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Cuspidal Representations Associated to $(GL(n), O(n))$ over Finite Fields and p-Adic Fields*

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When $\bar{r}$ is an irreducible cuspical representation of $\overline{G} = GL(n, q)$ and $\overline{H}$ is an orthogonal group associated to a symmetric matrix in $\overline{G}$ then the space of $\overline{H}$-fixed vectors for $\bar{r}$ is shown to have dimension at most one. Such a representation $\bar{r}$ induces an irreducible supercuspidal representation π of $G = GL(n, E)$, where E is a p-adic field whose residue field has order q. The space of those linear forms on the space of π which are invariant under an orthogonal group is computed. For the corresponding group of orthogonal similitudes, it is shown that the dimension of the space of invariant linear forms is always at most one.

1. INTRODUCTION

Suppose E is a field and V is an n-dimensional quadratic space over E, with $n > 1$. When $G = GL(n, E)$ and H is the orthogonal group $O(V)$, one can consider the following problem. If π is an irreducible (complex linear) representation of G, what is the dimension $[\pi | H : 1]$ of the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms on the space of π? The most well

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* The authors gratefully acknowledge H. Jacquet for suggesting this research. We also mention that the referee has pointed out that some of our proofs might be simplified somewhat by appealing to the results of [14]. We have chosen, however, to leave the paper in its present form so that it is more accessible to experts in various fields.

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known example occurs when $E = \mathbb{R}$ and H is the standard orthogonal group $O(n, \mathbb{R})$. Then $[\pi | H : 1] \leq 1$ for all irreducible representations π of G or, in other words, the pair (G, H) is a Gelfand pair [11]. Then the G-module $L^2(H \setminus G)$ has a multiplicity-free decomposition as a direct sum of irreducible representations. The irreducible constituents in this decomposition are precisely the irreducible representations π for which $[\pi | H : 1] = 1$. Therefore, in this example, the dimensions $[\pi | H : 1]$ play a central role in the harmonic analysis of the symmetric space $H \setminus G$.

In this paper, we consider the above problem in two cases: (1) when E is a finite field of odd characteristic and π is a cuspidal representation, and (2) when E is a finite extension of a p-adic field $\mathbb{Q}_p$, with $p \neq 2$, and π is a supercuspidal representation of lowest level. In case (1), though (G, H) is not a Gelfand pair, we show in Theorem 1 that $[\pi | H : 1] \leq 1$ for all irreducible *cuspidal* representation π. Case (2) is treated in Theorem 3. When n is odd, once again $[\pi | H : 1]$ is at most one. However, when n is even, it is possible to construct irreducible supercuspidal representations π such that $[\pi | H : 1] = 2$. In cases (1) and (2), the representations π may be realized as induced representations and we show in Theorems 2 and 3 the precise relationship between the inducing representation and $[\pi | H : 1]$.

Let H' be the group of orthogonal similitudes. For $h \in H'$, let $\lambda(h)$ be the similitude ratio of h. For χ a character of $E^\times$, we can associate a character $\chi(\lambda(h))$ on H'. In Theorem 4, it is shown that for the above supercuspidal representations π, for any character χ, the dimension $[\pi | H' : \chi]$ of $\mathrm{Hom}_{H'}(\pi, \chi \circ \lambda)$ is always at most one.

As with the example of $O(n, \mathbb{R}) \setminus GL(n, \mathbb{R})$, the results in this paper have a direct application to the harmonic analysis on $H \setminus G$. Our true motivation, however, lies in certain conjectures of Jacquet [9] which we now briefly recall. The Shimura correspondence between automorphic forms of half integral weight and automorphic forms of integral weight has been interpreted in [5] as a correspondence between automorphic cuspidal representations of the double cover of $GL(2)$ and automorphic cuspidal representations of $GL(2)$. Jacquet has shown [9] that if E is a number field, then an automorphic cuspidal representation π of $GL(2, E_{\mathbb{A}})$ is in the image of the Shimura correspondence precisely when it is (H', χ)-distinguished with respect to some orthogonal similitude group H' and some character χ of $E_{\mathbb{A}}^\times / E^\times$. In [9], it is conjectured that the image of the analogous correspondence from the double cover of $GL(n)$ to $GL(n)$ has a similar characterization (see also [12]). Without recalling the definition of "distinguished," we simply mention that if π is (H', χ)-distinguished and has the factorization $\otimes_v \pi_v$ over the places of E then the components π_v are characterized by the property that $[\pi_v | H'_v : \chi_v] = 1$.

In Theorem 5, we give a precise criterion which must be satisfied by supercuspidal representations of lowest level arising as local components

π_v in the image of the Shimura correspondence (under the conjecture of [9]). Note that in [4], Flicker and Kazhdan have conjectured that the local components π_v in the image of the Shimura correspondence are "metic." In the case π_v is supercuspidal, being metic is equivalent to having a central character μ with $\mu(-1) = 1$. It is clear the supercuspidal representations satisfying the criterion in Theorem 5 are exactly the metic representations. Thus the conjectures in [4, 9] are compatible. We also remark that Theorem 2 may be regarded as an analogue of Jacquet's conjectures for finite fields. This point of view is explained more fully in Section 2.

2. FINITE FIELDS

In this section, G will denote the group $GL(n, \overline{\mathbb{F}}_q)$, where $n > 1$ and q is an odd prime power and $\overline{\mathbb{F}}_q$ is an algebraic closure of the field $\mathbb{F}_q$. Up to conjugacy by $GL(n, q)$, the orthogonal groups in n variables which are defined over $\mathbb{F}_q$ are constructed as follows. Let $\kappa \in G$ be the diagonal matrix $\mathrm{diag}(1, \ldots, 1, \epsilon)$, where $\epsilon = 1$ if n is odd and ϵ is either 1 or a fixed nonsquare $\delta \in \mathbb{F}_q^\times$ when n is even. Given ϵ, we let H be the orthogonal group consisting of the points in G fixed by the involution $\theta(g) = \kappa\, {}^t g^{-1} \kappa^{-1}$. In general, if J is an algebraic subgroup of G defined over $\mathbb{F}_q$, then J^F denotes the group of points in J fixed by the Frobenius map $F((g_{ij})) = (g_{ij}^q)$.

THEOREM 1. *If $\pi : G^F \to GL(V)$ is an irreducible, cuspidal representation of $G^F = GL(n, q)$ on a complex vector space V, then the space V^{H^F} of H^F-fixed vectors in V has dimension at most one.*

We emphasize that Theorem 1 does not hold for arbitrary irreducible representations of G^F, or, in other words, the pair (G^F, H^F) is not a Gelfand pair. By Frobenius Reciprocity, this is the same as saying that the representation of G^F induced from the trivial representation of H^F is not multiplicity free. (See [8].)

The representations (π, V) in Theorem 1 for which V^{H^F} is nonzero will be referred to as *spherical representations*. Such representations may be precisely described in terms of Deligne–Lusztig characters as follows. We define below a maximal torus T in G such that $T^F \cup \{0\}$ has the structure of a field with q^n elements. If π is an irreducible cuspidal representation of G^F then the character of π is $\pm R^G_{T,\nu}$, where $R^G_{T,\nu}$ is the Deligne–Lusztig character associated to some character ν of T^F in general position.

THEOREM 2. *An irreducible, cuspidal representation of G^F with Deligne–Lusztig character $\pm R^G_{T,\nu}$ is spherical precisely when $\nu(-1) = 1$ or, equivalently, when there is a character μ of T^F such that $\nu = \mu^2$.*

Our proof appeals to a general formula due to Lusztig [10] which may be used to compute the dimension of the space of H^F-fixed vectors for representations associated to a general symmetric space (G^F, H^F) over a finite field.

Let $w_0 \in G$ be the permutation matrix associated to the permutation $(12 \dots n)$. Thus the ijth entry of w_0 is 1 if $j - 1 \equiv 1 \pmod n$ and otherwise it is 0. Let A denote the group of diagonal matrices in G. Given $\alpha \in \overline{\mathbb{F}}_q^\times$, we let $D(\alpha)$ denote the diagonal matrix $\mathrm{diag}(\alpha, F(\alpha), \dots, F^{n-1}(\alpha)) \in A$. It is easy to check that if $a \in A$ then $w_0 a w_0^{-1} = F(a)$ precisely when $a = D(\alpha)$ for some $\alpha \in \mathbb{F}_{q^n}^\times$.

Choose $v_1, \dots, v_n \in \mathbb{F}_{q^n}$ which are linearly independent over $\mathbb{F}_q$. Take g_0 to be the matrix in G whose ijth entry is $F^{j-1}(v_i)$. To see that g_0 is indeed invertible, one may argue as follows. Suppose $x = (x_1, \dots, x_n)$, viewed as a column vector, is in the null space of g_0. Then we have the dependence relation $x_1 + x_2 F + \cdots + x_{n-1} F^{n-2} + x_n F^{n-1} = 0$. Linear independence of characters of $\mathbb{F}_{q^n}$ implies $x = 0$ and thus $g_0 \in G$. We also remark that $F(g_0)^{-1} g_0 = w_0$.

We now use the element g_0 to define an F-stable torus T of G by $T = g_0 A g_0^{-1}$. If $t \in T$ and $a = g_0^{-1} t g_0 \in A$ then $t \in T^F$ exactly when $a = D(\alpha)$ for some $\alpha \in \mathbb{F}_{q^n}^\times$. It follows that $T^F \approx \mathbb{F}_{q^n}^\times$.

To apply Lusztig's formula, we must first recall several notations from [10]. First of all, $\mathscr{J}$ is defined in [10, 1.5] as the set of all θ-stable maximal tori S contained in a Borel subgroup B (depending on S) such that B is opposed to $\theta(B)$. Clearly $A \in \mathscr{J}$. Furthermore, according to [10, 1.5.a], the special orthogonal group H^0 (the identity component of H) acts transitively by conjugation on $\mathscr{J}$. Therefore, in the present context, $\mathscr{J}$ may be described simply as the set of all tori of the form hAh^{-1} with $h \in H^0$.

In [10, 2.2], a homomorphism $\epsilon_S : S^F \cap H \to \{\pm 1\}$ is defined for each F-stable, θ-stable maximal torus S of G. What is relevant, for our purposes, is that $\epsilon_S(z) = 1$ if z lies in the center of G. Following the definition in [10, 1.2], we let Θ_T denote the F-stable variety consisting of those $f \in G$ such that the torus $f^{-1}Tf$ is θ-stable. Then we take $\Theta_T^F = \Theta_T \cap G^F$ and let $\Theta_{T,\nu}^F$ be the set of all $f \in \Theta_T^F$ such that $\nu(fxf^{-1}) = \epsilon_{f^{-1}Tf}(x)$ for all $x \in H \cap f^{-1}T^F f$.

LEMMA 1. *The set* $\Theta_{T,\nu}^F$ *is empty unless* $\nu(-1) = 1$, *in which case* $\Theta_{T,\nu}^F = \Theta_T^F$.

Proof. According to Lemma 10.4 of [10], if $f \in \Theta_{T,\nu}^F$ then $f^{-1}Tf \in \mathscr{J}$. Hence there exists $h \in H^0$ such that $f^{-1}Tf = hAh^{-1}$. Thus $H \cap f^{-1}T^F f \subset h(H \cap A)h^{-1}$. But $H \cap A$ consists of the matrices of the form $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$, where $\alpha_i = \pm 1$. It follows that if $t \in h(H \cap A)h^{-1}$ then $t^2 = 1$. Since $T^2 \cup \{0\}$ is a field of odd characteristic, $\{\pm 1\}$ is the set of roots of $X^2 - 1$ in T^F. Consequently, if $f \in \Theta_{T,\nu}^F$ then $H \cap f^{-1}T^F f = \{\pm 1\}$. Since $\epsilon_{f^{-1}T^F f}(\pm 1) = 1$, our claim follows. ∎

It is clear from the definition of Θ_T that Θ_T^F is a union of double cosets in $T^F \setminus G^F / H^F$.

LEMMA 2. *The set Θ_T^F consists of a single double coset in $T^F \setminus G^F / H^F$.*

Proof. Suppose $f \in \Theta_T^F$. Choosing ν such that $\nu(-1)$ and hence $\Theta_T^F = \Theta_{T,\nu}^F$, we may argue as in the proof of Lemma 1 above to deduce that there exists $h \in H^0$ such that $h^{-1}Ah = f^{-1}Tf = f^{-1}g_0 A g_0^{-1} f$. Since $g_0^{-1} f h^{-1}$ normalizes A, there exists a permutation matrix $w \in G$ and an element $a \in A$ such that $g_0^{-1} f h^{-1} = aw$. We now have $g_0 awh = f = F(f) = F(g_0)F(a)wF(h)$ which, in turn, leads to the equation

$$hF(h)^{-1} = w^{-1}a^{-1}w_0^{-1}F(a)w. \qquad (*)$$

Since the left hand side of $(*)$ lies in H^0, we obtain

$$\kappa = w^{-1}a^{-1}w_0^{-1}F(a)w\kappa w^{-1}F(a)w_0 a^{-1}w.$$

Using the commutativity of A, this simplifies to $w_0 a' w_0^{-1} = F(a')$, where $a' = a^2 w\kappa w^{-1}$. Thus $a' = D(\alpha)$, for some $\alpha \in \mathbb{F}_{q^n}^{\times}$.

We would like to determine when α is the square of an element of $\mathbb{F}_{q^n}^{\times}$. Let N denote the norm map from $\mathbb{F}_{q^n}^{\times}$ to $\mathbb{F}_q^{\times}$. We claim that $\alpha \in (\mathbb{F}_{q^n}^{\times})^2$ precisely when $N(\alpha) \in (\mathbb{F}_q^{\times})^2$. To justify this, recall that N is surjective and, by Hilbert's Theorem 90, its kernel is $(\mathbb{F}_{q^n}^{\times})^{q-1} \approx \mathbb{F}_{q^n}^{\times}/\mathbb{F}_q^{\times}$. Clearly, $\alpha \in (\mathbb{F}_{q^n}^{\times})^2$ implies $N(\alpha) \in (\mathbb{F}_q^{\times})^2$. Now suppose $N(\alpha) = \beta^2$ for some $\beta \in \mathbb{F}_q^{\times}$. Then $\beta = N(\beta_0)$, for some $\beta_0 \in \mathbb{F}_{q^n}^{\times}$. Consequently, $\alpha\beta_0^{-2} \in (\mathbb{F}_{q^n}^{\times})^{q-1} \subset (\mathbb{F}_{q^n}^{\times})^2$. Hence, $\alpha \in (\mathbb{F}_{q^n}^{\times})^2$ and we have shown that $\alpha \in (\mathbb{F}_{q^n}^{\times})^2$ if and only if $N(\alpha) \in (\mathbb{F}_q^{\times})^2$.

We will now show that the square class of α in $\mathbb{F}_{q^n}^{\times}$ is determined by n and κ but not by the choice of f. In light of the remarks of the previous paragraph, it suffices to determine the square class of $N(\alpha)$ in $\mathbb{F}_q^{\times}$. Note that $N(\alpha) = \det D(\alpha) = \epsilon\mu^2$, where $\epsilon = \det \kappa$ and $\mu = \det a$. Taking determinants in eq. $(*)$, we see that $F(\mu) = (-1)^{n+1}\mu$. When n is odd, this implies that $\mu \in \mathbb{F}_q^{\times}$ and thus $N(\alpha) \in (\mathbb{F}_q^{\times})^2$. When n is even, $\mu^2 \in \delta(\mathbb{F}_q^{\times})^2$ and $N(\alpha) \in \delta \in (\mathbb{F}_q^{\times})^2$. In general, $N(\alpha) \in \delta^{n+1}\epsilon(\mathbb{F}_q^{\times})^2$.

Once n and κ are chosen, we may fix, independent of f, an element $\alpha_1 \in \mathbb{F}_{q^n}^{\times}$ such that $N(\alpha_1) = \delta^{n+1}\epsilon$. We may also fix an element $\alpha_0 \in \mathbb{F}_{q^{2n}}^{\times}$ such that $\alpha_0^2 = \alpha_1$. At this point, we have shown that $\alpha \in \alpha_1(\mathbb{F}_{q^n}^{\times})^2$, where the latter coset does not depend on the choice of f. We may therefore choose $\alpha_2 \in \mathbb{F}_{q^{2n}}^{\times}$ such that $\alpha = \alpha_1\alpha_2^2$. Since a, $w\kappa w^{-1}$, and $D(\alpha)$ are diagonal matrices which satisfy the relation $a^2 w\kappa w^{-1} = D(\alpha)$, we may take square roots in the latter equation to obtain $a = D(\alpha_0)D(\alpha_2)ws_0w^{-1}t$, where $s_0 \in A$ is chosen (independent of f) such that $s_0^2 = \kappa^{-1}$ and t is a diagonal matrix whose diagonal entries are ± 1. We now have $f = g_0 awh = g_0 D(\alpha_2)D(\alpha_0)ws_0w^{-1}twh$. Since $g_0 D(\alpha_2)g_0^{-1} \in T^F$, $s_0^{-1}ws_0 \in H$, and

$w^{-1}tw \in A \cap H$, we deduce that f lies in the double coset $T^F f_0 H$, where $f_0 = g_0 D(\alpha_0) s_0$ is independent of f. Note that $f_0^{-1}F(f_0) \in H^0$. Since H^0 is a closed connected subgroup of G, Lang's Theorem [2, p. 32] says that $h \mapsto h^{-1}F(h)$ defines a surjective map from H^0 onto itself. In particular, there exists $h_0 \in H^0$ such that $h_0 F(h_0)^{-1} = f_0^{-1}F(f_0)$ or, equivalently, $f_0 h_0 \in G^F$. We now have $f \in T^F f_0 H \cap G^F = T^F f_0 h_0 H \cap G^F = T^F f_0 h_0 H^F$. This shows that Θ_T^F coincides with the double coset $T^F f_0 h_0 H^F$. ▮

Proof of Theorem 1 *and* 2. Suppose $\pi : G^F \to GL(V)$ is an irreducible, cuspidal representation with Deligne–Lusztig character $\pm R_{T,\nu}^G$. Then according to [10, 10.6a], the dimension of V^{H^F} is identical to the number of double cosets in $T^F \setminus \Theta_{T,\nu}^F / H^F$. Lemma 1 implies that this dimension is zero unless $\nu(-1) = 1$. Lemma 2 implies that if $\nu(-1) = 1$ then the dimension is one. This proves Theorems 1 and 2. ▮

We remark that Theorem 2 can be considered as a finite field analogue of the conjecture in [9]. Over a finite field, the double cover of $GL(n, q)$ splits over $GL(n, q)$. The analogue of Shimura correspondence over the finite field is then a map from the irreducible representations of $GL(n, q)$ to itself. Heuristically, it maps a cuspidal representation $\pm R_{T,\mu}^G$ to $\pm R_{T,\mu^2}^G$. The analogue of the conjecture in [4] for the Shimura correspondence is then trivially true, and Theorem 2 (together with the following lemma) implies the analogue of the conjecture in [9].

Let H' be the group of orthogonal similitudes, consisting of points in G with $\theta(g) = \lambda(g)^{-1} g$.

LEMMA 3. *Let π be a cuspidal representation of G. If $[\pi|H : 1] = 1$, then there is a character χ of $\mathbb{F}_q^\times$ with $[\pi|H' : \chi] = 1$.*

Proof. It is clear that $[\pi|H' : \chi] \le [\pi|H : 1]$. We will show that if $[\pi|H : 1] = 1$ then there exists a nonzero vector v_0 in the space of π and a character χ such that the identity $\pi(h)v_0 = \chi(\lambda(h))v_0$ holds for all $h \in H'$.

From Theorem 2, we see that the central character of π has the form μ^2 for some character μ. Let V_0 be the (one-dimensional) space of H-fixed vectors in the space of π and fix a nonzero vector $v \in V_0$. When n is odd, $H' = ZH$ when Z is the center of G. If we take $\chi = \mu$ and $v_0 = v$, then the required identity holds.

When n is even, $H' = ZH \cup h_0 ZH$, for some $h_0 \in H'$ whose similitude ratio $\delta = \lambda(h_0)$ is a nonsquare in $\mathbb{F}_q^\times$. Then $h_0^{-1} H h_0 = H$ and, consequently, $\pi(h_0)v$ is fixed by H. Since V_0 is one-dimensional, $\pi(h_0)v = av$ for some $a \in \mathbb{F}_q^\times$. Since $h_0^2 \in \delta H$, it follows that $\pi(h_0^2)v = \mu^2(\delta)v$. Thus $a^2 = \mu^2(\delta)$ and $a = \zeta(\delta)\mu(\delta)$, where ζ is either the trivial character or the quadratic character on $\mathbb{F}_q^\times$. Taking $v_0 = v$ and $\chi = \mu\zeta$, the necessary identity is again satisfied. ∎

3. ORBITS OF SYMMETRIC MATRICES

For the remainder of this paper, we let $G = GL(n, E)$, where $n > 1$ and E is a finite extension of a p-adic field $\mathbb{Q}_p$ for some odd prime p. We consider the action $g \cdot \eta = g\eta {}^tg$ of G on the set S of symmetric matrices in G. The orbits are characterized by the discriminant and the Hasse invariant and they correspond to the classes of rank n quadratic spaces over E up to change of basis. (See [13, Theorem 4.4.2].)

To be more explicit, we fix some notations. Let ϖ be a prime element of the ring of integers $\mathfrak{O}$ of E and let $\mathfrak{P} = \varpi\mathfrak{O}$ be the maximal ideal of $\mathfrak{O}$. Fix $\delta \in \mathfrak{O}^\times$ such that the image of δ in the residue field $\mathfrak{O}/\mathfrak{P}$ is not a square. Then $\{1, \delta, \varpi, \delta\varpi\}$ is a set of representatives for the square classes $E^\times/(E^\times)^2$. The discriminant $\operatorname{disc}(\eta)$ of $\eta \in S$ is the image of the determinant of η in $E^\times/(E^\times)^2$. If $\eta \in S$, then the Hasse invariant $\operatorname{Hasse}(\eta)$ of η is $\prod_{i \le j}(a_i, a_j)$, where $\operatorname{diag}(a_1, \dots, a_n)$ is a diagonal matrix in the orbit of η and $(\ ,\ )$ is the Hilbert symbol. When $n > 2$ there are eight G-orbits in S, one for each possibility of the pair $(\operatorname{disc}(\eta), \operatorname{Hasse}(\eta))$. When $n = 2$ there are only seven orbits, since it is impossible to have both $\operatorname{disc}(\eta) = -1$ and $\operatorname{Hasse}(\eta) = -1$. (We warn the reader that the Hasse invariant is often defined as a product over $i < j$, instead of $i \le j$. Though these two definitions are not equivalent, either may be used to classify quadratic forms. If one adopts the latter definition, then in the table for odd n below the exponents $(n+1)/2$ should be changed to $(n-1)/2$. The table for even n is unaffected.)

If $K = GL(n, \mathfrak{O})$ then the problem of determining the K-orbits in S is the same as determining the rank n quadratic spaces over E up to integral change of basis. We recall O'Meara's solution to this problem, essentially as stated in [6]. Fix a prime element $\varpi \in E$. Then the K-orbits are parametrized by the set $\mathscr{A}$ of sequences $\alpha = (\alpha_1, \dots, \alpha_m)$ of triples $\alpha_i = (a_i, n_i, \epsilon_i)$ of the following type. We require that $a_1 > \cdots > a_m$ is a decreasing sequence of integers, $n_1 + \cdots + n_m = n$ is a partition of n by positive integers, and $\epsilon_i \in \{1, \delta\}$. For each such sequence $\alpha \in \mathscr{A}$, we define a diagonal matrix $\varpi^\alpha \in S$ by $\varpi^\alpha = \varpi^{\alpha_1} \oplus \cdots \oplus \varpi^{\alpha_m}$, where $\varpi^{\alpha_i} = \varpi^{a_i} \operatorname{diag}(1, \dots, 1, \epsilon_i) \in GL(n_i, E)$. The set $\{\varpi^\alpha : \alpha \in \mathscr{A}\}$ is a set of representatives for the K-orbits in S.

If $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathscr{A}$, we let $l(\alpha) = m$. We say that the symmetric matrix ϖ^α is *relevant* if $l(\alpha) = 1$. For relevant matrix ϖ^α determines a ZK-orbit which consists of the elements $k\varpi^{\alpha}{}^tk$ with $k \in ZK$. It is necessary for us to have an inventory of these orbits. Let $\xi = \operatorname{diag}(1, \dots, 1, \delta)$. It is elementary to see that every relevant matrix lies in the Z-orbit of one of the four relevant matrices: $1, \varpi, \xi, \varpi\xi$.

When n is odd, these matrices have distinct discriminants and hence they lie in distinct G-orbits. Of the eight G-orbits in S, four contain a unique ZK-orbit with a relevant matrix and these are represented by the columns in the table:

(n odd)	1	ϖ	ξ	$\varpi\xi$
disc	1	ϖ	δ	$\varpi\delta$
Hasse	1	$\zeta^{(n+1)/2}$	1	$\zeta^{(n+1)/2}$

Here ζ is the Hilbert symbol $(-1, \varpi)$. (This is 1 precisely when -1 is a square in E.) The other four G-orbits contain no relevant matrices.

When n is even, the situation is more complicated. We have

(n even)	1	ϖ	ξ	$\varpi\xi$
disc	1	1	δ	δ
Hasse	1	$\zeta^{n/2}$	1	$-\zeta^{n/2}$

If $\zeta^{n/2} = 1$ then 1 and ϖ represent two distinct ZK-orbits which lie in a common G-orbit. (To see that these are indeed distinct ZK-orbits, assume that $\varpi^i k^t (\varpi^i k) = \varpi$, for some integer i and some $k \in K$. Taking determinants produces a contradiction.) There are two G-orbits which contain a unique ZK-orbit with relevant matrices. The remaining G-orbits contain no relevant matrices. The case of $\zeta^{n/2} = -1$ is similar.

When $\eta \in S$, we let $R(\eta)$ denote the number of ZK-orbits which lie in the G-orbit of η and which contain some relevant matrices. The following result follows directly from the above discussion:

PROPOSITION 1. *Assume* $\eta \in S$ *and put* $x = \operatorname{disc}(\eta)$ *and* $y = \operatorname{Hasse}(\eta)$.

(1) *If* n *is odd then* $R(\eta) = 1$ *if* (x, y) *equals* $(1,1)$, $(\varpi, \zeta^{(n+1)/2})$, $(\delta, 1)$ *or* $(\varpi\delta, \zeta^{(n+1)/2})$. *Otherwise,* $R(\eta) = 0$ *when* n *is odd.*

(2) *Suppose* n *is even. Then* $R(\eta) = 2$ *if* $(x, y) = (1,1) = (1, \zeta^{n/2})$ *or if* $(x, y) = (\delta, 1) = (\delta, -\zeta^{n/2})$. *Otherwise,* $R(\eta) = 1$ *if* $x = 1$ *or* δ *and* $R(\eta) = 0$ *if* $x = \varpi$ *or* $\overline{\eta}\delta$.

There is one specific example worth mentioning, because it is important in applications. Let η be the permutation matrix associated to the permutation $(n, 1)(n-1, 2) \cdots$. (This is the "long element" of the Weyl group of G and the associated orthogonal group is split.) Then $\operatorname{disc}(\eta) = (-1)^{\lfloor n/2 \rfloor}$ and $\operatorname{Hasse}(\eta) = 1$. (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) It follows that $R(\eta) = 1$, when n is odd and $R(\eta) = 2$ when n is even.

4. PROJECTIONS OF ORTHOGONAL GROUPS

Associated to each symmetric matrix $\eta \in S$ is the orthogonal group $O(\eta)$ consisting of all $g \in G$ such that $g\eta{}^tg = \eta$, and the orthogonal similitude group $GO(\eta)$ consisting of all $g \in G$ such that $g\eta{}^tg = \lambda(g)\eta$ where $\lambda(g)$ is a scalar. We have $gO(\eta)g^{-1} = O(g\eta{}^tg)$. Let K_1 be the principal congruence subgroup $1 + M(n, \mathfrak{P})$ of K. Then every subgroup J of G projects to a subgroup $\bar{J}$ of the group $\overline{G} = GL(n, \mathfrak{O}/\mathfrak{P})$ by first taking the intersection $J \cap K$ and then reducing entries modulo $\mathfrak{P}$. Thus $\bar{J} \approx (J \cap K)/(J \cap K_1)$. We also adopt the notation $\bar{x}$ for the image of $x \in \mathfrak{O}$ in the residue field $\mathfrak{O}/\mathfrak{P}$. Given $\alpha \in \mathscr{A}$, we will abbreviate the group $O(\varpi^{\alpha})$ as H_α. In this section, we consider the groups $\overline{H}_\alpha$.

Let L be the subgroup of lower triangular matrices in K_1.

LEMMA 4. *If $\eta \in G$ is a diagonal matrix with $|\eta_{11}| \le \cdots \le |\eta_{nn}|$ then $x \mapsto x\eta{}^tx$ defines a bijection of L onto $S \cap \eta K_1$.*

Proof. Assume $\eta \in G$ and $y \in S \cap \eta K_1$ are fixed, with η as in the hypothesis of the lemma. It suffices to show that there exists a unique $x \in L$ which satisfies $x\eta{}^tx = y$. Since the latter equation is an identity of symmetric matrices, it suffices to consider the ijth entries with $i \ge j$. The entries x_{ij} (with $i \ge j$) of the desired matrix x must satisfy the equations $y_{ij} = \sum_{s \le j} x_{is}\eta_{ss}x_{js}$. When $i = j = 1$, this reduces to $x_{11}^2 = \eta_{11}^{-1}y_{11}$. Therefore, x_{11} must equal $\sqrt{\eta_{11}^{-1}y_{11}}$, where $\sqrt{u}$ denotes the unique square root of $u \in 1 + \mathfrak{P}$ which lies in $1 + \mathfrak{P}$. The other entries x_{ij} (with $i \ge j$) are defined as follows in terms of previously defined entries x_{ij} with smaller values of $i + j$. If $i > j$, then

$$x_{ij} = \eta_{jj}^{-1}x_{jj}^{-1}\left(y_{ij} - \sum_{s<j} x_{is}\eta_{ss}x_{js}\right)$$

and when $i = j$ we use

$$x_{ii} = \sqrt{\eta_{ii}^{-1}\left(y_{ii} - \sum_{s<i} x_{is}\eta_{ss}x_{is}\right)}\,.$$

It is easy to verify that since $\eta^{-1}y \in K_1$, the matrix $x \in L$ just constructed lies in L. Furthermore, since the equations used in the construction have unique solutions, the matrix x is the unique solution in L of $x\eta{}^tx = y$. ▮

LEMMA 5. *If $\eta \in K$ is diagonal then $\overline{O(\eta)} = O(\bar{\eta})$ and $\overline{GO(\eta)} = GO(\bar{\eta})$.*

Proof. The inclusion $\overline{O(\eta)} \subset O(\bar{\eta})$ is obvious. To prove the reverse inclusion, we assume we are given $k \in K$ such that $\bar{k} \in O(\bar{\eta})$. Since $\overline{\eta^{-1}k\eta{}^tk} = 1$, we have $\eta^{-1}k\eta{}^tk \in K_1$. Thus we may apply Lemma 4 to obtain $x \in K_1$ (in fact, $x \in L$) such that $x\eta{}^t x = k\eta{}^t k$. It follows that $\bar{k} = \bar{x}^{-1}\bar{k} \in \overline{O(\eta)}$. Thus $O(\bar{\eta}) \subset \overline{O(\eta)}$.

Similarly, we have the inclusion $\overline{GO(\eta)} \subset GO(\bar{\eta})$. When n is odd, $GO(\bar{\eta}) = \overline{(Z \cap K)}O(\bar{\eta})$, where Z is the center of G. Our claim $\overline{GO(\eta)} = GO(\bar{\eta})$ follows from the previous statement for the orthogonal group. When n is even, as in the proof of Lemma 3, $GO(\bar{\eta}) = (\overline{Z \cap K})O(\bar{\eta}) \cup h_0(\overline{Z \cap K})O(\bar{\eta})$. From the previous statement for the orthogonal group, we only need to show there exists $h_0' \in GO(\eta) \cap K$ with $\bar{h}_0' \in h_0(\overline{Z \cap K})O(\bar{\eta})$. From the description in Section 3, η and $\delta\eta$ lie in the same K-orbit. Thus there exists $h_0' \in K$ with $h_0'\eta{}^t h_0' = \delta\eta$. Such an h_0' satisfies the above condition. ▮

We emphasize that Lemma 5 does not extend to arbitrary diagonal matrices $\eta \in G$. For example, if η is the scalar matrix ϖ then $\overline{O(\eta)} \neq O(\bar{\eta})$.

We now restrict our attention to those groups $\bar{H}_\alpha$ for which ϖ^α is not relevant, that is, $l(\alpha) > 1$. If $\alpha \in \mathscr{A}$ satisfies $l(\alpha) > 1$ and $n = n_1 + \cdots + n_m$ is the associated partition of n, then we define the (proper) parabolic subgroup P_α of G as the group of lower triangular block matrices $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in G$ with $a \in GL(n - n_m, E)$ and $d \in GL(n_m, E)$. The unipotent radical of P_α consists of the block matrices $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ and it is denoted by N_α.

LEMMA 6. *For all* $\alpha \in \mathscr{A}$ *with* $l(\alpha) > 1$, *the group* $\bar{N}_\alpha$ *is a subgroup of* $\bar{H}_\alpha$.

Proof. Given a block matrix $u = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in N_\alpha \cap K$, it suffices to show that there exist blocks a, b, and d such that the block matrix $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies in $H_\alpha \cap K$ and $\bar{x} = \bar{u}$. As a block matrix, ϖ^α assumes the form $\left(\begin{smallmatrix} \varpi^\beta & 0 \\ 0 & \varpi^{\alpha_m} \end{smallmatrix}\right)$, with $\beta = (\alpha_1, \dots, \alpha_{m-1})$. The matrix equation $x\varpi^\alpha x^* = \varpi^\alpha$ is equivalent to three equations involving the blocks:

$$d\varpi^{\alpha_m}d^* = \varpi^{\alpha_m} - c\varpi^\beta c^*, \tag{1}$$

$$a\varpi^\beta c^* + b\varpi^{\alpha_m}d^* = 0, \tag{2}$$

$$a\varpi^\beta a^* + b\varpi^{\alpha_m}b^* = \varpi^\beta. \tag{3}$$

The right hand side of Eq. (1) is symmetric and lies in $\varpi^{\alpha_m}(1 + M(n_m, \mathfrak{P}))$. Hence, Lemma 4 gives a solution $d \in 1 + M(n_m, \mathfrak{P})$. Equation (2) may be solved for b and used to eliminate b from Eq. (3). Equation (3) then becomes

$$a^{-1}\varpi^\beta(a^*)^{-1} = \varpi^\beta + \varpi^\beta c^*(d^*)^{-1}(\varpi^{\alpha_m})^{-1}d^{-1}c\varpi^\beta.$$

The right hand side lies in $\varpi^{\beta}(1 + M(n - n_m, \mathfrak{P}))$ and thus Lemma 4 gives a solution $a \in 1 + M(n - n_m, \mathfrak{P})$. Substituting a back into the expression for b obtained from Eq. (2), we see that the entries of b lie in $\mathfrak{P}$. We have now constructed the desired matrix x. ▮

5. SUPERCUSPIDAL REPRESENTATIONS

Fix an irreducible representation r of ZK on a complex vector space W such that the restriction of r to K_1 is trivial. Restricting r to K gives a representation $\bar{r}$ of $\overline{G} \approx K/K_1$ on W. We assume that the latter representation is cuspidal in the sense that no nonzero vector in W is fixed by the unipotent radical of any (proper) parabolic subgroup of $\overline{G}$. This assumption implies that r induces an irreducible, supercuspidal representation $\pi = \operatorname{ind}_{ZK}^{G}(r)$. The representation π is realized as follows. Let $\mathscr{F}$ be the space of functions $f: G \to W$ such that $f(kg) = r(k)f(g)$, for all $k \in ZK$ and $g \in G$. This is a G-module with respect to the action of G by right translations. The representation π is defined as the submodule $\mathscr{F}_c$ of functions $f \in \mathscr{F}$ with compact support modulo Z.

Fix a symmetric matrix $\eta \in S$ and let $H = O(\eta)$. The purpose of this section is to give a formula for the dimension $[\pi | H : 1]$ of $\operatorname{Hom}_H(\pi, 1)$. The formula involves the constant $R(\eta)$, computed in Proposition 1, as well as the dimension $[\bar{r} | \overline{H}(\eta) : 1]$ of $\operatorname{Hom}_{\overline{H}(\eta)}(\bar{r}, 1)$, where $\overline{H}(\eta)$ is an orthogonal group in $\overline{G} = GL(n, \mathfrak{O}/\mathfrak{P})$ defined as follows. We define $\bar{\epsilon}_0 \in \mathfrak{O}/\mathfrak{P}$ to be 1 if $\operatorname{disc}(\eta)$ is the square class of 1 or ϖ and $\bar{\epsilon}_0 = \bar{\delta}$ otherwise. Take $\bar{\eta}_0 \in \overline{G}$ to be the diagonal matrix $\operatorname{diag}(1, \ldots, 1, \bar{\epsilon}_0)$. Then $\overline{H}(\eta)$ is defined as the group of $\bar{g} \in \overline{G}$ such that $\bar{g}\bar{\eta}_0 {}^t\bar{g} = \bar{\eta}_0$. Our main result is:

THEOREM 3. $[\pi | H : 1] = R(\eta)[\bar{r} | \overline{H}(\eta) : 1]$.

When n is odd, Theorem 3 implies $\operatorname{Hom}_H(\pi, 1)$ always has dimension at most one. When n is even, this dimension is always at most two. We recall from Section 3, that in the special case where η is the permutation matrix associated to the longest Weyl element, then $R(\eta) = 1$ if n is odd and $R(\eta) = 2$ if n is even. We remark also that $[\bar{r} | \overline{H}(\eta) : 1]$ is independent of η.

To discuss invariant linear forms on $\mathscr{F}$, it is convenient to use the language of l-sheaves from [1]. We are interested in computing $\operatorname{Hom}_H(\mathscr{F}_c, 1) = \operatorname{Hom}_H(\pi, 1)$, that is, the space of linear forms Λ on $\mathscr{F}_c$ which are H-invariant in the sense that $\Lambda(\pi(h)f) = \Lambda(f)$, for all $h \in H$ and $f \in \mathscr{F}_c$. This is the same as the space of H-invariant distributions on the l-sheaf $\mathscr{F}$.

Given $g \in G$, let $\mathscr{F}_g$ denote the space of $f \in \mathscr{F}$ whose support is contained in the double coset $ZKgH$ and $\mathscr{F}_{g,c} = \mathscr{F}_g \cap \mathscr{F}_c$. Since $ZK \backslash G/H$ is a discrete space $\mathscr{F}_c$ decomposes as a direct sum $\oplus_g \mathscr{F}_{g,c}$, where g ranges over a set of representatives for $ZK \backslash G/H$. Each summand is stable under the action of H and thus $\operatorname{Hom}_H(\mathscr{F}_c, 1)$ decomposes as a direct sum of the spaces $\operatorname{Hom}_H(\mathscr{F}_{g,c}, 1)$ of H-invariant distributions on $\mathscr{F}_g$.

The following result uses standard techniques due to Mackey:

LEMMA 7. *Let J be a closed subgroup of G and let $\mathscr{J}_{g,c}$ be the space of $f \in \mathscr{F}_c$ with support contained in the double coset $ZKgJ$. Then the map which sends $\Lambda \in \operatorname{Hom}_{ZK \cap gJg^{-1}}(r, 1)$ to the linear form $\overline{\Lambda} \in \operatorname{Hom}_J(\mathscr{J}_{g,c}, 1)$ defined by*

$$\overline{\Lambda}(f) = \sum_{h \in (g^{-1}ZKg \cap J)\backslash J} \Lambda(f(gh))$$

is an isomorphism.

Proof. Let r_g be the representation of $J_g = g^{-1}ZKg \cap J$ on W given by $r_g(h) = r(ghg^{-1})$. We also consider the induced representation $\operatorname{ind}_{J_g}^J(r_g)$ consisting of the compactly supported functions $\varphi : J \to W$ such that $\varphi(kh) = r_g(k)\varphi(h)$, for all $k \in J_g$ and $h \in J$. There is an isomorphism of J-modules $\mathscr{J}_{g,c} \approx \operatorname{ind}_{J_g}^J(r_g)$ given by sending $f \in \mathscr{J}_{g,c}$ to the function f_g given by $f_g(h) = f(gh)$. Therefore, the space $\operatorname{Hom}_J(\mathscr{J}_{g,c}, 1)$ is equivalent to $\operatorname{Hom}_J(\operatorname{ind}_{J_g}^J(r_g), 1)$. By Frobenius Reciprocity (as stated in Proposition 2.29 of [1]), the latter space is isomorphic to $\operatorname{Hom}_{J_g}(r_g, 1)$ which is identical to $\operatorname{Hom}_{ZK \cap gJg^{-1}}(r, 1)$. It follows that $\operatorname{Hom}_{ZK \cap gJg^{-1}}(r, 1)$ and $\operatorname{Hom}_J(\mathscr{J}_{g,c}, 1)$ are isomorphic. The explicit form of the isomorphism follows from the explicit form of the Frobenius Reciprocity isomorphism in [1]. ▮

Proof of Theorem 3. Taking $J = H$ in Lemma 7, we obtain

$$\begin{aligned} \operatorname{Hom}_H(\pi, 1) &\approx \bigoplus_{g \in ZK\backslash G/H} \operatorname{Hom}_{K \cap gHg^{-1}}(r, 1) \\ &= \bigoplus_{g \in ZK\backslash G/H} \operatorname{Hom}_{\overline{O(g\eta^t g)}}(\bar{r}, 1). \end{aligned}$$

We now compute the summands in the latter decomposition. The double coset space $ZK \backslash G/H$ embeds via $g \mapsto g\eta^t g$ in the space of ZK-orbits in S. The image of this embedding consists of the ZK-orbits contained in the G-orbit of η. For each such ZK-orbit, we may choose a representative $g_\alpha \in G$ such that $g_\alpha \eta^t g_\alpha = \varpi^\alpha$, for some $\alpha \in \mathscr{A}$. The resulting indices α

comprise a subset $\mathscr{A}_0$ of $\mathscr{A}$. We have

$$\mathrm{Hom}_H(\pi, 1) \approx \bigoplus_{\alpha \in \mathscr{A}_0} \mathrm{Hom}_{\overline{H}_\alpha}(\bar{r}, 1).$$

If $\alpha \in \mathscr{A}_0$ and $l(\alpha) > 1$ then the summand indexed by α is zero. (Indeed, if $\mathrm{Hom}_{\overline{H}_\alpha}(\bar{r}, 1)$ were nonzero then there would exist a nonzero $\overline{H}_\alpha$-fixed vector $w \in W$. But then, according to Lemma 6, w would be fixed by $\overline{N}_\alpha$ which contradicts the cuspidality of $\bar{r}$.) Letting $\mathscr{A}_1$ denote the collection of those $\alpha \in \mathscr{A}_0$ with $l(\alpha) = 1$, we have

$$\mathrm{Hom}_H(\pi, 1) \approx \bigoplus_{\alpha \in \mathscr{A}_1} \mathrm{Hom}_{\overline{H}_\alpha}(\bar{r}, 1).$$

We now make several remarks about the latter direct sum. According to Lemma 5 and Theorem 1, each summand has dimension at most one. In addition, the number of summands, that is, the cardinality of $\mathscr{A}_1$, is the same as the quantity $R(\eta)$ computed in Proposition 1. The possible values of $R(\eta)$ are 0, 1, and 2. Note that if $R(\eta) = 2$, say $\mathscr{A}_1 = \{\alpha', \alpha''\}$, then the orthogonal groups $H_{\alpha'}$ and $H_{\alpha''}$ are identical (nor merely isomorphic). It follows that if $R(\eta) = 2$ then the summands associated to α' and α'' are identical. Thus the dimension of $\mathrm{Hom}_{\overline{H}_\alpha}(\bar{r}, 1)$ is 0 or 2, when $R(\eta) = 2$. Theorem 3 now follows. ∎

6. SIMILITUDE GROUPS

In the previous section, we saw that it was possible for the dimension $[\pi | H : 1]$ to exceed one. This defect of $H \backslash G$ significantly complicates its harmonic analysis. For this reason, and other technical reasons, it is often more convenient and more natural to replace H with the associated group of orthogonal similitudes. Let H' denote the group $GO(\eta)$ of orthogonal similitudes associated to η. (Thus H' consists of those $g \in G$ such that $g \eta {}^t g = zg$ for some $z = \lambda(g) \in Z$.) When χ is a character of $E^\times$, there is an associated character $\chi \circ \lambda$ of H' which, by abuse of notation, we also denote as χ when there is no risk of confusion. We now prove:

THEOREM 4. *For every irreducible, supercuspidal representation* $\pi = \mathrm{ind}_{ZK}^G(r)$ *and every character* χ *of* $E^\times$, *the dimension* $[\pi | H' : \chi]$ *of* $\mathrm{Hom}_{H'}(\pi, \chi)$ *is at most one.*

Proof. We follow the approach of the proof of Theorem 3 with some minor modifications. Let $\mathscr{F}'_g$ denote the space of $f \in \mathscr{F}$ with support in KgH' and $\mathscr{F}'_{g,c} = \mathscr{F}'_g \cap \mathscr{F}_c$. Then $\mathrm{Hom}_{H'}(\mathscr{F}'_c, \chi)$ decomposes as a direct sum of the spaces $\mathrm{Hom}_{H'}(\mathscr{F}'_{g,c}, \chi)$ as g varies over a set of representatives for $K \backslash G/H'$. We note that Lemma 7 still holds if we replace the trivial representation of J by a character of J. Taking $J = H'$ in Lemma 7, we get

$$\mathrm{Hom}_{H'}(\pi, \chi) \approx \bigoplus_{g \in K \backslash G/H'} \mathrm{Hom}_{ZK \cap gH'g^{-1}}(r, \chi_g),$$

where $\chi_g(k) = \chi(\lambda(g^{-1}kg))$. Since

$$\mathrm{Hom}_{ZK \cap gH'g^{-1}}(r, \chi_g) \subset \mathrm{Hom}_{\overline{O(g\eta{}^tg)}}(\bar{r}, 1), \qquad (**)$$

each summand has dimension at most one. We have a natural embedding of $ZK \backslash G/H$ in $K \backslash G/H'$. If the summand associated to a double coset in KgH' is nonzero, then this double coset must contain a double coset $ZKg_\alpha H$ with $\alpha \in \mathscr{A}_1$. Our assertion is clear except in the case in which $\mathscr{A}_1$ has two elements, say α' and α''. According to Section 3, α' and α'' may be chosen so that there exist $g_{\alpha'}, g_{\alpha''} \in G$ such that either: (a) $g_{\alpha'}\eta{}^tg_{\alpha'} = 1$ and $g_{\alpha''}\eta{}^tg_{\alpha''} = \varpi$, or (b) $g_{\alpha'}\eta{}^tg_{\alpha'} = \xi$ and $g_{\alpha''}\eta{}^tg_{\alpha''} = \varpi\xi$. In either case, $g_{\alpha'}^{-1}g_{\alpha''} \in H'$, and thus $Kg_{\alpha'}H' = Kg_{\alpha''}H'$. This shows that in the decomposition $(**)$ there is at most one nonzero summand. Our claim now follows. ▮

Using Lemma 3, a more precise statement follows easily:

THEOREM 5. *Given an irreducible supercuspidal representation* $\pi = \mathrm{ind}_{ZK}^{G}(r)$, *the central character* μ *is even* ($\mu(-1) = 1$) *if and only if there is a group of orthogonal similitudes* H' *and a character* χ *such that* $[\pi | H' : \chi] = 1$.

Proof. Assume $[\pi | H' : \chi] = 1$. Then we may choose a linear form $l \in \mathrm{Hom}_{H'}(\pi, \chi)$ and a vector v in the space of π such that $l(v)$ is nonzero. Then $\mu(-1)l(v) = l(\pi(-1)v) = \chi(\lambda(-1))l(v)$. As $\lambda(-1) = 1$, we get $\mu(-1) = 1$.

Now assume $\mu(-1) = 1$. Then $\bar{r}$ has central character $\bar{\mu}$ with $\bar{\mu}(-1) = 1$. By Theorem 2 and Lemma 3, for any group of orthogonal similitudes $\overline{H}'$ over a finite field, there is a character $\bar{\chi}$ with $[\bar{r} | \overline{H}' : \bar{\chi}] = 1$. Thus for any diagonal matrix $\eta \in K$, we can choose a character χ such that $\mathrm{Hom}_{ZK \cap GO(\eta)}(r, \chi)$ is nontrivial. In fact such a χ is determined by the following conditions: χ is trivial on $1 + \mathfrak{P}$ and the restriction of χ to the residue field agrees with $\bar{\chi}$.

From the identity before $(**)$ in the proof of Theorem 4, we have

$$\mathrm{Hom}_{GO(\eta)}(\pi, \chi) \supset \mathrm{Hom}_{ZK \cap GO(\eta)}(r, \chi).$$

Thus let $H' = GO(\eta)$ for the above η and we have $[\pi|H' : \chi] > 0$. Our claim now follows from Theorem 4. ∎

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