equals:

$$\sum_{h \in N'_U \backslash Sp_n(F), B \in F^n} \int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, hg) \Phi(A_0, B) \mu(u) du. \tag{43}$$

From (38), we see

Lemma 5.3. *If $B \neq B_0$,*

$$\int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, g) \Phi(A_0, B) \mu(u) du = 0.$$

Thus the left hand side of (41) equals:

$$\sum_{h \in N'_U \backslash Sp_n(F)} \int_{U(F) \backslash U(\mathbf{A})} \omega'_\psi(u, hg) \Phi(A_0, B_0) \mu(u) du. \tag{44}$$

Integrating over the normal subgroup U^2U^3 gives a factor which equals the volume of $U^2U^3(F) \backslash U^2U^3(\mathbf{A})$; with suitable choice of measure, this volume equals 1. As $(U^2U^3 \backslash U) \cong U^1 \cong N$, from (36) we get (44) equals:

$$\sum_{h \in N'_U \backslash Sp_n(F)} \int_{N(F) \backslash N(\mathbf{A})} \omega'_\psi(1_{2n+1}, hg) \Phi(A_0n, B_0) \theta(n) dn. \tag{45}$$

Here we used the fact $\mu(u_1(n)) = \theta(n)$.

Since $A_0n = \begin{pmatrix} n & \\ & n^* \end{pmatrix} A_0$, from (35), we see (45) equals:

$$\sum_{h \in N'_U \backslash Sp_n(F)} \int_{N(F) \backslash N(\mathbf{A})} \omega'_\psi(1_{2n+1}, \begin{pmatrix} n & \\ & n^* \end{pmatrix} hg) \Phi(A_0, B_0) \theta(n^{-1}) dn. \tag{46}$$

As $\theta'(\begin{pmatrix} n & \\ & n^* \end{pmatrix}) = \theta(n)$ and $N \cong N'_U \backslash N'$, we can unwind the above sum and integral to get

$$\sum_{h \in N' \backslash Sp_n(F)} \int_{N'_U \backslash N'(\mathbf{A})} \omega'_\psi(1_{2n+1}, nhg) \Phi(A_0, B_0) \theta'(n^{-1}) dn. \tag{47}$$

On the other hand, from (2) the right hand side of (41) equals:

$$\sum_{h \in N' \backslash Sp_n(F)} \int_{N'(\mathbf{A})} \tilde{f}(n^{-1}hg) \theta'(n) dn. \tag{48}$$

The Proposition 5.1 follows from comparing (47) and (48). □

It follows immediately from Proposition 5.1, (15) and (39) that:

Proposition 5.4. *When Φ and \tilde{f} satisfy (40), we have $I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}) = I_{G'}(\tilde{f} : N', \theta', N', (\theta')^{-1})$.*

6. Proof of the result

We first show how to define the map ϵ_v that associates to each $f_v \in C_c^\infty(G_v)$ a function $\tilde{f}_v \in \mathcal{S}(G'_v)$. Given $f_v \in C_c^\infty(G_v)$, from Lemma 5.3 in [MR1], there exists a $\Phi_v = \epsilon_{1,v}(f_v) \in \mathcal{S}(Z_1^+(F_v))$ such that:

$$\int_{R'(F_v) \backslash R(F_v)} \omega_\psi(r^{-1}g, 1_{2n}) \Phi_v(E_1) \chi(r) dr = \int_{R(F_v)} f_v(r^{-1}g) \chi(r) dr. \tag{49}$$

We remark that this statement is rather easy to prove; the above equation determines the function Φ_v on the closed variety $S(F_v)$, we extend it to a function on $Z_1^+(F_v)$, (and that is the reason for the restriction $f_v \in C_c^\infty(G_v)$).

Let $\Phi'_v \in \mathcal{S}(Z_2^+(F_v))$ be defined by

$$\epsilon'_v : \Phi_v \mapsto \Phi'_v, \quad \Phi'_v(A) = \omega_\psi(w_0) \Phi(w_0^{-1}A)$$

for $A \in Z_2^+(F_v)$. From Lemma 5.5 in [MR1], there is a function $\tilde{f}_v = \epsilon_{2,v}(\Phi'_v) \in \mathcal{S}(G'_v)$ such that

$$\begin{aligned} & \int_{N'_U(F_v)\backslash N'(F_v)} \omega'_\psi(1_{2n+1}, ng)\Phi'_v(E_2)\theta'(n^{-1})dn \\ &= \int_{N'(F_v)} \tilde{f}_v(n^{-1}g)\theta'(n)dn. \end{aligned} \tag{50}$$

We remark that this statement is again easy to prove: the left hand side of (50) gives an N' -equivariant function that is a Schwartz function on $N'\backslash G'$, thus the existence of \tilde{f}_v .

We will define $\epsilon_v(f_v)$ through this association $f_v \mapsto \Phi_v \mapsto \Phi'_v \mapsto \tilde{f}_v$. Next we prove the equivalent of the fundamental lemma in this case. Let v be a non-archimedean place, let $\Phi_{0,v}$ and $\Phi'_{0,v}$ be the characteristic functions of the integer lattices of $Z_1^+(F_v)$ and $Z_2^+(F_v)$; when f_v and \tilde{f}_v are Hecke functions, let

$$\begin{aligned} \Phi_{f,v}(A) &= \int_{G_v} f_v(g)\omega_\psi(g, 1_{2n})\Phi_{0,v}(A)dg, \quad A \in Z_1^+(F_v); \\ \Phi'_{\tilde{f},v}(A) &= \int_{G'_v} \tilde{f}_v(g)\omega'_\psi(1_{2n+1}, g)\Phi'_{0,v}(A)dg, \quad A \in Z_2^+(F_v). \end{aligned}$$

Proposition 6.1. *At almost all places v , when f_v and \tilde{f}_v are Hecke functions:*

$$\int_{R'(F_v)\backslash R(F_v)} \omega_\psi(r^{-1}g, 1_{2n})\Phi_{f,v}(E_1)\chi(r)dr = \int_{R(F_v)} f_v(r^{-1}g)\chi(r)dr; \tag{51}$$

$$\begin{aligned} & \int_{N'_U(F_v)\backslash N'(F_v)} \omega'_\psi(1_{2n+1}, ng)\Phi'_{\tilde{f},v}(E_2)\theta'(n^{-1})dn \\ &= \int_{N'(F_v)} \tilde{f}_v(n^{-1}g)\theta'(n)dn; \end{aligned} \tag{52}$$

Moreover when $\tilde{f}_v = \lambda_v(f_v)$ under the Hecke algebra homomorphism, $\epsilon'_v(\Phi_{f,v}) = \Phi'_{\tilde{f},v}$.

Proof. Let $f_{0,v}$ and $\tilde{f}_{0,v}$ be the unit elements of the Hecke algebras. For a general Hecke function f (or \tilde{f}), we have $f = f * f_0$ (and $\tilde{f} = \tilde{f} * \tilde{f}_0$). Thus from the definition of $\Phi_{f,v}$ and $\Phi'_{\tilde{f},v}$, we only need to show the identities (51) and (52) when $f_v = f_{0,v}$, $\tilde{f}_v = \tilde{f}_{0,v}$. In these cases $\Phi_{f,v} = \Phi_{0,v}$ and $\Phi'_{\tilde{f},v} = \Phi'_{0,v}$. The proof of the identities in this case can be done by direct computation. For example both sides of (52) gives left N' equivariant and right K' invariant functions, (here K' is the maximal compact subgroup of G'); from Iwasawa

decomposition, we only need to check the two functions equal over the diagonal elements. A simple computation shows both functions vanish over diagonal elements not in K' and equals 1 over identity. The detail of the proof is given in Lemmas 7.3 and 7.2 in [MR1].

Over almost all places, w_0 is an integral matrix; we have $\epsilon'_v(\Phi_{0,v}) = \Phi'_{0,v}$. The last statement of the Proposition is just another statement of the unramified Howe duality, see [H],[R],[Wa]. \square

We are now ready to prove Theorem 2.1.

Proof. Let S_0 be a large enough finite set of places, such that if $v \notin S_0$ the statements in Proposition 6.1 hold. For $f = \otimes_{v \in S_0} f_v \otimes_{v \notin S_0} f_v \in \mathcal{S}(G(\mathbf{A}))$ where f_v are Hecke functions when $v \notin S_0$, let $\tilde{f} = \otimes_{v \in S_0} \epsilon_v(f_v) \otimes_{v \notin S_0} \lambda_v(f_v)$. We prove the identity (16). From Proposition 4.4, (49) and (51), we get when $\Phi = \otimes_{v \in S_0} \epsilon_{1,v}(f_v) \otimes_{v \notin S_0} \Phi_{f,v}$,

$$I_G(f : R, \chi, U, \mu) = I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}). \tag{53}$$

From Propositions 3.1 and 6.1, we see when $\Phi' = \otimes_{v \in S_0} \epsilon'_v(\Phi_v) \otimes_{v \notin S_0} \Phi'_{\tilde{f},v}$,

$$I_{\tilde{G}}(\Phi : U \times N', \mu \times (\theta')^{-1}) = I_{\tilde{G}}(\Phi' : U \times N', \mu \times (\theta')^{-1}). \tag{54}$$

From Proposition 5.4, (50) and (52), we get

$$I_{\tilde{G}}(\Phi' : U \times N', \mu \times (\theta')^{-1}) = I_{G'}(\tilde{f} : N', \theta', N', (\theta')^{-1}). \tag{55}$$

The identity (16) follows. \square

We remark that the same argument also shows the relative trace identities between the distribution (21) and the distributions $I_G(f : R, \chi, U, \mu)$ and $I_{G'}(\tilde{f} : N', \theta', N', (\theta')^{-1})$.

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Zhengyu Mao
 Department of Mathematics and Computer Science
 Rutgers University at Newark
 Newark, NJ 07102, USA
 zmao@andromeda.rutgers.edu

Stephen Rallis
 Department of Mathematics
 The Ohio State University
 Columbus, OH 43210, USA
 haar@math.ohio-state.edu

