A BESSEL IDENTITY FOR THE THETA CORRESPONDENCE

EHUD MOSHE BARUCH, EREZ LAPID, AND ZHENGYU MAO

Abstract. We establish a spectral identity between global Bessel distributions with respect to generic cuspidal representations of an odd orthogonal group and the metaplectic cover of a symplectic group which are related by the theta correspondence. We also provide analogous local identities for square integrable representations.

Contents

1. Introduction

In a seminal paper [Shi73], Shimura obtained a famous correspondence $f \mapsto F$ between modular forms of half-integral weight and those with integral weight. A well-known result of Waldspurger [Wal81] relates the Fourier coefficients of f to central values of the twisted L-functions of F. This relation has seen many applications and it was explicated and extended by several authors [Koh85, KZ81, Shi93, KS93, Koj00, Koj99, KM96] (to mention a few). The most general formula in this context is given in [BM07].

Nowadays, the Shimura correspondence is often viewed representation theoretically in the framework of the *theta correspondence*. The basic idea, which is due to Howe [How79], is to relate two classical groups G, G' (or covers thereof) as a *dual reductive pair* inside a bigger metaplectic group and to use the Weil representation of the latter to obtain a correspondence between (a subset of) the representations of G and G'. (In Shimura's case the dual pair is $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$.) Over the years, the theta correspondence became one of the most useful and well developed tools in the study of representations of classical groups and their interrelations, both globally (automorphic representations) and locally. The properties of these correspondences are well studied in the literature.

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The result of [BM07] uses the *relative trace formula* which was introduced by Jacquet [Jac87a] to study Waldspurger's formula from a different point of view. A closely related approach is due to Iwaniec [Iwa87]. The trace formula approach can be formulated for other cases of dual reductive pairs. In this paper we focus on the dual pair (G, G') consisting of the split odd orthogonal group G and the metaplectic cover G' of the symplectic group, both of rank n. We consider this pair both in the p-adic and the number field case. The representation theory of the two groups are intimately related through the theta correspondence. Our main result is a spectral identity between representations which are related by the theta correspondence. We will only consider *generic* representations, and in the local case we restrict to square-integrable representations.

To describe the local result, let F be a p-adic field and let $\mathcal{S}(G(F))$ and $\mathcal{S}(G'(F))$ denote the space of compactly supported locally constant (and *genuine* in the case of G') functions on $G(F)$, $G'(F)$ respectively. Let π and π' be irreducible generic square integrable representations of $G(F)$ and $G'(F)$ respectively (with π' genuine) which correspond under the Howe duality. We define in §2.2 the *Bessel distributions* $\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}$ and $\mathcal{B}^{\pi'}_{\mathcal{W}',\mathcal{W}'}$ on $\mathcal{S}(G(F))$ and $\mathcal{S}(G'(F))$ respectively.

We say that f and f' *match* if their respective orbital integral are compatible. (see [MR04, Proposition 6.1] and §4.3).

We prove the following identity of Bessel distributions.

Theorem 1. *Under the above assumptions we have*

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f) = \mathcal{B}^{\pi'}_{\mathcal{W}',\mathcal{W}'}(f')$$

for all matching pairs $f \leftrightarrow f'$.

We expect that if π is generic and square-integrable then π' is automatically square-integrable. (An analogous result for *even* orthogonal groups was proved by Muic and Savin [MS00].)

In the global case, we obtain a similar identity of Bessel distributions with respect to cuspidal automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ which are related by the (global) theta correspondence. For a precise statement see Theorem 2. The result was announced in [Jac05]. It is a typical spectral identity in the context of the relative trace formula and a direct generalization of the case $n = 1$ considered by Jacquet. In contrast, our approach here does not use the relative trace formula. Instead we use the theta correspondence.

Our main tool are formulas of Furusawa [Fur95] relating the Fourier coefficients of theta lifts and Bessel coefficients. These formulas generalize the case $n = 1$ which was worked out by Waldspurger and they already play a major role in his work. The main new ingredient in deriving the spectral identity is to use the obvious, but crucial, fact that the adjoint of the theta lift is simply the theta lift in the converse direction. In the local case, we use the machinery of the Weil representation in order to derive an explicit Howe duality (in both directions) between π and π' realized in their corresponding Whittaker models. Once again, this idea goes back idea to Waldspurger [Wal91] who used it in the case $n = 1$. It was used by Jiang-Soudry [JS03] in the supercuspidal case (for general n). As in the global case the spectral identity reduces to an adjointness relation between the explicit

Howe duality maps in both directions. This approach is simpler than those used in [BM03] and [BM05], where more detailed analysis of the Bessel distribution was required.

Ultimately, we would like to obtain a precise formula as in [BM07] relating the Fourier coefficients of forms on the metaplectic group to the central value of L-functions of their theta lifts. This will require, in particular, extending our local results, and in fact, their formulations, to the non-square-integral case. We hope to get back to this problem in the future.

We remark that the geometric comparison of the relative trace formula at hand was carried out in [MR04], once again using the theta correspondence. The results here can be viewed as the spectral counterpart of [ibid.], although they are formally independent of each other.

Finally, we mention that there are additional closely related (and equally important) results of Waldspurger in the context of the theta correspondence ([Wal85]). They also admit a relative trace formula interpretation [Jac87b, Jac86], which in turn admit higher rank generalization. The geometric comparison of these trace formula as well as others resulting from theta correspondence was carried out in [MR05, MR04, MR99b, MR99a]. It is likely that our approach can be applied to obtain the spectral identities underlying these comparisons.

1.1. Notation and Preliminaries.

Until §5 F is a local non-archimedean field of characteristic zero and $\mathcal{O}$ is its ring of integers. We fix a non-trivial character ψ of F.

For a vector space W, we use $\langle v_1, \dots, v_m\rangle$ to denote the span of vectors $v_1, \dots, v_n$ in W. Denote by $M_{a,b}$ the space of $a \times b$ matrices.

Let $V = M_{2n+1,1}(F)$, with the standard basis $e_1, \dots, e_{2n+1}$, and the symmetric bilinear form $\langle\cdot,\cdot\rangle_V$ given by $\langle e_i, e_{2n+2-j}\rangle_V = \delta_{i,j}$. Then V has a splitting $V = V_+ \oplus V_- \oplus \langle e_{n+1}\rangle$, where $V_+ = \langle e_1, \dots, e_n\rangle$ and $V_- = \langle e_{-1}, \dots, e_{-n}\rangle$, (where we set $e_{-i} = e_{2n+2-i}$). We let $G = \mathrm{SO}(V)$ be the special orthogonal group of $(V, \langle\cdot,\cdot\rangle_V)$ (acting on the left). We denote by $\mathcal{S}(G)$ the space of locally constant compactly supported functions on G.

Let $V' = M_{1,2n}(F)$, with standard basis $f_1, \dots, f_{2n}$ and the anti-symmetric form $\langle\cdot,\cdot\rangle_{V'}$ given by $\langle f_i, f_{2n+1-j}\rangle_{V'} = \delta_{i,j}$ for $i \le n$. Then V' has a splitting $V' = V'_+ \oplus V'_-$, where $V'_+ = \langle f_1, \dots, f_n\rangle$ and $V'_- = \langle f_{-1}, \dots, f_{-n}\rangle$, (where we set $f_{-j} = f_{2n+1-j}$). Denote by $\mathrm{Sp}(V')$ the symplectic group of $(V', \langle\cdot,\cdot\rangle_{V'})$ acting on the right.

Let $G' = \mathrm{Mp}(V')$ be the metaplectic cover of $\mathrm{Sp}(V')$. (Cf. [Kud, Chapter I] and [MVW87] for basic facts and conventions about G'.) An element in G' has the form $(g, \pm 1)$ with $g \in \mathrm{Sp}(V')$. Multiplication in G' is given by $(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1 g_2, \check{c}(g_1, g_2)\varepsilon_1\varepsilon_2)$ where $\check{c}$ is the cocycle defined in [RR93]. We write $\widetilde{g'} = (g', 1)$ for any $g' \in \mathrm{Sp}(V')$. (Of course, $g' \mapsto \widetilde{g'}$ is not a homomorphism.) We denote by $\mathcal{S}(G')$ the space of locally constant compactly supported functions on G' which are *genuine*, i.e., such that $\phi(g, -1) = -\phi(g, 1)$ for all $g \in \mathrm{Sp}(V')$.

Denote by e and e' the identity elements of G and G' respectively.

Let P (resp. P') be the Siegel parabolic subgroup of G (resp. $\mathrm{Sp}(V')$) and let U (resp. U') be its unipotent radical. We identify the Levi subgroups M and M' of P and P' with GL_n

via

$$m(g)=\begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix},\ \ m'(g)=\begin{pmatrix} g & \\ & g^* \end{pmatrix}\quad g\in\mathrm{GL}_n\,.$$

where we define

$$g^*=w_n(g^t)^{-1}w_n\quad g\in\mathrm{GL}_n$$

where w_n is the matrix with ones on the non-principal diagonal and zeros elsewhere.

Denote by N, N' and N'' the maximal unipotent subgroups of $\mathrm{SO}(V)$, $\mathrm{Sp}(V')$ and GL_n respectively, consisting of upper triangular matrices with unit diagonal. Thus, $N=m(N'')\ltimes U$ and $N'=m'(N'')\ltimes U'$. Note that by the property of the cocycle, for all $n\in N'$ and $g'\in\mathrm{Sp}(V')$ we have $\widetilde{n'}\widetilde{g'}=\widetilde{n'g'}$ and $\widetilde{g'}\widetilde{n'}=\widetilde{g'n'}$. In particular $n'\mapsto\widetilde{n'}$ embeds N' in G'.

Denote by T'' the maximal tori of G'' consisting of diagonal matrices. Let $T=m(T'')$ and $T'=m'(T'')$ be the maximal tori of G and $\mathrm{Sp}(V')$ respectively. We will also consider T' as a subgroup of G' as the covering splits over T'. We fix good maximal compact subgroups K, K' and K'' of G, G' and GL_n respectively, so that the Iwasawa decomposition $G=NTK$ holds for G (and similarly for G', G'').

When $t\in T$, we write $te_i=t_ie_i$ for $i=1,\dots,n$. Similarly, when $t'\in T'$, we write $f_it'=t'_if_i$, $i=1,\dots,n$. We enumerate the simple roots of G by $\alpha_i(t)=t_i/t_{i+1}$, $i=1,\dots,n$ where $t_{n+1}=1$. Similarly, $\alpha'_i(t')=t'_i/t'_{i+1}$, $i=1,\dots,n$ where $t'_{n+1}=1$. We use $\delta(t)$ and $\delta'(t')$ to denote the modulus functions of the Borel subgroups of G and $\mathrm{Sp}(V')$ respectively. Thus $\delta(t)=\prod_{i=1}^n|t_i|^{2(n+\frac12-i)}$ and $\delta(t')=\prod_{i=1}^n|t'_i|^{2(n+1-i)}$.

Let Z be a symplectic space over F with a polarization $Z=Z_+\oplus Z_-$. We write a typical element of $\mathrm{Sp}(Z)$ as $\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ where $A\in\mathrm{Hom}(Z_+,Z_+)$, $B\in\mathrm{Hom}(Z_+,Z_-)$, $C\in\mathrm{Hom}(Z_-,Z_+)$ and $D\in\mathrm{Hom}(Z_-,Z_-)$. Consider the Weil representation ω_ψ of the group $\mathrm{Mp}(Z)$ (with respect to the Rao cocycle defined by the splitting). It can be realized on $\mathcal{S}(Z_+)$ as follows. (Cf. [Kud, Chapter I].) For $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$, let

$$r(g)\phi(z)=\int_{Z_-/\ker C}\psi(\frac12\langle zA,zB\rangle+\frac12\langle z'C,z'D\rangle-\langle zB,z'C\rangle)\phi(zA+z'C)\ d_gz' \tag{1}$$

where the measure d_g on $Z_-/\ker C$ is uniquely determined by the property that $r(g)$ is unitary on $L^2(Z_+)$. Then we take $\omega_\psi(g)\phi(z)=\beta_\psi(g)[r(g)\phi](z)$ where $\beta_\psi(g)$ is a certain root of unity defined in [Kud, Theorem 4.5]. [1]

In our case we take

$$Z=\mathrm{Hom}(V,V')\simeq V^*\otimes V'\simeq V\otimes V'\simeq M_{2n+1,2n} \tag{2}$$

where we identify V^* with V through $\langle\cdot,\cdot\rangle_V$. On the level of matrices $G\times\mathrm{Sp}(V')$ acts on Z by

$$z(g,h)=g^{-1}zh\quad g\in G,h\in\mathrm{Sp}(V'),z\in Z.$$

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[1] The notation in [Kud] suggests that β depends on a choice of a standard basis. However, in fact, it only depends on the splitting.

We define the Weil representation ω_ψ using the polarization $Z_\pm = V \otimes V'_\pm$. Later on we will denote $Z_\pm$ by $Z^1_\pm$ to distinguish it from another polarization that will be used. We denote the ensuing model of the Weil representation on $\mathcal{S}(Z_+)$ by ω^1_ψ.

We endow the additive group of F with the self-dual Haar measure with respect to ψ. We will use the following convention for the normalization of Haar measures on unipotent groups. For each root α we choose a splitting of U_α in the following way. The one dimensional unipotent subgroup U_α corresponds to a unique pair (i,j) with $1 \le j \le n$ and $i \le 0$ or $i > j$, such that U_α consists of matrices $u(x)$ with $\langle e_i, u(x)e_{-j}\rangle = x$. The map $u(x) \mapsto x$ identifies U_α with F, and we take the Haar measure on U_α accordingly. If U is a T-stable subgroup of N then it directly spanned by the U_α's contained in it (in any order) [Hum75, §28.1]. We take the product measure as our choice of Haar measure on U. Similarly we fix a choice of Haar measures for subgroups of N'. In particular, for a one dimensional unipotent subgroup U'_α of N', it corresponds to a unique pair (i,j) with $1 \le j \le n$ and $i < 0$ or $i \ge j$, such that U'_α consists of matrices $u'(x)$ with $\langle f_i, f_j u'(x)\rangle = x$. Define $du'(x) = dx$ on U'_α. We will also fix arbitrary Haar measures on G and G'.

2. Bessel distributions

2.1. **Whittaker model and inner product.** Define non-degenerate characters χ on N and χ' on N' by

$$\chi(u) = \psi(u_{1,2} + u_{2,3} + \ldots + u_{n,n+1}), \ \ u \in N;$$

$$\chi'(u') = \psi^{-1}(u'_{1,2} + u'_{2,3} + \ldots + u'_{n-1,n} + \frac{u'_{n,n+1}}{2}), \ \ u' \in N'.$$

Let $\Omega(G)$ be the G-space of smooth functions $W : G \to \mathbb{C}$ such that $W(ug) = \chi(u)W(g)$ for all $u \in N$, $g \in G$, with G acting by right translation. Suppose that π is an irreducible generic representation of G. That is, π can be realized as a subspace $\mathcal{W}(\pi)$ of $\Omega(G)$. The space $\mathcal{W}(\pi)$ is uniquely determined by the equivalence class of π and is called the *Whittaker model* of π.

Similarly if π' is an irreducible generic representation of G', it can be realized as a subspace $\mathcal{W}(\pi')$ of the G'-space $\Omega(G')$, where $\Omega(G')$ consists of smooth genuine functions $W' : G' \to \mathbb{C}$ such that $W'(\widetilde{u'}g') = \chi'(u')W'(g')$ for all $u' \in N'$, $g' \in G'$. Once again, the space $\mathcal{W}(\pi')$ is a uniquely determined by the equivalent class of π' [Szp07].

From now on assume that π and π' are square integrable. Then from [LM09, Theorem 3.1, Lemma 2.6, Corollary 3.4], cf. also [CS80, §6], we get

Lemma 1. *There exist $\lambda > 0$ such that for any $W \in \mathcal{W}(\pi)$ there exists $\phi \in \mathcal{S}(F^n)$ such that*

$$|W(t)| \le \delta(t)^{\frac{1}{2}}\phi(\alpha_1(t), \ldots, \alpha_n(t))\prod_{i=1}^{n}|\alpha_i(t)|^\lambda, \ \ t \in T. \tag{3}$$

The form

$$(W_1, W_2)_{\mathcal{W}(\pi)} := \int_{N\backslash G} W_1(g)\overline{W_2(g)}\ dg \tag{4}$$

is absolutely convergence and defines a G-invariant inner product on $\mathcal{W}(\pi)$.

Similarly, there exist $\mu > 0$ such that for any $W' \in \mathcal{W}(\pi')$ there exists $\phi' \in \mathcal{S}(F^n)$ such that

$$|W'(\widetilde{t'})| \le \delta'(t')\phi'(\alpha'_1(t'), \ldots, \alpha'_n(t')) \prod_{i=1}^{n} |\alpha'_i(t')|^{\mu} \quad t' \in T'. \tag{5}$$

The form

$$(W'_1, W'_2)_{\mathcal{W}(\pi')} = \int_{N'\backslash G'} W'_1(g')\overline{W'_2(g')}\, dg' \tag{6}$$

defines a G'-invariant inner product on $\mathcal{W}(\pi')$.

Proof. The inner product formulas were established in [LM09]. The inequalities were not explicitly given, so we give a derivation here. We consider the inequality (3) of $W(t)$, the other inequality for $W'(\widetilde{t'})$ is established similarly.

Use the notations in [LM09]. Let B be the Borel subgroup of G containing N. From [LM09, Theorem 3.1], $W(t)$ is a finite combination of functions of the form $\delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t)$ where P is a parabolic subgroup of G containing B, χ is a character of T so that the χ-generalized eigenspace of the Jacquet module $J_P(\pi)$ of π with respect to P contains a supercuspidal component, $\phi_{P,\chi} \in \mathfrak{F}_{P,\chi}$. We only need to establish the bound of (3) for any such function $\delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t)$, as it is clear the sum of two functions of t of the form (3) is again bounded by a function of the form (3).

From [LM09, (2.2) and Lemma 2.6(3)] $\delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t) = \delta_B^{\frac{1}{2}}(t)\phi'_{B,\chi}(t)$ for a function $\phi'_{B,\chi} \in \mathfrak{F}_{B,\chi}$. From [LM09, Lemma 2.6(2)], $\phi'_{B,\chi}$ is bounded by $q^{-\langle \operatorname{Re}\chi, H(t)\rangle} \prod_{i=1}^{n} \phi'_i(\alpha_i(t))$ with ϕ'_i being Schwartz functions. From the square integrability of π we have $q^{-\langle \operatorname{Re}\chi, H(t)\rangle} \le \prod_{i=1}^{n} |\alpha_i(t)|^{\lambda'}$ with $\lambda' > 0$ ([Cas]). Thus we get the bound (3) for $\delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t)$. □

2.2. **Local Bessel period.** We define the *Bessel functional* of π on the space $\mathcal{W}(\pi)$. The Bessel subgroup is by definition

$$R = \{g \in G : ge_2 = e_2,\ ge_3 = e_3 \pmod{\langle e_2\rangle}, \ldots, ge_{n+1} = e_{n+1} \pmod{\langle e_2, \ldots, e_n\rangle}\}.$$

Note that $R = (R \cap N)H$ where H is the subgroup of G which fixes $e_2, \ldots, e_{n+1}$ and fixes $e_{-2}, \ldots, e_{-n}$ modulo $\langle e_{-1}\rangle$. Explicitly

$$H = \{m(A_{\xi,\eta}) : \xi \in F^{n-1}, t \in F^*\} \text{ where } A_{\xi,\eta} = \begin{pmatrix} \eta & & & \\ \xi_1 & 1 & & \\ \vdots & & \ddots & \\ \xi_{n-1} & & & 1 \end{pmatrix}.$$

Define a character χ_R on R by

$$\chi_R(r) = \psi(r_{2,3} + \cdots + r_{n,n+1}),\ \ r \in R. \tag{7}$$

Set

$$\mathcal{P}(W)=\int_{R\cap N\backslash R} W(r)\chi_R(r)^{-1}\,dr=\int_H W(h)\,dh \tag{8}$$

where dr is a Haar measure on $R\cap N\backslash R$ and dh is a suitable Haar measure on H.

Lemma 2. *The integral in* (8) *is absolutely convergent and defines an* (R,χ_R)*-equivariant functional on* π*.*

Proof. The second part of the Lemma is clear. To show convergence we write the integral as

$$\int_{F^*}\int_{F^{n-1}} W(m(A_{\xi,\eta}))\,d\xi\,|\eta|^{1-n}\,d^*\eta \tag{9}$$

Let $A_{\xi,\eta}=ntk$ be the Iwasawa decomposition with $n\in N''$, $k\in K''$ and $t=t_{\xi,\eta}=\operatorname{diag}(t_1,\dots,t_n)$. Then

$$\prod_{j=k+1}^{n}|t_j|=\max(1,|\xi_k|,\dots,|\xi_{n-1}|)\quad k=1,\dots,n-1 \text{ and } \prod_{j=1}^{n}|t_j|=|\eta|. \tag{10}$$

Note that $\delta(t)=|t_1|^{n-\frac12}\prod_{i=2}^{n}|\alpha_i(t)|^{m_i}$ with $m_i\geq 0$. On the other hand,

$$|t_1|=\frac{|\prod_{i=1}^{n}t_i|}{|\prod_{i=2}^{n}t_i|}=\frac{|\eta|}{\max(1,|\xi_1|,\dots,|\xi_{n-1}|)}\leq|\eta|.$$

Thus, by (3), the integrand in (9) is bounded by

$$\phi(\alpha_1(t),\dots,\alpha_n(t))|\eta|^{\frac12}$$

for some Schwartz function $\phi\in\mathcal{S}(F^n)$. From the support condition of ϕ we can impose upper bounds on $|\alpha_i(t)|$, $i=1,\dots,n$. Hence we get upper bounds on $|t_i|$, $i=1,\dots,n$, and therefore on $|\xi_i|$, $i=1,\dots,n-1$ and $|\eta|$ as well. Thus the integral (9) is majorized by the convergent integral

$$\int_{|\xi_i|<C_i}\int_{|\eta|<C_n} C|\eta|^{\frac12}\,d\xi\,d^*\eta$$

for some positive constants $C,C_1,\dots,C_n$. □

2.3. **Definition of local Bessel distribution.** Fix a sequence K_n of congruence subgroups in K. We say that an orthonormal basis $\mathfrak{B}$ of π is *admissible* if $\mathfrak{B}\cap\pi^{K_n}$ spans π^{K_n} for any n. Similarly for π'.

We define the local relative Bessel distribution of π by

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f)=\sum_{W_i}\mathcal{P}(\pi(f)W_i)\overline{W_i(e)}\quad f\in\mathcal{S}(G)$$

where the sum is over an admissible orthonormal basis of $\mathcal{W}(\pi)$. By the usual argument this does not depend on a choice of the orthonormal basis. (Cf. [Jac]).

Note that $\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}$ does not depend on the choice of the Haar measure on G since rescaling it affects both $\pi(f)$ and the inner product formula (4) in the same way.

Similarly we define the local Bessel distribution of π' by

$$\mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(f') = \sum_{W'_i} \pi'(f')W'_i(e')\overline{W'_i(e')} \quad f' \in \mathcal{S}(G')$$

where $\{W'_i\}$ is an admissible orthonormal basis of $\mathcal{W}(\pi')$.

As before, $\mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(f')$ does not depend on the choice of Haar measure on G'. (It only depends on the measure on N' chosen above.)

3. Local theta correspondence

We now wish to realize the local Howe duality explicitly in terms of an integral transform on the corresponding Whittaker models. This was done in [JS03, §2] in case where π is supercuspidal, and is modeled on the global computations of Furusawa ([Fur95]). We need to extend this to the square integrable case. The only issue is convergence.

Recall that G and G' comprise a dual pair inside $\mathcal{M} = \mathrm{Mp}(Z)$. Let ω_ψ be the Weil representation of $\mathcal{M}$. The *Howe duality conjecture* associates to an irreducible admissible representation π' of G' a unique irreducible admissible representation π of G (or 0). It is known to hold for odd residual characteristic [Wal90]. Even without assuming the Howe duality conjecture, we can proceed as in [JS03], and say that irreducible representations π of G and π' of G' *correspond under the Howe duality* if the space

$$\mathrm{Hom}_{G\times G'}(\omega_{\psi^{-1}}, \pi \otimes \pi') \tag{11}$$

is nontrivial. For such representations π and π', we construct two explicit maps in (11), or more precisely, in the isomorphic space

$$\mathrm{Hom}_{G\times G'}(\omega_\psi, \overline{\mathcal{W}(\pi)} \otimes \overline{\mathcal{W}(\pi')}). \tag{12}$$

3.1. **Explicit theta correspondence I.** Recall the realization $(\omega^1_\psi, \mathcal{S}(Z^1_+))$ of the Weil representation according to the G-invariant splitting $Z^1_\pm = \mathrm{Hom}(V, V'_\pm)$. We sometimes identify Z^1_+ with $M_{2n+1,n}$. The explicit action of ω^1_ψ is described by formulas [MR04, (3.1)–(3.3)]. In particular:

$$\omega^1_\psi(g, \widetilde{m'(h)})\Phi(X) = |\det(h)|^{n+\frac{1}{2}} \frac{\gamma_\psi(1)}{\gamma_\psi((\det h)^{2n+1})} \Phi(g^{-1}Xh), \tag{13}$$

$$\omega^1_\psi(e, \widetilde{\left(\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right)})\Phi(X) = \Phi(X)\psi(\mathrm{Tr}({}^tX w_{2n+1} X B w_n)/2). \tag{14}$$

Here $\gamma_\psi(a)$ is a certain root of unity (Weil's constant).

Let $E_1 \in Z^1_+$ be given by $E_1 e_i = E_1 e_{-1} = 0$, and $E e_{-i-1} = f_i$, $i = 1, \dots, n$. Thus, E_1 corresponds to $\sum_{i=1}^n e_{i+1} \otimes f_i$ under the isomorphism $Z \simeq V \otimes V'$ (cf. (2)). Define a map in

$$\mathrm{Hom}_{G\times G'}(\omega_\psi, \mathrm{Hom}(\mathcal{W}(\pi), \overline{\mathcal{W}(\pi')}))$$

(realized by $\Phi \mapsto (W \mapsto \overline{\theta_\Phi(W)}))$ by setting for $\Phi \in \mathcal{S}(Z^1_+)$:

$$\overline{(\theta_\Phi W)(g')} = \int_{R_1\backslash G} \omega^1_\psi(g, g')\Phi(E_1)W(g)\, dg \tag{15}$$

where R_1 is the stabilizer of $e_1, \dots, e_{n+1}$. It is the stabilizer of e_{n+1} in U, i.e. the derived group of U. Note that R_1 is T-stable and is endowed with a measure as in §1.1.

The following is an extension of Corollary 2.2 of [JS03] from supercuspidal representation case to square integrable representation case.

Proposition 1. *The integral* (15) *converges absolutely and defines a nonzero map in* $\mathrm{Hom}_{G\times G'}(\omega_\psi^1, \mathrm{Hom}(\mathcal{W}(\pi), \overline{\mathcal{W}(\pi')}))$.

Proof. For the convergence, we may assume, upon replacing Φ by $\omega_\psi^1(e, g')\Phi$ that $g' = e'$. Using Iwasawa decomposition, we have to show the convergence of

$$\int_{\mathrm{GL}_n}\int_{R_1\backslash U} |\omega_\psi^1(vm(h), e')\Phi_k(E_1)W(m(h)k)||\det h|^{-n}\, dv\, dh$$

for any $k \in K$ where $\Phi_k = \omega_\psi(k, e')\Phi$. Without loss of generality we can assume that $k = e$. From (13), the above integral can be written as

$$\int_{\mathrm{GL}_n}\int_{M_{n,1}(F)} |\Phi(h^{-1}e_2, \dots, h^{-1}e_n, e_{n+1} + h^{-1}v)W(m(h))||\det h|^{-n}\, dv\, dh.$$

By changing the variable we can write this as

$$\int_{\mathrm{GL}_n}\int_{M_{n,1}(F)} |\Phi(h^{-1}e_2, \dots, h^{-1}e_n, e_{n+1} + v)W(m(h))||\det h|^{1-n}\, dv\, dh.$$

Using the Iwasawa decomposition again, now for GL_n, we write $h = bk''$ where b is in the Borel subgroup of GL_n and $k'' \in K''$. Let t be the diagonal part of b and let $x_{i,j}$, $j > i$ be the upper triangular entries of b^{-1}. The Haar measure is given by

$$\prod_{i=1}^n |t_i|^{i-1}\, dt \ \otimes_{1\le i<j\le n} dx_{i,j}\ dk''.$$

Again, we may suppress k'', so it is enough to show the absolute convergence of the integral over b. From (3), the latter integral is majorized by

$$\iiiint \phi(t_2^{-1}, \dots, t_n^{-1}, v, x_{i,j}, \alpha_1(t), \dots, \alpha_n(t)) \prod_{i=1}^n |t_i|^{\frac{1}{2}} \prod_{i=1}^n |\alpha_i(t)|^{\lambda}\, dv \otimes_{1\le i<j\le n} dx_{i,j}\ dt$$

where ϕ is a Schwartz function in $n - 1 + n + \binom{n}{2} + n$ variables and $\lambda > 0$. Note that the integrand is compactly supported in the variables $t_2, \dots, t_n \in F^*$. Therefore, the integral is majorized by the convergent integral

$$\int \phi'(t_1)|t_1|^{\lambda+\frac{1}{2}}\, d^*t_1$$

for an appropriate $\phi \in \mathcal{S}(F^*)$.

To show that θ is non-zero we can assume that $W(e) \neq 0$ and let Φ be a non-negative function supported in a small neighborhood Ξ of E_1 such that $\Phi(E_1) = 1$. Then the set $\{g \in R_1\backslash G : \omega_\psi^1(g, e')\Phi(E_1) \neq 0\}$ consists of g with $g^{-1}E_1 \in \Xi$, which is an arbitrarily small neighborhood of R_1. Therefore, $\theta_\Phi W(e') \neq 0$.

To show that $\overline{(\theta_\Phi W)} \in \overline{\Omega(G')}$, we establish:

$$\overline{(\theta_\Phi W)(\widetilde{n'g'})} = \chi'^{-1}(n')\overline{(\theta_\Phi W)(g')},\ n' \in N'. \tag{16}$$

Case 1: $n' = m'(u')$ with $u' \in N''$ the maximal unipotent of GL_n, then from (13):

$$\begin{aligned}\overline{(\theta_\Phi W)(\widetilde{m'(u')}g')} &= \int_{R_1\backslash G} \omega^1_\psi(g,g')\Phi(E_1u')W(g)\ dg\\ &= \int_{R_1\backslash G} \omega^1_\psi(n_u^{-1}g,g')\Phi(E_1)W(g)\ dg\end{aligned}$$

where n_u is any element in N such that its $(1,j)$-th entries are 0 for $j>1$, and $(i+1,j+1)$-th entry is $u_{i,j}$ for $1 \le i < j \le n$. Since $W(n_ug) = \chi'^{-1}(m(u))W(g)$, a change of variable $g \mapsto n_ug$ gives (16).

Case 2: $n' \in U'$. Then from (14), $\omega_\psi(g,\widetilde{n'}g')\Phi(E_1) = \psi(n_{n,n+1}/2)\omega_\psi(g,g')\Phi(E_1)$, while $\chi'(n') = \psi(-n_{n,n+1}/2)$. We again get (16).

Combining the two cases, we get (16) and $\overline{(\theta_\Phi W)}$ lies in $\overline{\Omega(G')}$. □

Let R_0 be the stabilizer of $e_2,\dots,e_{n+1}$ and we recall that R is the Bessel subgroup. Note that R_0, R_1 and R are unimodular, and $R_1\backslash R_0 \cong R\cap N\backslash R$. By (8) we get

$$\mathcal{P}(W) = \int_{R_1\backslash R_0} W(r)\ dr \tag{17}$$

for an appropriate Haar measure on $R_1\backslash R_0$.

Lemma 3. *We have*

$$\overline{(\theta_\Phi W)(e')} = \mathcal{P}(\pi(f)W) \tag{18}$$

whenever f satisfies

$$\int_R f(rg)\chi_R(r)\ dr = \int_{R_0\backslash R} \omega^1_\psi(rg,e')\Phi(E_1)\chi_R(r)\ dr. \tag{19}$$

Proof. From (17),

$$\begin{aligned}\overline{(\theta_\Phi W)(e')} &= \int_{R_0\backslash G} \omega^1_\psi(g,e')\Phi(E_1)\mathcal{P}(\pi(g)W)\ dg\\ &= \int_{R\backslash G}\int_{R_0\backslash R} \omega^1_\psi(rg,e')\Phi(E_1)\chi_R(r)\mathcal{P}(\pi(g)W)\ dr\ dg\\ &= \int_{R\backslash G}\int_R f(rg)\mathcal{P}(\pi(rg)W)\ dr\ dg.\end{aligned}$$

□

3.2. **Explicit theta correspondence II.** To go in the other direction we work with the mixed model realization of the Weil representation as in [MVW87]. Namely, decompose

$$Z_+^1 = \big((V_- \oplus \langle e_{n+1}\rangle) \otimes V'_+\big) \oplus \big(V_+ \otimes V'_+\big)$$

and let $Z_+^2 = V_- \otimes V' \oplus e_{n+1} \otimes V'_+$. Define $T : \mathcal{S}(Z_+^1) \to \mathcal{S}(Z_+^2)$ to be the partial Fourier transform with respect to $V_+ \otimes V'_+$ given by

$$T(\Phi)(X,Y) = \int_{V_+\otimes V'_+} \Phi(X,W)\psi(\langle Y,W\rangle)\, dW$$

where $X \in (V_- \oplus \langle e_{n_1}\rangle) \otimes V'_+$, $Y \in V_- \otimes V'_-$. Here we identify $V_+ \otimes V'_+$ with F^{n^2} through the basis $e_i \otimes f_j$ and endow it with the product measure.

Define the realization ω_ψ^2 on $\mathcal{S}(Z_2^+)$ by

$$\omega_\psi^2(g)T(\Phi) = T(\omega_\psi^1(g)\Phi), \quad \Phi \in \mathcal{S}(Z_1^+).$$

Explicitly, identifying Z_+^2 with $M_{n,2n} \oplus M_{n,1}$ and letting $\Phi' = \Phi_1 \otimes \Phi_2 \in \mathcal{S}(Z_+^2)$ where $\Phi_1 \in \mathcal{S}(M_{n,2n}(F))$ and $\Phi_2 \in \mathcal{S}(M_{n,1}(F))$, we have ([MR04, (3.13)–(3.16)])

$$\omega_\psi^2(m(u),g')[\Phi_1 \otimes \Phi_2](X,x) = [\Phi_1 \otimes \omega_\psi(g')\Phi_2]((u^*)^{-1}Xg',x), \tag{20}$$

$$\omega_\psi^2(u',e')[\Phi_1 \otimes \Phi_2](E_2) = \psi(-u'_{n,n+1})[\Phi_1 \otimes \Phi_2](E_2), \tag{21}$$

where $u \in N''$ and $u' \in U$ and $E_2 = e_{n+1}\otimes f_n - \sum_{i=1}^n e_{-i}\otimes f_{-i}$ (identified with $[(0_n,-1_n),f_n]$).

Since π' is unitary, $\overline{\mathcal{W}(\pi')}$ is the contragredient of $\mathcal{W}(\pi')$. Thus the space (12) is isomorphic to $\operatorname{Hom}_{G\times G'}(\omega_\psi^2, \operatorname{Hom}(\mathcal{W}(\pi'), \overline{\mathcal{W}(\pi)}))$. For $\Phi' \in \mathcal{S}(Z_+^2)$ define a map $\Phi' \mapsto (W' \mapsto \overline{\theta'_{\Phi'}(W')})$ by setting:

$$\overline{(\theta'_{\Phi'}W')(g)} = \int_{U'\backslash G'} \omega_\psi^2(g,g')\Phi'(E_2)W'(g')\, dg'. \tag{22}$$

Recall that the measure on U' is fixed in §1.1. We remark that

$$\omega_\psi^2(g,\widetilde{u'}g')\Phi'(E_2) = \chi'(u')^{-1}\omega_\psi^2(g,g')\Phi'(E_2), \ \ u \in U', \tag{23}$$

so that the integrand is left U'-invariant. The following is a restatement of part of [JS03, Corollary 2.1]. It holds for any generic π' without restriction.

Proposition 2. *The integral in* (22) *is absolutely convergent. It defines a nonzero map in* $\operatorname{Hom}_{G\times G'}(\omega_\psi^2, \operatorname{Hom}(\mathcal{W}(\pi'), \overline{\mathcal{W}(\pi)}))$.

Proof. For completeness we provide a proof for some main facts required for the Proposition. We first consider the convergence. By replacing Φ' with $\omega_\psi^2(g,e')\Phi'$, we can assume that $g = e$. By Iwasawa decomposition, to show absolute convergence, it suffices to show that the integrand is a Schwartz function in $m' \in M'$. From (20)

$$\omega_\psi^2(e,\widetilde{m'})\Phi'(E_2) = \gamma|\det(x)|^{1/2}\Phi'((0_n,-x^*),*) \text{ for } m' = m'(x) \in M',$$

where γ is a root of unity and we don't specify the second argument of Φ'. It follows directly from (5) that $\omega_\psi^2(e,\widetilde{m'})\Phi'(E_2)\cdot W'(\widetilde{m'})$ is a Schwartz function in m'.

In order to show that the map θ' is non-zero, we argue as before. We can assume that $W'(e') \neq 0$ and let Φ' be a non-negative function supported in a small neighborhood of E_2 such that $\Phi'(E_2) = 1$. Then it is easy to see that $\{g' \in U'\backslash G' : \omega_\psi^2(e, g')\Phi'(E_2) \neq 0\}$ is an arbitrarily small neighborhood of U'. Therefore, $\theta'_{\Phi'}\overline{W'(e)} \neq 0$.

Finally we show the integral defines a function in $\overline{\Omega(G)}$. We claim for $n \in N$,

$$\overline{(\theta'_{\Phi'}W')(ng)} = \chi(n)^{-1}\overline{(\theta'_{\Phi'}W')(g)}. \tag{24}$$

Case 1: $n = m(u)$ with $u \in N''$. Then by (20)

$$\begin{aligned}\overline{(\theta'_{\Phi'}W')(ng)} &= \int_{U'\backslash G'} \omega_\psi^2(m(u)g, g')\Phi'(E_2)W'(g')\ dg' \\ &= \int_{U'\backslash G'} \omega_\psi^2(g, \widetilde{m'(u^{-1})}g')\Phi'(E_2)W'(g')\ dg' \\ &= \int_{U'\backslash G'} \omega_\psi^2(g, g')\Phi'(E_2)W'(\widetilde{m'(u)}g')\ dg' \\ &= \chi'(m(u)) \int_{U'\backslash G'} \omega_\psi^2(g, g')\Phi'(E_2)W'(g')\ dg'.\end{aligned}$$

Since $\chi'(m(u)) = \chi(n)^{-1}$ in this case, we get the claim.

Case 2: $n \in U$. Then $\chi(n) = \psi(n_{n,n+1})$. From (21),

$$\omega_\psi^2(ng, g')\Phi'(E_2) = \psi(-n_{n,n+1})\omega_\psi^2(g, g')\Phi'(E_2),$$

we get the claim again.

Combining cases 1 and 2, we have shown the condition (24). Thus the integral defines a function in $\overline{\Omega(G)}$. ☐

Lemma 4. *We have*

$$\overline{(\theta'_{\Phi'}W')(e)} = (\pi'(f')W')(e') \tag{25}$$

whenever f' *satisfies*

$$\int_{N'} f'(\widetilde{n'}g')\chi'(n')\ dn' = \int_{U'\backslash N'} \omega_\psi^2(e, \widetilde{n'}g')\Phi'(E_2)\chi'(n')\ dn'. \tag{26}$$

Proof. The integral on the right-hand side of (26) makes sense by (23). We have

$$\begin{aligned}\overline{(\theta'_{\Phi'}W')(e)} &= \int_{U'\backslash G'} \omega_\psi^2(e, g')\Phi'(E_2)W'(g')\ dg' \\ &= \int_{N'\backslash G'}\int_{U'\backslash N'} \omega_\psi^2(e, \widetilde{n'}g')\Phi'(E_2)\chi'(n')W'(g')\ dn'\ dg' \\ &= \int_{N'\backslash G'}\int_{N'} f'(\widetilde{n'}g')W'(\widetilde{n'}g')\ dn'\ dg'\end{aligned}$$

as required. ☐

4. Proof of Theorem 1

4.1. **A relation between ω_ψ^1 and ω_ψ^2.** We have chosen two models $(\omega_\psi^i, \mathcal{S}(Z_+^i))$, $i=1,2$ for the Weil representation and an intertwining operator $\Phi \mapsto T(\Phi)$ between them.

Proposition 3. *When* $\Phi' = T(\Phi)$,

$$\int_{U'\backslash N'} \omega_\psi^2(g, \widetilde{n'}g')\Phi'(E_2)\chi'(n')\ dn' = \int_{R_1\backslash N} \omega_\psi^1(ng, g')\Phi(E_1)\chi(n)\ dn. \tag{27}$$

Proof. The elements in Z_+^2 will be denoted by (A,B) where A is a $n\times 2n$ matrix and B is a vector in V'_+. The elements in Z_+^1 will be denoted by $(2n+1)\times n$ matrices.

Clearly we only need to establish the identity when g and g' are the identity elements. From (20), the left-hand side of (27) is

$$\int_{N''} T(\Phi)((0,-u^*), f_n)\chi''(u)^{-1}\ du$$

where

$$\chi''(u) = \psi(u_{1,2}+\ldots+u_{n-1,n}).$$

If $f_{-i}u^* = f_{-i} + \sum_{j=1}^{i-1} a_{i,j} f_{-j}$, then explicitly the above integral equals:

$$\int_{F^{n(n-1)/2}} T(\Phi)(E_2 - \sum_{j=1}^{n}\sum_{i=j+1}^{n} a_{i,j} e_{-i}\otimes f_{-j})\,\psi(\sum_{i=1}^{n-1} a_{i+1,i})\ \otimes_{1\le j<i\le n} da_{i,j},$$

where as always $da_{i,j}$ is the self dual measure on F. Applying the Fourier inversion formula, this becomes

$$\int_{F^{n(n+1)/2}} \Phi(\sum_{i=1}^{n} e_{i+1}\otimes f_i + \sum_{i=1}^{n}\sum_{j=i}^{n} b_{i,j} e_i\otimes f_j)\,\psi(-\sum_{i=1}^{n} b_{i,i})\ \otimes_{1\le i\le j\le n} db_{i,j}.$$

Let $u\in N$ such that the $(i,j+1)-$th entry of u^{-1} is $b_{i,j}$ for $i=1,\ldots,n$ and $j=i,\ldots,n$, then clearly the above integrand is just $\Phi(u^{-1}E_1)\chi(u)$. We get the integral equals

$$\int_{R_1\backslash N} \omega_\psi^1(u, e')\Phi(E_1)\chi(u)\ du.$$

This proves the proposition. □

Remark 1. The proposition and its proof carry over verbatim to the archimedean case.

4.2. **Adjointness property.** If $\Phi' = T(\Phi)$ as before, then the images of Φ' and Φ in $\overline{\mathcal{W}(\pi)} \otimes \overline{\mathcal{W}(\pi')}$ should agree up to a scalar multiple. Namely for any $W \in \mathcal{W}(\pi)$ and $W' \in \mathcal{W}(\pi')$, we have $(W', \theta_\Phi W)_{\mathcal{W}(\pi')} = c(W, \theta_{\Phi'} W')_{\mathcal{W}(\pi)}$ for some constant c. We can be more precise about this statement:

Proposition 4. *Suppose that* Φ *and* Φ' *are related by* (27). *Then* $(W', \theta_\Phi W)_{\mathcal{W}(\pi')} = (W, \theta_{\Phi'} W')_{\mathcal{W}(\pi)}$ *for all* $W\in\mathcal{W}(\pi)$ *and* $W'\in\mathcal{W}(\pi')$.

Proof. We have

$$\begin{aligned}(W',\theta_\Phi W)_{\mathcal{W}(\pi')} &= \int_{N'\backslash G'}\int_{R_1\backslash G}\omega_\psi^1(g,g')\Phi(E_1)W(g)\ dg\ W'(g')\ dg' \\ &= \int_{N'\backslash G'}\int_{N\backslash G}\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)W(ng)\ dg\ W'(g')\ dg'.\end{aligned}$$

On the other hand

$$(W,\theta'_{\Phi'}W')_{\mathcal{W}(\pi)} = \int_{N\backslash G}W(g)\int_{U'\backslash G'}\omega_\psi^2(g,g')\Phi'(E_2)W'(g')\ dg'\ dg,$$

which is

$$\int_{N\backslash G}W(g)\int_{N'\backslash G'}\int_{U'\backslash N'}\omega_\psi^2(g,\widetilde{n'}g')\Phi'(E_2)\chi'(n')\ dn'\ W'(g')\ dg'\ dg.$$

Applying the relation (27), the integral becomes

$$\int_{N\backslash G}W(g)\int_{N'\backslash G'}\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)\chi(n)\ dn\ W'(g')\ dg'\ dg.$$

By Fubini's Theorem, it remains to show the convergence of

$$\int_{N'\backslash G'}\int_{N\backslash G}|\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)\chi(n)\ dn||W(g)W'(g')|\ dg\ dg'. \tag{28}$$

As before, using Iwasawa decomposition, this amounts to the convergence of

$$\int_{T'}\int_{T}|\int_{R_1\backslash N}\omega_\psi^1(nt,\widetilde{t'})\Phi(E_1)\chi(n)\ dn||W(t)W'(\widetilde{t'})|\delta(t)^{-1}\delta'(t')^{-1}\ dt\ dt'. \tag{29}$$

The inner integration can be substituted by integrating over the maximal unipotent of GL_{n+1}. We have

$$\omega_\psi^1(ut,\widetilde{t'})\Phi(E_1) = \gamma|\det(t')|^{n+\frac{1}{2}}\Phi(\sum_{i=1}^n t_i^{-1}u^{-1}e_{i+1}\otimes t'_i f_i)$$

where γ is a root of unity which depends on t, t' but not on u.

Thus, the argument of Φ is the matrix

$$\begin{pmatrix} t_1^{-1}x_{1,2}t'_1 & t_1^{-1}x_{2,3}t'_2 & \dots & t_1^{-1}x_{1,n+1}t'_n \\ t_2^{-1}t'_1 & t_2^{-1}x_{2,3}t'_2 & \dots & t_2^{-1}x_{2,n+1}t'_n \\ & t_3^{-1}t'_2 & \dots & t_3^{-1}x_{3,n+1}t'_n \\ & & \ddots & \vdots \\ & & & t'_n \end{pmatrix}$$

where $n^{-1} = (x_{i,j})$. Using the change of variables

$$x_{i,j} \mapsto \frac{t_i}{t'_{j-1}}x_{i,j} \quad 1 \le i < j \le n+1$$

the inner integral becomes $\prod_{i=1}^{n}|t_i|^{n+1-i}|t'_i|^{n+\frac{1}{2}-i}$ times

$$\int_{F^{\binom{n+1}{2}}} \Phi(\begin{pmatrix} x_{1,2} & x_{2,3} & \dots & x_{1,n+1} \\ t_2^{-1}t'_1 & x_{2,3} & \dots & x_{2,n+1} \\ & t_3^{-1}t'_2 & \dots & x_{3,n+1} \\ & & \ddots & \vdots \\ & & & t'_n \end{pmatrix})\psi(\frac{t_1}{t'_1}x_{1,2} + \dots + \frac{t_n}{t'_n}x_{n,n+1})\ dx_{i,j}.$$

This integral can be written as

$$\phi(y_1, \dots, y_n, z_1, \dots, z_n)$$

where $\phi \in \mathcal{S}(F^{2n})$ is obtained by restricting a partial Fourier transform of Φ to a certain subspace, and $y_i = \frac{t_i}{t'_i}$, $z_i = \frac{t'_i}{t_{i+1}}$, $i = 1, \dots, n$ (with $t_{n+1} = 1$).

From the bounds (3) and (5), it suffices to show the convergence of

$$\int_{T'}\int_{T} \phi(y_1, \dots, y_n, z_1, \dots, z_n) \prod_{i=1}^{n} |\alpha_i(t)|^{\lambda} |\alpha'_i(t')|^{\mu} |t_i/t'_i|^{\frac{1}{2}}\ dt\ dt'$$

for $\lambda, \mu > 0$. (Note that the support conditions of the Whittaker functions are already incorporated into the support of ϕ.) Using the variables $y_1, \dots, y_n, z_1, \dots, z_n$ the integral becomes

$$\int_{(F^*)^n}\int_{(F^*)^n} \phi(\underline{y}, \underline{z})|y_1|^{\lambda+\frac{1}{2}} \prod_{i=2}^{n} |y_i|^{\lambda+\mu+\frac{1}{2}}|z_i|^{\lambda+\mu}\ d^*y_1 \dots\ d^*y_n\ d^*z_1 \dots\ d^*z_n$$

which converges. $\square$

4.3. **Bessel distribution identity.** We say f and Φ match if (19) is satisfied. We say f' and Φ' match if (26) is satisfied.

Corollary 1. *Assume π and π' are irreducible generic square integrable representations of G and G' that correspond under Howe duality. Suppose that Φ and Φ' are related by (27), and f and f' match Φ and Φ' respectively. Then*

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f) = \mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(f').$$

Proof. From Lemma 3:

$$\overline{\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f)} = \sum_i \overline{\mathcal{P}(\pi(f)W_i)}W_i(e) = \sum_i (\theta_\Phi W_i)(e')W_i(e) = \sum_{i,j} (\theta_\Phi W_i, W'_j)W'_j(e')W_i(e).$$

Similarly, from Lemma 4:

$$\overline{\mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(f')} = \sum_j \overline{\pi'(f')W'_j(e')}W'_j(e') = \sum_j (\theta'_{\Phi'}W'_j)(e)W'_j(e') = \sum_{i,j} (\theta'_{\Phi'}W'_j, W_i)W'_j(e')W_i(e).$$

It remains to invoke Proposition 4. $\square$

We are now ready to prove Theorem 1. First, we recall the notion of matching functions and related concepts.

An $N'' \times N''$ orbit $\{n''_1 g'' n''_2\}$ of $g'' \in \mathrm{GL}_n$ is *relevant* if $n''_1 g'' n''_2 = g''$ implies $\chi''(n''_1) = \chi''(n''_2)$. Similarly an $R \times N$ orbit $\{rgn\}$ of $g \in G$ is relevant if $rgn = g$ implies $\chi_R(r) = \chi(n)^{-1}$; an $N' \times N'$ orbit $\{n'_1 g' n'_2\}$ of $g' \in \mathrm{Sp}(V')$ is relevant if $n'_1 g' n'_2 = g'$ implies $\chi'(n'_1) = \chi'(n'_2)^{-1}$.

Let S_l be a complete set of representatives of relevant orbits in GL_l. In [MR04, §4], two injective maps s and t are defined from $\cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$ to G and $\mathrm{Sp}(V')$ respectively, so that the images of s and t give complete sets of representatives of relevant orbits in G and $\mathrm{Sp}(V')$. For $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$, define orbital integrals:

$$I_{s(x)}(f) = \int_R \int_{N \cap s(x)^{-1} R s(x) \backslash N} f(rs(x)n) \chi_R(r) \chi(n) \, dn \, dr,$$

$$J_{t(x)}(f') = \int_{N'} \int_{N' \cap t(x)^{-1} N' t(x) \backslash N'} f'(\widetilde{n'_1 t(x) n'_2}) \chi'(n'_1) \chi'(n'_2) \, dn'_1 \, dn'_2.$$

Define transfer factors $\Delta(x)$ for $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$ as in [MR04, §6]. We say f and f' match if $J_{t(x)}(f') = \Delta(x) I_{s(x)}(f)$ for all $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$.

By [MR04, Proposition 6.1] to any $f \in \mathcal{S}(G)$ there exists a matching $\tilde{f} \in \mathcal{S}(G')$ More precisely, the $\tilde{f}$ is constructed explicitly in the following manner. First by [ibid., Lemma 5.2] there exists Φ that relates to f through (19). Next, by [ibid., Lemma 5.6], (see [ibid., (5.5)]) there exists $\tilde{f} \in \mathcal{S}(G')$ such that the relation

$$\int_{N'} \tilde{f}(\widetilde{n'g'}) \chi'(n') \, dn' = \int_{R_1 \backslash N} \omega_\psi^1(n, g') \Phi(E_1) \chi(n) \, dn$$

holds for all $g' \in G'$. From this and (27) we see that $\tilde{f}$ and $T(\Phi)$ match in the sense of (26). By Corollary 1 we infer that

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f) = \mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(\tilde{f}).$$

On the other hand $\mathcal{B}^{\pi'}_{\mathcal{W}',\mathcal{W}'}(f')$ is a bi-(N', χ')-equivariant distribution on $\mathcal{S}(G')$ and therefore, it depends only on the orbital integrals of f' by [GK75]. Theorem 1 follows.

Remark 2. In [MR04, Theorem 7.1], it is established that at almost all places (the odd places where ψ is unramified), when f is an element in the Hecke algebra of G, f' the corresponding element in Hecke algebra of G', the functions f and f' match.

5. Global Bessel distribution identity

In this section, we consider the global counterpart of Theorem 1. Let F be a number field and $\mathbb{A}$ its ring of adèles. We retain the notation V, V', $Z_+ = Z^1_+$, Z^2_+ from §§1.1,3 and denote by $G'_{\mathbb{A}}$ the metaplectic cover of $\mathrm{Sp}(V'_{\mathbb{A}})$. (We refer to [JS07] for the precise definition and standard facts about the metaplectic group over the adèles.) Define $\mathcal{S}(G_A)$ to be the space of Schwartz-Bruhat functions on G_A; similarly for $\mathcal{S}(G'_{\mathbb{A}})$. We define theta functions $\theta^{\phi}_{\psi}(g, g')$ on $G_{\mathbb{A}} \times G'_{\mathbb{A}}$ by:

$$\theta^{\phi}_{\psi}(g, g') = \sum_{z \in Z^+_F} \omega_{\psi}(g, g')\phi(z), \ \phi \in \mathcal{S}(Z_{+,\mathbb{A}}).$$

Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal representation of $G_{\mathbb{A}}$ realized in $L^2(G_F \backslash G_{\mathbb{A}})$. We denote by $\theta_{\psi}(\pi)$ the (possibly zero) representation of $G'_{\mathbb{A}}$ spanned by the functions

$$\Theta^{\phi}_{\psi}[\varphi](g') = \int_{G_F \backslash G_{\mathbb{A}}} \theta^{\phi}_{\psi}(g, g')\varphi(g) \, dg, \ \varphi \in \pi, \ \phi \in \mathcal{S}(Z_{+,\mathbb{A}}).$$

Similarly for an irreducible genuine cuspidal representation $\widetilde{\pi}$ of $G'_{\mathbb{A}}$ realized in $L^2(G'_F \backslash G'_{\mathbb{A}})$, we denote by $\theta'_{\psi}(\widetilde{\pi})$ the representation of $G_{\mathbb{A}}$ spanned by functions of the form:

$$\widetilde{\Theta}^{\phi}_{\psi}[\widetilde{\varphi}](g) = \int_{G'_F \backslash G'_{\mathbb{A}}} \theta^{\phi}_{\psi}(g, g')\widetilde{\varphi}(g') \, dg, \ \widetilde{\varphi} \in \widetilde{\pi}, \ \phi \in \mathcal{S}(Z_{+,\mathbb{A}}).$$

For any automorphic form φ on G_A define the Whittaker coefficient by

$$\mathcal{W}(\varphi)(g) = \int_{N_F \backslash N_{\mathbb{A}}} \varphi(ng)\chi(n)^{-1} \, dn$$

and the Bessel functions $\mathcal{P}(\varphi)$ by

$$\mathcal{P}(\varphi)(g) = \int_{R_F \backslash R_{\mathbb{A}}} \varphi(rg)\chi_R(r)^{-1} \, dr.$$

Similarly, the Whittaker coefficient $\mathcal{W}'(\varphi')$ of an automorphic form φ' on $G'_{\mathbb{A}}$ is given by

$$\mathcal{W}'(\varphi')(g') = \int_{N'_F \backslash N'_{\mathbb{A}}} \varphi'(\widetilde{n'}g')\chi'(n')^{-1} \, dn'.$$

We write $\mathcal{W}(\varphi)$ for $\mathcal{W}(\varphi)(e)$ and similarly for $\mathcal{P}(\varphi)$ and $\mathcal{W}'(\varphi')$.

Consider the space $\mathcal{A}^{\mathrm{gen}}_{\psi}(G)$ of cusp forms which are orthogonal to the space of automorphic forms with vanishing ψ-Whittaker coefficient. (In fact, $\mathcal{A}^{\mathrm{gen}}_{\psi}(G)$ does not depend on ψ.) Define $\mathcal{A}^{\mathrm{gen}}_{\psi}(G')$ similarly. By local uniqueness of Whittaker models these spaces are multiplicity free [PS79]. Let $\Xi^{\mathrm{gen}}_{\psi}$ be the set of irreducible cuspidal representations π of $G_{\mathbb{A}}$ in $\mathcal{A}^{\mathrm{gen}}_{\psi}(G)$ for which $\theta_{\psi}(\pi)$ is non-zero. Similarly for ${\Xi'}^{\mathrm{gen}}_{\psi}$. The following results were established by Furusawa and extended by Jiang-Soudry.

Proposition 5. *(*[Fur95]*, cf.* [JS07]*)*

1. *Let* $\varphi \in \pi$, $\phi = \otimes \phi_v \in \mathcal{S}(Z^1_{+,\mathbb{A}})$, $\widetilde{\varphi} = \Theta^{\phi}_{\psi}[\varphi]$ *and* $\varphi' = \overline{\widetilde{\varphi}}$*. Then* $\overline{\mathcal{W}'(\varphi')} = \mathcal{P}(\pi(f)\varphi)$ *where* $f = \otimes f_v$ *such that* f_v *and* ϕ_v *are related through* (19).
2. *Let* $\varphi' \in \pi'$, $\phi' = \otimes \phi'_v \in \mathcal{S}(Z^2_{+,\mathbb{A}})$ *and* $\overline{\varphi} = \widetilde{\Theta}^{\phi'}_{\psi}[\varphi']$*. Then* $\overline{\mathcal{W}(\varphi)} = \mathcal{W}'(\pi'(f')\varphi')$ *where* $f' = \otimes f'_v$ *such that* f'_v *and* ϕ'_v *are related through* (26).

(3) *θ_ψ defines a bijection between Ξ_ψ^{gen} and ${\Xi'}_\psi^{\mathrm{gen}}$ whose inverse map is $\theta'_{\psi^{-1}}$. Moreover, Ξ_ψ^{gen} is the set of irreducible representations π in $\mathcal{A}_\psi^{\mathrm{gen}}(G)$ such that $L(\frac{1}{2},\pi)\neq 0$, or equivalently, $\mathcal{P}\not\equiv 0$ on π, while ${\Xi'}_\psi^{\mathrm{gen}}$ consists of all irreducible representations in $\mathcal{A}_\psi^{\mathrm{gen}}(G')$.*

Note that the first two parts are the global analogues of the results in section 3.

Let us now consider $\widetilde{\pi}=\theta_\psi(\pi)$ with $\pi\in\Xi_\psi^{\mathrm{gen}}$ and let $\pi'=\overline{\widetilde{\pi}}$. Thus, by the above proposition we have $\theta'_\psi(\pi')=\overline{\pi}$.

Define distributions on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$ with respect to π and π' as follows:

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f)=\sum_i \mathcal{P}(\pi(f)\varphi_i)\overline{\mathcal{W}(\varphi_i)},$$

$$\mathcal{B}^{\pi'}_{\mathcal{W}',\mathcal{W}'}(f')=\sum_i \mathcal{W}'(\pi'(f')\varphi'_i)\overline{\mathcal{W}'(\varphi'_i)}$$

where φ_i is an admissible orthonormal basis of π, that is, the restricted tensor product of admissible bases of π_v, and similarly for φ'_i.

Theorem 2. *Given $f=\otimes f_v\in\mathcal{S}(G_{\mathbb{A}})$ there exists a matching $f'=\otimes f'_v\in\mathcal{S}(G'_{\mathbb{A}})$ (that is, f'_v matches f_v for all v) such that for any $\pi\in\Xi_\psi^{\mathrm{gen}}$ we have*

$$\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f)=\mathcal{B}^{\pi'}_{\mathcal{W}',\mathcal{W}'}(f') \tag{30}$$

where $\pi'=\overline{\theta_\psi(\pi)}$.

Proof. As in the proof of Corollary 1, if f_v is related to ϕ_v by (19) for all v then we have

$$\begin{aligned}\overline{\mathcal{B}^{\pi}_{\mathcal{P},\mathcal{W}}(f)}&=\sum_i\overline{\mathcal{P}(\pi(f)\varphi_i)}\mathcal{W}(\varphi_i)\\&=\sum_i\mathcal{W}'(\overline{\Theta^{\phi}_{\psi}[\varphi_i]})\mathcal{W}(\varphi_i)=\sum_{i,j}(\overline{\Theta^{\phi}_{\psi}[\varphi_i]},\varphi'_j)_{G'_{\mathbb{A}}}\mathcal{W}(\varphi_i)\mathcal{W}'(\varphi'_j).\end{aligned}$$

Similarly if f'_v is related to ϕ'_v by (26) for all v then

$$\begin{aligned}\overline{\mathcal{B}^{\pi'}_{\mathcal{W},\mathcal{W}}(f')}&=\sum_j\overline{\pi'(f')\mathcal{W}'(\varphi'_j)}\mathcal{W}'(\varphi'_j)\\&=\sum_j\mathcal{W}(\overline{\widetilde{\Theta}^{\phi'}_{\psi}[\varphi'_j]})\mathcal{W}'(\varphi'_j)=\sum_{i,j}(\overline{\widetilde{\Theta}^{\phi'}_{\psi}[\varphi'_j]},\varphi_i)_{G_{\mathbb{A}}}\mathcal{W}(\varphi_i)\mathcal{W}'(\varphi'_j).\end{aligned}$$

By the same argument as in the proof of Theorem 1, given $f=\otimes f_v\in\mathcal{S}(G_A)$ we construct $f'=\otimes f'_v$ using [MR04] by first constructing ϕ_v related to f_v and then choosing f'_v related to $T(\phi_v)$. We remark that in [ibid.] f_v is restricted to be compactly supported, but the proof works for any $f_v\in\mathcal{S}(G(F_v))$ (in the archimedean case).

It remains to establish the adjointness relation

$$(\overline{\Theta^{\phi}_{\psi}[\varphi_i]},\varphi'_j)_{G'_{\mathbb{A}}}=(\overline{\widetilde{\Theta}^{T(\phi)}_{\psi}[\varphi'_j]},\varphi_i)_{G_{\mathbb{A}}}.$$

This immediately follows from the relation

$$\theta_\psi^\phi = \theta_\psi^{T(\phi)}$$

which is in turn a consequence of the Poisson summation formula (cf. the proof of [MR05, Proposition 3.1]). □

Remark 3. Assuming the validity of the Gelfand-Kazhdan localization principle in the archimedean case, every bi-(N', ψ')-equivariant distribution on $G'_{\mathbb{A}}$ would be determined by its orbital integrals. Therefore, we would be able to rephrase Theorem 2 by simply saying that the relation (30) holds for any matching f, f'.

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Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: `embaruch@math.technion.ac.il`

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel
E-mail address: `erezla@math.huji.ac.il`

Department of Mathematics & Computer Science, Rutgers University, Newark, NJ 07102, USA
E-mail address: `zmao@rutgers.edu`