

A GENERALIZED KOHNEN-ZAGIER FORMULA FOR MAASS FORMS

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ABSTRACT. We prove an explicit identity relating the central values of the twist L -functions of weight 0 Maass forms to the Fourier coefficients of weight $\pm\frac{1}{2}$ Maass forms. We also give a bound on Fourier coefficients of weight $\pm\frac{1}{2}$ Maass forms.

1. INTRODUCTION

The Shimura correspondence gives a mapping between spaces of integral weight and half integral weight holomorphic cusp forms. Waldspurger was the first to establish a relation between the central L -values of the integral weight form and certain Fourier coefficients of the corresponding half integral weight form. Such relationship is explicated in many cases, see for examples [KZ],[BM1]. We call such explicit identities between central values and Fourier coefficients, Kohnen-Zagier type formulas. Such formulas have many important arithmetic applications. There is much less literature on Shimura correspondence for Maass forms. In [KaS] the correspondence is studied for the case of level 1 Maass forms and a formula is proven relating central L -values of weight zero Maass forms to certain sums of squares of Fourier coefficients of weight $\frac{1}{2}$ Maass forms. Biro ([Bi]), Khuri-Makdisi ([Kh]) and Kojima ([Ko1],[Ko2], [Ko3]) studied generalization of this work.

In this paper, we establish a Kohnen-Zagier type formula for Maass forms of odd square free level, in the following generalized sense. Given a Maass form of weight 0, we associate through Shimura correspondence a set of Maass forms of weight $\pm\frac{1}{2}$ (as oppose to just one form). We then give an explicit relation between the central values of twisted L -functions and the Fourier coefficients of the set of weight $\pm\frac{1}{2}$ forms. This approach allows us to get a formula for central values of L -functions twisted by all fundamental discriminants without

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restriction. Our result is based on the general formula proved in [BM1] for automorphic forms. As another application of the result in [BM1] and the main proposition in this paper, we give a bound on Fourier coefficient for $\pm\frac{1}{2}$ weight Maass cusp forms of general level, refining a result in [Du].

1.1. Preliminaries. We recall some facts on integral weight and half integral weight Maass forms.

Weight 0 Maass forms and L -function

Let N be an odd square free positive integer. Let $S_0(N)$ be the space of Maass cusp forms of level N on the upper half plane, so that when $f(z) \in S_0(N)$, it is a cusp form satisfying

$$f(\gamma z) = f(z), \quad \forall \gamma \in \Gamma_0(N).$$

For k a real number, define the Laplace operator

$$\Delta_k = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - ik y \frac{\partial}{\partial x}.$$

Then $S_0(N)$ is invariant under the Laplace operator Δ_0 . Let $f(z)$ be a Maass cusp form with $\Delta_0 f = -(\frac{1}{4} - s^2)f$, it has a Fourier expansion

$$f(x + iy) = \sum_{n \in \mathbb{Z} - \{0\}} a(n) W_{0,s}(4\pi|n|y) e(nx).$$

Here $e(z) = e^{2\pi iz}$ and $W_{k,m}$ is the Whittaker function given by:

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(-m - k + \frac{1}{2})} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(m - k + \frac{1}{2})} M_{k,-m}(z)$$

and

$$M_{k,m}(z) = e^{-z/2} \sum_{r=0}^{\infty} \frac{\Gamma(2m+r)\Gamma(m-k+r+\frac{1}{2})}{r!\Gamma(2m+r+1)\Gamma(m-k+\frac{1}{2})} z^{m+r+\frac{1}{2}}.$$

The Whittaker function $W_{k,m}$ does not vanish on the real line, a fact we will use throughout the paper. We also note in the case $k = 0$ the identity $W_{0,s}(4\pi|n|y) = (4|n|y)^{\frac{1}{2}} K_s(2\pi|n|y)$, where K_s is the K -Bessel function.

The Hecke operators for $S_0(N)$ are defined as follows: for $f(z) \in S_0(N)$ as above and $(p, N) = 1$,

$$T_p(f)(z) = \sum_{n \in \mathbb{Z} - \{0\}} b(n) W_{0,s}(4\pi|n|y) e(nx),$$

where $b(n) = a(pn) + p^{-1}a(n/p)$; here we assume $a(m) = 0$ when m is not an integer. (We remark that some authors normalize Hecke operators differently, see for example [KaS]). A

Maass-Hecke form is a simultaneous eigenform for T_p with $(p, N) = 1$ and Δ_0 . For $f(z)$ a Maass-Hecke form, it is either an even form ($a(n) = a(-n)$) or an odd form ($a(n) = -a(-n)$). We let $\eta_f = 1$ if f is even and $\eta_f = -1$ if f is odd.

One can define the Maass newforms in the same way newforms are defined in the holomorphic case ([AL]). Let $S_0^-(N)$ be the space generated by the forms $f(dz)$ where $d > 1$, $d|N$ and $f(z)$ is in $S_0(N/d)$; let $S_0^+(N)$ be the orthogonal complement of $S_0^-(N)$ in $S_0(N)$. We call $f(z)$ a Maass newform if it is a Maass-Hecke form in $S_0^+(N)$ with Fourier coefficient $a(1) = 1$.

Let $f(z)$ be a Maass newform. When D is a fundamental discriminant, define the twisted L -function

$$L(f, D, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{s-\frac{1}{2}}} \left(\frac{D}{n}\right).$$

Weight $\pm\frac{1}{2}$ Maass forms

Let M be an odd positive integer and $S_{\pm\frac{1}{2}}(4M, \chi)$ be the space of weight $\pm\frac{1}{2}$, level $4M$ Maass cusp forms with even character χ of $(\mathbb{Z}/4M)^*$. Then if $g(z) \in S_{\pm\frac{1}{2}}(4M, \chi)$, it is a cusp form satisfying for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$g(\gamma z) = J(\gamma, z)^{\pm 1} \chi(d) g(z), \quad \forall \gamma \in \Gamma_0(4M);$$

where $J(\gamma, z) = \theta(\gamma z)/\theta(z)$, with $\theta(z) = y^{1/4} \sum_{n=-\infty}^{\infty} e(n^2 z)$.

The Laplace operator $\Delta_{\pm\frac{1}{2}}$ acts on $S_{\pm\frac{1}{2}}(4M, \chi)$. If $g(z) \in S_{\pm\frac{1}{2}}(4M, \chi)$ is an eigenform with $\Delta_{\pm\frac{1}{2}} g = -(\frac{1}{4} - t^2)g$, then $g(z)$ has Fourier expansion:

$$(1.1) \quad g(x + iy) = \sum_{n \in \mathbb{Z} - \{0\}} c(n) W_{\pm\frac{1}{4} \operatorname{sgn}(n), t}(4\pi|n|y) e(nx).$$

The Hecke operators for $S_{\pm\frac{1}{2}}(4M, \chi)$ is defined as follows ([Sh, Theorem 1.7]): for $(p, 4M) = 1$,

$$T_{p^2}(g)(z) = \sum_{n \in \mathbb{Z} - \{0\}} d(n) W_{\pm\frac{1}{4} \operatorname{sgn}(n), t}(4\pi|n|y) e(nx).$$

where

$$d(n) = c(p^2 n) + \chi(p) \left(\frac{\pm n}{p}\right) p^{-\frac{3}{2} \pm \frac{1}{2}} c(n) + \chi(p^2) p^{-2 \pm 1} c(n/p^2).$$

Again $c(m) = 0$ if m is not an integer.

1.2. Main results. Let $f(z)$ be a Maass cusp form of weight 0 and odd square free level N , with $\Delta_0 f = -(\frac{1}{4} - s^2)f$. Let T be the set of prime divisors of N . Let χ be an even character of $(\mathbf{Z}/(4\prod_{l \in T} l))^*$. We denote the l -primary component of χ by $\chi_{(l)}$. Let T'_χ be the subset of T consisting of primes l where $\chi_{(l)}(-1) = -1$. Let T_χ be the subset of T where $\chi_{(l)} \equiv 1$. For simplicity, from now on we will make the further assumption that $T_\chi \cup T'_\chi = T$; namely

$$(C0) \quad \chi_{(l)} \equiv 1 \text{ when } l \in T - T'_\chi.$$

Let M be an odd integer with the same set of prime divisors T . We consider the space $S_{\pm\frac{1}{2}}(f, 4M, \chi)$ of forms $g(z) \in S_{\pm\frac{1}{2}}(4M, \chi)$ satisfying $\Delta_{\pm\frac{1}{2}}g = -(\frac{1-s^2}{4})g$ and:

$$(C1) \text{ when } (p, 4M) = 1, T_{p^2}g(z) = \lambda(p)g(z) \text{ with } \lambda(p) = \chi(p)a(p).$$

Minimal Level of $g(z)$

Theorem 1.1. *Let χ be an even character of $(\mathbf{Z}/4N)^*$ satisfying (C0). Let $N(\chi) = N \prod_{l \in T'_\chi} l$, $\epsilon = \pm\frac{1}{2}$, then $S_\epsilon(f, 4M, \chi) \neq \{0\}$ if and only if*

- (1) $\epsilon = \frac{1}{2}\eta_f$, and
- (2) $N(\chi)$ divides M .

A set of weight $\pm\frac{1}{2}$ forms associated to $f(z)$

Now consider $S(f, \chi) = S_\epsilon(f, 4M, \chi)$ when $\epsilon = \frac{1}{2}\eta_f$ and $M = N(\chi)$, or rather its Kohnen subspace $S^+(f, \chi)$ consisting of $g(z)$ with

$$(C2) \quad c(n) = 0 \text{ unless } \chi_{(2)}(-1)n \equiv 0, 1 \pmod{4}.$$

Theorem 1.2. *The space $S^+(f, \chi)$ is one dimensional.*

We can then associate to $f(z)$ a set of weight $\pm\frac{1}{2}$ forms $g_{f,\chi}(z)$, where $g_{f,\chi}(z)$ are generators of the one dimensional spaces $S^+(f, \chi)$.

Vanishing of Fourier coefficients of $g_\chi(z)$

Fix a character χ as above and let $g(z) = g_{f,\chi}(z)$ be a generator of $S^+(f, \chi)$. The Fourier coefficients $c(n)$ of $g(z)$ vanish in many occasions:

Proposition 1.3. *Let w_p be the eigenvalue of Atkin-Lehner involution of f at p . Let $c(n)$ be the coefficients in expansion (1.1) of $g_{f,\chi}(z)$. When $\left(\frac{n}{p}\right) = -\chi_{(p)}(-1)w_p$ for some $p \in T$ or $l|n$ for some $l \in T'_\chi$, $c(n) = 0$.*

Here $l|n$ means $l|n$ and $l^2 \nmid n$.

A Kohnen-Zagier type formula for Maass forms

When D is a fundamental discriminant and the Fourier coefficient $c(D)$ of $g(z)$ does not vanish for the reason described by the above proposition, we have the following Kohnen-Zagier type identity relating the Fourier coefficient to the twisted L -value of $f(z)$:

Theorem 1.4. *Let $g(z) = g_{f,\chi}(z)$ with Fourier expansion (1.1). For a fundamental discriminant D with $\left(\frac{D}{p}\right) \neq -\chi_{(p)}(-1)w_p$ for all $p \in T$ and $l \nmid D$ when $l \in T'_\chi$:*

$$(1.2) \quad \frac{|c(D)|^2}{\langle g, g \rangle} = \frac{L(f, D, \frac{1}{2})}{\langle f, f \rangle} (\pi|D|)^{-1} \Gamma\left(\frac{1}{2} - \frac{\text{sgn}(D)\eta_f}{4} - \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\text{sgn}(D)\eta_f}{4} + \frac{s}{2}\right) 2^{\nu(N) - \mu(D, N)} \prod_{p \in T'_\chi} \frac{p}{p+1}.$$

Here $\nu(N)$ is number of prime divisors of N , $\mu(D, N)$ is the number of common prime divisors of D and N .

The above formula gives an explicit relation between central values $L(f, D, \frac{1}{2})$ and Fourier coefficients of $\pm\frac{1}{2}$ weight Maass forms. Notice that for each fundamental discriminant D , we can find a character χ with $\left(\frac{D}{p}\right) \neq -\chi_{(p)}(-1)w_p$ for all $p \in T$ and satisfying (C0), thus a corresponding $g = g_{f,\chi}$ with its Fourier coefficient $c(D)$ satisfying the equation (1.2).

We remark that our results can be extended to any even character χ , as well as to the case N is odd but not square free. See [M2] for details in the holomorphic modular form case.

A bound on Fourier coefficients of weight $\pm\frac{1}{2}$ Maass forms

Let $g(z)$ be a weight $\pm\frac{1}{2}$ Maass cusp form of general level in $S_{\pm\frac{1}{2}}(4M)$ with $\Delta_{\pm\frac{1}{2}}g = -(\frac{1}{4} - t^2)g$ (here M is not necessarily square free or odd). Recall the Fourier expansion (1.1) for $g(z)$. We prove a bound for Fourier coefficients of $g(z)$ in terms of $|t|$ and level M .

When f is a weight 0 Maass newform of level N with $\Delta_0 f = -(\frac{1}{4} - s^2)f$, Theorem 1 in [BHM] gives the following bound for $L(f, D, \frac{1}{2})$:

$$(1.3) \quad L(f, D, \frac{1}{2}) \ll_\epsilon ((1 + |s|)N|D|)^\epsilon (1 + |s|)^B N^C |D|^{\frac{1}{2} - \frac{1}{8}(1-2\theta)},$$

for any $\epsilon > 0$ and D a fundamental discriminant; here $\theta = \frac{7}{64}$, $B = \frac{75+12\theta}{16}$ and $C = \frac{9}{16}$. Based on this estimate, we get:

Theorem 1.5. *Let $g(z)$ be a weight $\pm\frac{1}{2}$ Maass cusp form in $S_{\pm\frac{1}{2}}(4M)$ normalized so that $\langle g, g \rangle = 1$. For any n square free integer,*

$$(1.4) \quad |c(n)| \ll_{\epsilon} ((1 + |t|)M|n|)^{\epsilon} e^{\pi\frac{|t|}{2}} (1 + |t|)^{-\operatorname{sgn}(n)\frac{\eta}{4} + \frac{B+1}{2}} M^{C+2} |n|^{-\frac{1}{4} - \frac{1}{16}(1-2\theta)}.$$

This bound improves the n -aspect estimate of [Du]: $|c(n)| \ll |n|^{-\frac{2}{7}}$ and gives a bound in the M (level) aspect.

1.3. Structure of paper and notations. In introduction we intentionally avoided any discussion of automorphic forms, so that the results can be read and used without any reference to automorphic representation theory. However our proof of the results is based on relating Maass forms to automorphic forms on the groups GL_2 and $\overline{\mathrm{SL}}_2$ (the two fold cover of SL_2).

We will use $(g, \pm 1)$ to denote an element in $\overline{\mathrm{SL}}_2$ and write multiplication in that group as $a \cdot b$. Thus $(g_1, 1) \cdot (g_2, 1) = (g_1 g_2, c(g_1, g_2))$ where $c(g_1, g_2)$ is the Rao cocycle. In the computation involved in this paper, the cocycle is either irrelevant (as in estimates) or equals 1.

Let I be the identity matrix in GL_2 or $\overline{\mathrm{SL}}_2$. Let $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\tilde{n}(x) = (n(x), 1)$. Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tilde{w} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right)$. Let $\underline{a} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $\underline{\underline{a}} = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1\right)$. For $-\pi < \theta \leq \pi$, let $k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. For $-\pi < \theta \leq 3\pi$, let $\tilde{k}(\theta) = [k(\theta), 1]$ for $\theta \in (-\pi, \pi]$ and $[k(\theta), -1]$ for $\theta \in (\pi, 3\pi]$. Let K be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of all $k(\theta)$'s. Let \tilde{K} be the subgroup of $\overline{\mathrm{SL}}_2(\mathbb{R})$ consisting of all $\tilde{k}(\theta)$'s. Denote by Z the center of GL_2 .

In section 2 we relate Maass forms with vectors in cuspidal representations. We recall the main result of [BM1] in section 3. In section 4 we prove the key result over real field. The results stated in introduction are proved in section 5. In the appendix we prove an integral formula that is used in section 4.

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2. MAASS CUSP FORMS AND CUSPIDAL REPRESENTATIONS

2.1. Weight 0 Maass forms. Let $\mathbf{A}_{\mathbb{Q}}$ be the adèle ring of \mathbb{Q} . For a prime p , let \mathcal{O}_p be the ring of integers in \mathbb{Q}_p and $K_{0,p}$ be the subgroup of $\mathrm{GL}_2(\mathcal{O}_p)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with c in the prime ideal of \mathcal{O}_p .

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0(N) \right\}$. Let $S_0(N)$ be the space of Maass cusp form on $\Gamma_0(N)$ (of level N), and with trivial character. Let $f \in S_0(N)$ be a Maass form. Then f determines a vector in the space of automorphic forms on $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$ by $f \mapsto \varphi = s(f)$. The map $s(f)$ is defined as follows. For $g_{\infty} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ considered as an element $(g_{\infty}, I, I, \dots)$ in $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$, let

$$\varphi(g_{\infty}) = f\left(\frac{ai+b}{ci+d}\right).$$

Moreover $\varphi(\gamma g k) = \varphi(g)$ when $\gamma \in \mathrm{GL}_2(\mathbb{Q})Z(\mathbf{A}_{\mathbb{Q}})$, and $k \in \prod_{p|N} \mathrm{GL}_2(\mathcal{O}_p) \prod_{p|N} K_{0,p}$. These relations determine a unique $\varphi = s(f)$.

When $f(z)$ is a Maass newform, $\varphi = s(f)$ lies in the space $V(\pi)$ of an irreducible cuspidal representation π of $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$, with trivial central character. Then $V(\pi) \cong \otimes V(\pi_v)$ where π_v are irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_v)$, $V(\pi_v)$ is the space of π_v when v is finite, $V(\pi_{\infty})$ is a K -submodule of the space $\hat{V}(\pi_{\infty})$ of π_{∞} consisting of K -finite vectors. Moreover under the isomorphism we have $\varphi = \otimes_v \varphi_v$ with $\varphi_v \in V(\pi_v)$.

When v is a p -adic place, the representation π_v and the vector φ_v are the same as those described in [BM1, §9.1]. Consider now the case $v = \infty$. When $f(z)$ is an even Maass form, then $\pi_{\infty} = \pi_{s,+}$ with $V(\pi_{s,+})$ being the space of K -finite functions ϕ on $\mathrm{GL}_2(\mathbb{R})$ satisfying:

$$(2.1) \quad \phi(n(x)\underline{a}zg) = |a|^{s+1/2}\phi(g), \quad z \in Z.$$

When $f(z)$ is an odd Maass form, then $\pi_{\infty} = \pi_{s,-}$ with $V(\pi_{s,-})$ being the space of K -finite functions ϕ on $\mathrm{GL}_2(\mathbb{R})$ satisfying:

$$(2.2) \quad \phi(n(x)\underline{a}zg) = |a|^{s+1/2} \mathrm{sgn}(a)\phi(g), \quad z \in Z.$$

In both cases π_{∞} is either a unitary principal series or a complementary series representation, meaning s is either a purely imaginary number or a real number satisfying $-\frac{1}{2} < s < \frac{1}{2}$. The vector φ_{∞} is the unique (up to scalar multiple) K -fixed vector in $\pi_{s,\pm}$.

2.2. Half integral weight Maass forms. Let M be an odd positive integer and χ be an even character of $(\mathbb{Z}/4M)^*$. Let $\tilde{A}'_{\pm 1/2}(4M, \chi)$ be the space generated by vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v$ in the space of cuspidal automorphic forms on $\overline{\mathrm{SL}}_2(\mathbf{A}_{\mathbb{Q}})$ satisfying the following conditions:

(2.2.0) $\tilde{\varphi}$ lies in $V(\tilde{\pi}) \cong \otimes V(\tilde{\pi}_v)$ where $\tilde{\pi}_v$ are irreducible representations of $\overline{\mathrm{SL}}_2(\mathbb{Q}_v)$, $V(\tilde{\pi}_v)$ is the space of $\tilde{\pi}_v$ when v is finite, $V(\tilde{\pi}_{\infty})$ is a submodule of $\hat{V}(\tilde{\pi}_{\infty})$ the space of $\tilde{\pi}_{\infty}$ consisting of \tilde{K} -finite vectors.

(2.2.1) When $v = \infty$, $\tilde{\varphi}_\infty$ is a weight $\pm\frac{1}{2}$ vector in the space of a unitary principal series or a complementary series representation $\tilde{\pi}_\infty$. The representation $\tilde{\pi}_\infty$ is of the form $\tilde{\pi}_{s,\pm}$ with $V(\tilde{\pi}_{s,\pm})$ being the space of \tilde{K} -finite genuine functions ϕ of $\overline{\mathrm{SL}}_2(\mathbb{R})$ satisfying:

$$(2.3) \quad \phi(\tilde{n}(x) \cdot \underline{a} \cdot g) = |a|^{s+1} \tilde{\gamma}(a)^{\pm 1} \phi(g).$$

Here s is either a purely imaginary number or a real number with $-\frac{1}{2} < s < \frac{1}{2}$; and $\tilde{\gamma}(a) = 1$ if $a > 0$ and $\tilde{\gamma}(a) = i$ if $a < 0$. The weight $\pm\frac{1}{2}$ vector $\tilde{\varphi}_\infty \in V(\tilde{\pi}_{s,\pm})$ satisfies

$$(2.4) \quad \tilde{\varphi}_\infty(\tilde{n}(x) \cdot \underline{a} \cdot \tilde{k}(\theta)) = a^{s+1} e^{\pm i\theta/2} \tilde{\varphi}_\infty(I)$$

for $a > 0$ and $-\pi < \theta \leq 3\pi$.

(2.2.2) When v is p -adic, $\tilde{\varphi}_v$ is as described in [BM1, (9.2.2)-(9.2.4)]. In the case of (9.2.4), take $k = 0$.

[W2, Proposition 3] establishes a bijection from $S_{\pm\frac{1}{2}}(4M, \chi)$ to $\tilde{A}'_{\pm\frac{1}{2}}(4M, \chi)$. The bijection is given by $g(z) \mapsto \tilde{\varphi} = t(g)$, where $t(g)$ is the unique function on $\overline{\mathrm{SL}}_2(\mathbf{A}_\mathbb{Q})$ that is genuine, continuous and left invariant under $\mathrm{SL}_2(\mathbb{Q})$ and satisfies:

$$t(g)\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, 1\right) = y^{1/4} e^{\pm\frac{1}{2}i\theta} g(x + yi),$$

where $y > 0, x \in \mathbb{R}$ and $-\pi < \theta \leq \pi$.

2.3. Whittaker functional and Fourier coefficient. The Whittaker functionals for the space of π of GL_2 and $\tilde{\pi}$ of $\overline{\mathrm{SL}}_2$ are defined as follows: for ψ a nontrivial additive character of $\mathbf{A}_\mathbb{Q}/\mathbb{Q}$ and $\varphi \in V(\pi)$:

$$(2.5) \quad W_\varphi = \int_{\mathbf{A}_\mathbb{Q}/\mathbb{Q}} \varphi(n(x)) \psi(-x) dx.$$

For $\lambda \in \mathbb{Q}^*$, $\tilde{\varphi} \in V(\tilde{\pi})$:

$$(2.6) \quad \tilde{W}_{\tilde{\varphi}}^\lambda = \int_{\mathbf{A}_\mathbb{Q}/\mathbb{Q}} \tilde{\varphi}(\tilde{n}(x)) \psi(-\lambda x) dx.$$

We choose ψ as follows: if $x \in \mathbb{R}$, $\psi(x) = e^{2\pi i x}$; at a rational prime p , if $x \in \mathbb{Q}_p$, choose $\hat{x} \in \mathbb{Q}$ so that $|x - \hat{x}|_p \leq 1$, and set $\psi(x) = e^{-2\pi i \hat{x}}$. By definition, when $\varphi = s(f)$

$$(2.7) \quad W_\varphi = \int_0^1 f(x+i) \psi(-x) dx = a(1) W_{0,s}(4\pi).$$

We remark when f is a newform, $a(1) = 1$. Similarly when $\tilde{\varphi} = t(g)$ as in §2.2 such that $\tilde{\varphi}_\infty \in V(\tilde{\pi}_{s,\pm})$, for $D \in \mathbb{Z} - \{0\}$:

$$(2.8) \quad \tilde{W}_{\tilde{\varphi}}^D = \int_0^1 g(x+i)\psi(-Dx) dx = c(D)W_{\pm\frac{1}{4}\operatorname{sgn}(D), \frac{s}{2}}(4\pi|D|).$$

3. A FORMULA FOR L -VALUES

A key result we use is [BM1, Theorem 4.3], which we now recall. Notations will be as in [BM1]; for example $\|\cdot\|_v$ is the valuation at place v .

Let ψ be a fixed nontrivial additive character of $\mathbf{A}_{\mathbb{Q}}/\mathbb{Q}$, D be a fundamental discriminant. Let $\psi^D(x) = \psi(Dx)$; let χ_D be the quadratic character of $\mathbf{A}_{\mathbb{Q}}^*$ associated to D . Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$ with trivial central character, and $\tilde{\pi} = \tilde{\pi}_D = \Theta(\pi \otimes \chi_D, \psi^D)$ be the irreducible cuspidal representation of $\overline{\mathrm{SL}}_2(\mathbf{A}_{\mathbb{Q}})$ obtained through theta correspondence ([W1], [W3]). Take two vectors $\varphi = \otimes \varphi_v$ and $\tilde{\varphi} = \otimes \tilde{\varphi}_v$ in $V(\pi)$ and $V(\tilde{\pi})$ respectively, then we have for S a large enough finite set of places,

$$(3.1) \quad \frac{|\tilde{W}_{\tilde{\varphi}}^D|^2}{\|\tilde{\varphi}\|^2} = \frac{|W_{\varphi}|^2 L(\pi \otimes \chi_D, 1/2)}{\|\varphi\|^2} \prod_{v \in S} E_v(\varphi_v, \tilde{\varphi}_v, \psi, D)$$

where

$$(3.2) \quad E_v(\varphi_v, \tilde{\varphi}_v, \psi, D) = \frac{e(\varphi_v, \psi)}{e(\tilde{\varphi}_v, \psi^D) L(\pi_v \otimes \chi_D, 1/2) |D|_v}.$$

Here W_{φ} and $\tilde{W}_{\tilde{\varphi}}^D$ are the Whittaker coefficients defined in the previous section; $\|\varphi\|$ and $\|\tilde{\varphi}\|$ are Petersson norms; and

$$(3.3) \quad e(\varphi_v, \psi) = \frac{\|\varphi_v\|^2}{|L_v(\varphi_v)|^2}$$

$$(3.4) \quad e(\tilde{\varphi}_v, \psi^D) = \frac{\|\tilde{\varphi}_v\|^2}{|\tilde{L}_v^D(\tilde{\varphi}_v)|^2}$$

with L_v and \tilde{L}_v^D being fixed local Whittaker functionals and the local norms given by:

$$(3.5) \quad \|\varphi_v\|^2 = \int_{\mathbb{Q}_v^*} |L_v(\pi_v \underline{a}) \varphi_v|^2 \frac{da}{|a|_v},$$

$$(3.6) \quad \|\tilde{\varphi}_v\|^2 = \sum_{\delta_i} \frac{|2|_v}{2} \int_{\mathbb{Q}_v^*} |\tilde{L}_v^{D\delta_i}(\tilde{\pi}_v(\underline{a}) \tilde{\varphi}_v)|^2 \frac{da}{|a|_v}.$$

The sum is taken over representatives of square classes of \mathbb{Q}_v^* ; the functionals $\tilde{L}^{D\delta_i}$ for $\delta_i \neq 1$ are determined up to a complex unit by the choice of \tilde{L}^D .

Note that when f is a weight 0 Maass newform,

$$(3.7) \quad L(f, D, s) = L(\pi_f \otimes \chi_D, s) / L(\pi_{f, \infty} \otimes \chi_D, s).$$

In order to combine the identity (3.1) with the identities (2.7) and (2.8) to get the explicit identity (1.2), we need to describe the local constants $E_v(\varphi_v, \tilde{\varphi}_v, \psi, D)$. We have computed $E_v(\varphi_v, \tilde{\varphi}_v, \psi, D)$ for v finite in [BM1]. The value of $E_\infty(\varphi_\infty, \tilde{\varphi}_\infty, \psi, D)$ is given in Proposition 4.1.

4. COMPUTATION OF A LOCAL CONSTANT

Recall we fixed a choice of ψ in §2.3. We will denote the restriction of ψ to \mathbb{R} through embedding $x \mapsto (x, 0, 0, \dots)$ again by ψ . Then $\psi(x) = e^{2\pi i x}$ is an additive character of \mathbb{R} . In this section D is a nonzero real number. Recall $\psi^D(x) = \psi(Dx)$. We prove:

Proposition 4.1. *Let s be either a real number with $0 < s < \frac{1}{2}$ or a purely imaginary number. Let $\eta = \pm 1$. Let φ_∞ be a weight 0 vector in $V(\pi_{s, \eta})$ and $\tilde{\varphi}_\infty$ be a weight $\frac{1}{2}\eta$ vector in $V(\tilde{\pi}_{s, \eta})$. Then $\frac{e(\varphi_\infty, \psi)}{e(\tilde{\varphi}_\infty, \psi^D)}$ equals*

$$(2\pi)^{-1} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} - \frac{s}{2}\right) \left| \frac{W_{\frac{\eta}{4} \operatorname{sgn}(D), \frac{s}{2}}(4\pi|D|)}{W_{0, s}(4\pi)} \right|^2.$$

We remark that from [JL],

$$(4.1) \quad L(\pi_{s, \eta} \otimes \chi_D, \frac{1}{2}) = (\pi)^{-1 + \frac{\operatorname{sgn}(D)\eta}{2}} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} - \frac{s}{2}\right).$$

The proof of the proposition involves explicit computation of Whittaker functionals and norms of the special vectors in the given representations. We will make use of an integral formula that is established in the appendix.

4.1. Ratio of Whittaker functionals. We fix Whittaker functionals as follows: for any $\varphi \in V(\pi_{s, \eta})$,

$$(4.2) \quad L^D(\varphi) = \int_{-\infty}^{\infty} \varphi(w_n(x)) \psi(-Dx) dx.$$

For $\tilde{\varphi} \in V(\tilde{\pi}_{s, \eta})$,

$$(4.3) \quad \tilde{L}^D(\tilde{\varphi}) = \int_{-\infty}^{\infty} \tilde{\varphi}(\tilde{w} \cdot \tilde{n}(x)) \psi(-Dx) dx.$$

For the weight 0 vector φ_∞ in $V(\pi_{s,\eta})$ with $\varphi_\infty(I) = 1$, we have:

$$\varphi_\infty(n(x)ak(\theta)) = |a|^{s+\frac{1}{2}} \operatorname{sgn}(a)^{\frac{\eta-1}{2}}.$$

Let $\Phi(x) = \varphi_\infty(w_n(x))$, then

$$(4.4) \quad \Phi(x) = (1+x^2)^{-\frac{1}{2}-s}.$$

Take $\tilde{\varphi}_\infty$ to be the function in $V(\tilde{\pi}_{s,\eta})$ with weight $\eta/2$ and $\tilde{\varphi}_\infty(I) = 1$, and let $\Phi'(x) = \tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x))$. Then from (2.4),

$$(4.5) \quad \Phi'(x) = (1+x^2)^{-\frac{1}{2}-\frac{s}{2}} e^{i\eta\theta(x)/2}.$$

Here $\theta(x) = -\frac{\pi}{2} + \tan^{-1}(x) \in (-\pi, 0)$.

Lemma 4.2.

$$\left| \frac{\tilde{L}^D(\tilde{\varphi}_\infty)}{L(\varphi_\infty)} \right|^2 = \left| |D|^{\frac{s-1}{2}} \pi^{-\frac{s}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{1}{2}+\operatorname{sgn}(D)\frac{\eta}{4})} \frac{W_{\frac{\eta}{4}\operatorname{sgn}(D),\frac{s}{2}}(4\pi|D|)}{W_{0,s}(4\pi)} \right|^2.$$

Proof. We need to consider the function:

$$(4.6) \quad Q_{t,k}(D) = \int_{-\infty}^{\infty} (1+x^2)^{-\frac{1}{2}-t} e^{ik\theta(x)} e^{-2\pi i D x} dx.$$

Clearly $L^D(\varphi_\infty) = \hat{\Phi}(D) = Q_{s,0}(D)$ and $\tilde{L}^D(\tilde{\varphi}_\infty) = \hat{\Phi}'(D) = Q_{\frac{s}{2},\frac{\eta}{2}}(D)$. It is well known that as functions of D , $|D|^{\frac{1}{2}-t} Q_{t,k}(D)$ satisfy the differential equations of Whittaker functions ([Go, p. 2.20]). More precisely we have:

$$|D|^{\frac{1}{2}-t} Q_{t,k}(D) = \beta_{t,k,\operatorname{sgn}(D)} W_{\frac{1}{2}k\operatorname{sgn}(D),t}(4\pi|D|).$$

Here $\beta_{t,k,\pm}$ is some function of (t, k) to be determined. To determine these functions, we let D approach 0.

$$\lim_{D \rightarrow 0^+} Q_{t,k}(D) = \beta_{t,k,+} (4\pi)^{\frac{1}{2}-t} \frac{\Gamma(2t)}{\Gamma(t - \frac{k}{2} + \frac{1}{2})}.$$

$$\lim_{D \rightarrow 0^-} Q_{t,k}(D) = \beta_{t,k,-} (4\pi)^{\frac{1}{2}-t} \frac{\Gamma(2t)}{\Gamma(t + \frac{k}{2} + \frac{1}{2})}.$$

On the other hand $\lim_{D \rightarrow 0} Q_{t,k}(D) = e^{-i\pi k/2} F(t, k)$ where

$$(4.7) \quad F(t, k) = \int_{-\infty}^{\infty} (1+x^2)^{-t-\frac{1}{2}} e^{ik \tan^{-1}(x)} dx.$$

From the appendix, we have

$$(4.8) \quad F(t, k) = \sqrt{\pi} \frac{\Gamma(t)\Gamma(t + \frac{1}{2})}{\Gamma(t + \frac{1}{2} - \frac{k}{2})\Gamma(t + \frac{1}{2} + \frac{k}{2})}.$$

Using the doubling formula

$$(4.9) \quad \Gamma(2s) = \pi^{-\frac{1}{2}} 2^{2s-1} \Gamma(s)\Gamma(s + \frac{1}{2}),$$

we get

$$\begin{aligned} \beta_{t,k,+} &= e^{-i\pi k/2} (4\pi)^{t-\frac{1}{2}} \pi 2^{1-2t} / \Gamma(t + \frac{k}{2} + \frac{1}{2}), \\ \beta_{t,k,-} &= e^{-i\pi k/2} (4\pi)^{t-\frac{1}{2}} \pi 2^{1-2t} / \Gamma(t - \frac{k}{2} + \frac{1}{2}). \end{aligned}$$

We are interested in

$$\left| \frac{\tilde{L}^D(\tilde{\varphi}_\infty)}{L(\varphi_\infty)} \right|^2 = \left| \frac{Q_{\frac{s}{2}, \frac{\eta}{2}}(D)}{Q_{s,0}(1)} \right|^2 = \|D\|^{\frac{s-1}{2}} \frac{\beta_{\frac{s}{2}, \frac{\eta}{2}, \text{sgn}(D)}}{\beta_{s,0,+}} \left| \frac{W_{\frac{\eta}{4} \text{sgn}(D), \frac{s}{2}}(4\pi|D|)}{W_{0,s}(4\pi)} \right|^2.$$

The Lemma follows from the fact $\left| \frac{\beta_{\frac{s}{2}, \frac{\eta}{2}, \text{sgn}(D)}}{\beta_{s,0,+}} \right|^2$ equals

$$\left| (4\pi)^{-\frac{s}{2}} 2^s \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{s}{2} + \frac{1}{2} + \text{sgn}(D)\frac{\eta}{4})} \right|^2 = \left| \pi^{-\frac{s}{2}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{s}{2} + \frac{1}{2} + \text{sgn}(D)\frac{\eta}{4})} \right|^2.$$

□

4.2. Ratio of inner products. Here we separate the case of s imaginary (the unitary principal series case) and s real (the complementary series case).

Lemma 4.3. *When s is imaginary,*

$$\frac{\|\varphi_\infty\|^2}{\|\tilde{\varphi}_\infty\|^2} = |D|.$$

Proof. When s is purely imaginary, the inner product formula for GL_2 is

$$(4.10) \quad \|\varphi_\infty\|^2 = \int_{-\infty}^{\infty} |L(\pi_\infty(\underline{a})\varphi_\infty)|^2 \frac{da}{|a|}.$$

Since

$$(4.11) \quad L(\pi_\infty(\underline{a})\varphi_\infty) = |a|^{1/2-s} L^a(\varphi_\infty) = |a|^{1/2-s} \hat{\Phi}(a),$$

we get that $\|\varphi_\infty\|^2$ equals:

$$\int_{-\infty}^{\infty} |\hat{\Phi}(a)|^2 da = \int_{-\infty}^{\infty} |\Phi(x)|^2 dx = \int_{-\infty}^{\infty} (1+x^2)^{-1} dx = \pi.$$

Since we fixed our choice of \tilde{L}^D and \tilde{L}^{-D} , the inner product formula for the $\overline{\text{SL}}_2$ case does not necessarily take the form of (3.6). When s is purely imaginary, from [BM2, (16.1)], the inner product formula is:

$$(4.12) \quad \|\tilde{\varphi}_\infty\|^2 = \int_{-\infty}^{\infty} |\tilde{L}^D(\tilde{\pi}_\infty(\underline{a})\tilde{\varphi}_\infty)|^2 \frac{da}{|a|} + \int_{-\infty}^{\infty} |\tilde{L}^{-D}(\tilde{\pi}_\infty(\underline{a})\tilde{\varphi}_\infty)|^2 \frac{da}{|a|}.$$

Note

$$(4.13) \quad \tilde{L}^D(\tilde{\pi}_\infty(\underline{a})\tilde{\varphi}_\infty) = |a|^{1-s} \tilde{L}^{a^2 D}(\tilde{\varphi}_\infty) = |a|^{1-s} \hat{\Phi}'(a^2 D).$$

Thus we get

$$\|\tilde{\varphi}_\infty\|^2 = |D|^{-1} \int_{-\infty}^{\infty} |\hat{\Phi}'(a)|^2 da = |D|^{-1} \int_{-\infty}^{\infty} |\Phi'(a)|^2 da = \pi |D|^{-1}.$$

The Lemma follows. □

Now consider the case of complementary series. Here $0 < s < \frac{1}{2}$.

From [BM2, (16.2)]

$$(4.14) \quad \|\tilde{\varphi}_\infty\|^2 = \int_{-\infty}^{\infty} |\tilde{L}^D(\tilde{\pi}_\infty(\underline{a})\tilde{\varphi}_\infty)|^2 \frac{da}{|a|} + \frac{\Delta_{s,\eta}(-D)}{\Delta_{s,\eta}(D)} \int_{-\infty}^{\infty} |\tilde{L}^{-D}(\tilde{\pi}_\infty(\underline{a})\tilde{\varphi}_\infty)|^2 \frac{da}{|a|}.$$

Here

$$(4.15) \quad \Delta_{s,\eta}(D) = \Gamma(s)(e^{-is\pi \operatorname{sgn}(D)/2} - \eta i e^{is\pi \operatorname{sgn}(D)/2})$$

(the definition [BM2, (15.6)] has an error in negative sign).

Lemma 4.4.

$$\|\tilde{\varphi}_\infty\|^2 = \Delta_{s,\eta}(D)^{-1} e^{-\eta i \pi/4} F\left(\frac{s}{2}, \frac{\eta}{2}\right) (2\pi)^s |D|^{s-1} \pi.$$

Proof. From equation (4.13) and (4.14) we get $\Delta_{s,\eta}(D) \|\tilde{\varphi}_\infty\|^2$ equals

$$\int_{-\infty}^{\infty} \Delta_{s,\eta}(D) |Da^2|^{-s+1} |\hat{\Phi}'(Da^2)|^2 \frac{da}{|a|} + \int_{-\infty}^{\infty} \Delta_{s,\eta}(-D) |Da^2|^{-s+1} |\hat{\Phi}'(-Da^2)|^2 \frac{da}{|a|}.$$

The above two integrals can be combined into one, resulting in the equation:

$$(4.16) \quad \Delta_{s,\eta}(D) \|\tilde{\varphi}_\infty\|^2 = |D|^{s-1} \int_{-\infty}^{\infty} \Delta_{s,\eta}(b) |b|^{-s} |\hat{\Phi}'(b)|^2 db.$$

We claim the right hand side of the above equation equals

$$(4.17) \quad (2\pi)^s |D|^{s-1} \int_{-\infty}^{\infty} A \tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x)) \overline{\tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x))} dx.$$

Here A is the intertwining operator from $\tilde{\pi}_{s,\eta}$ to $\tilde{\pi}_{-s,\eta}$.

From [BM2, (15.7)],

$$A\tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x)) = \int \hat{\Phi}'(b)\Delta_{s,\eta}(b)(2\pi|b|)^{-s}e^{2\pi ibx} db.$$

Thus (4.17) is just

$$|D|^{s-1} \int \int_{-\infty}^{\infty} \overline{\Phi'(x)}e^{2\pi ibx}\hat{\Phi}'(b)\Delta_{s,\eta}(b)|b|^{-s} dx db.$$

Since $\int \overline{\Phi'(x)}e^{2\pi ibx} dx = \overline{\hat{\Phi}'(b)}$, we get the above equals the right hand side of (4.16).

Let $\tilde{\varphi}'_\infty$ be the weight $\eta/2$ vector in $V(\tilde{\pi}_{-s,\eta})$ with $\tilde{\varphi}'_\infty(I) = 1$, then

$$A\tilde{\varphi}_\infty(g) = \int_{-\infty}^{\infty} \tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x) \cdot g) dx = \tilde{\varphi}'_\infty(g) \int_{-\infty}^{\infty} \tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x)) dx.$$

The integral is $\int_{-\infty}^{\infty} \Phi'(x)dx$ which equals $e^{-\eta i\pi/4}F(\frac{s}{2}, \frac{\eta}{2})$.

Thus from (4.16) and (4.17), we get

$$(4.18) \quad \Delta_{s,\eta}(D)\|\tilde{\varphi}_\infty\|^2 = e^{-\eta i\pi/4}F(\frac{s}{2}, \frac{\eta}{2})(2\pi)^s|D|^{s-1} \int_{-\infty}^{\infty} \tilde{\varphi}'_\infty(\tilde{w} \cdot \tilde{n}(x))\overline{\tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x))} dx.$$

Observe that

$$\int_{-\infty}^{\infty} \tilde{\varphi}'_\infty(\tilde{w} \cdot \tilde{n}(x))\overline{\tilde{\varphi}_\infty(\tilde{w} \cdot \tilde{n}(x))} dx = \int_{-\infty}^{\infty} |\Phi'(x)|^2 dx = \int_{-\infty}^{\infty} (1+x^2)^{-1} dx = \pi.$$

From (4.18) we get the Lemma. □

The GL_2 case is similar, we have

$$\|\varphi_\infty\|^2 = \delta_{s,\eta}(1)^{-1}F(s,0)(2\pi)^{2s}\pi.$$

here

$$(4.19) \quad \delta_{s,\eta}(1) = \Gamma(2s)(e^{-is\pi} + e^{is\pi}).$$

Thus when s is real,

$$(4.20) \quad \frac{\|\varphi_\infty\|^2}{\|\tilde{\varphi}_\infty\|^2} = |D|^{1-s} \frac{\Delta_{s,\eta}(D)}{\delta_{s,\eta}(1)} \frac{F(s,0)}{F(\frac{s}{2}, \frac{\eta}{2})} (2\pi)^s e^{\eta i\pi/4}.$$

From (4.15) and (4.19)

$$\begin{aligned} \frac{\delta_{s,\eta}(1)}{\Delta_{s,\eta}(D)} &= \frac{\Gamma(2s)}{\Gamma(s)} (e^{-is\pi \operatorname{sgn}(D)/2} + \eta i e^{is\pi \operatorname{sgn}(D)/2}) \\ &= 2 \frac{\Gamma(2s)}{\Gamma(s)} e^{\eta i\pi/4} \sin(\pi(\frac{1}{2} - \frac{s}{2} - \operatorname{sgn}(D)\frac{\eta}{4})). \end{aligned}$$

Since

$$\sin(\pi x) = \frac{\pi}{\Gamma(x)\Gamma(1-x)},$$

we get

$$(4.21) \quad e^{\eta i\pi/4} \frac{\Delta_{s,\eta}(D)}{\delta_{s,\eta}(1)} = (2\pi)^{-1} \frac{\Gamma(s)\Gamma(\frac{1}{2} - \frac{s}{2} - \operatorname{sgn}(D)\frac{\eta}{4})\Gamma(\frac{1}{2} + \frac{s}{2} + \operatorname{sgn}(D)\frac{\eta}{4})}{\Gamma(2s)}$$

From (4.8) and (4.9):

$$\begin{aligned} F\left(\frac{s}{2}, \frac{\eta}{2}\right) &= \sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2} + \frac{1}{2}\right) / \left(\Gamma\left(\frac{s}{2} + \frac{3}{4}\right)\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)\right) \\ &= \sqrt{\pi} \frac{\sqrt{\pi}2^{1-s}\Gamma(s)}{\sqrt{\pi}2^{\frac{1}{2}-s}\Gamma\left(s + \frac{1}{2}\right)} \\ &= \sqrt{2\pi} \frac{\Gamma(s)}{\Gamma\left(s + \frac{1}{2}\right)}. \end{aligned}$$

From (4.8)

$$F(s, 0) = \sqrt{\pi}\Gamma(s)/\Gamma\left(s + \frac{1}{2}\right).$$

Thus $\frac{F(s,0)}{F(s/2,\eta/2)} = 2^{-\frac{1}{2}}$. From (4.20) and (4.21), we get

Lemma 4.5. *When s is real,*

$$\frac{\|\varphi_\infty\|^2}{\|\tilde{\varphi}_\infty\|^2} = |D|^{1-s} (2\pi)^{s-1} 2^{-1/2} \frac{\Gamma(s)\Gamma(\frac{1}{2} - \frac{s}{2} - \operatorname{sgn}(D)\frac{\eta}{4})\Gamma(\frac{1}{2} + \frac{s}{2} + \operatorname{sgn}(D)\frac{\eta}{4})}{\Gamma(2s)}.$$

4.3. Proof of the proposition 4.1. We now finish the computation of $\frac{e(\varphi_\infty, \psi)}{e(\tilde{\varphi}_\infty, \psi^D)}$. When s is purely imaginary we use Lemmas 4.2 and 4.3. All we need to check is:

$$(4.22) \quad \left| |D|^{\frac{s}{2}} \pi^{-\frac{s}{2}} \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2} + \operatorname{sgn}(D)\frac{\eta}{4}\right)} \right|^2 = (2\pi)^{-1} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} - \frac{s}{2}\right).$$

From (4.9),

$$\left| \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2} + \operatorname{sgn}(D)\frac{\eta}{4}\right)} \right|^2 = (2\pi)^{-1} |2^s \Gamma\left(\frac{s}{2} + \frac{1}{2} - \operatorname{sgn}(D)\frac{\eta}{4}\right)|^2.$$

Since s is purely imaginary, the equation (4.22) holds and the proposition follows in this case.

When s is real, from Lemmas 4.2 and 4.5 we need to check the right hand side of (4.22) equals:

$$(2\pi)^{-1}2^{s-\frac{1}{2}}\left(\frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{1}{2}+\operatorname{sgn}(D)\frac{\eta}{4})}\right)_2\frac{\Gamma(s)\Gamma(\frac{1}{2}-\frac{s}{2}-\operatorname{sgn}(D)\frac{\eta}{4})\Gamma(\frac{1}{2}+\frac{s}{2}+\operatorname{sgn}(D)\frac{\eta}{4})}{\Gamma(2s)}.$$

The equality again follows from formula (4.9).

Thus in both the s real or imaginary case we get the identity in the proposition.

5. PROOF OF THEOREMS

PROOF OF THEOREMS 1.1 AND 1.2: We first show the necessity of the conditions in Theorem 1.1. If $g(z)$ is a nonzero form in $S_{\pm\frac{1}{2}}(f, 4M, \chi)$ satisfying the Hecke eigenvalue condition (C1), let $\tilde{\varphi} = t(g)$. Then $\tilde{\varphi}$ is a finite combination of forms $\tilde{\varphi}_i$ in the spaces $V(\tilde{\pi}_i)$ of irreducible cuspidal representations $\tilde{\pi}_i$ of $\overline{\mathrm{SL}}_2$. Then the local components $\tilde{\pi}_{i,p}$ at $p \nmid N$ are isomorphic to that of $\Theta(\pi, \psi)_p$. In other words, $\tilde{\pi}_i$ is in the near equivalence class of $\Theta(\pi, \psi)$.

Waldspurger's strong multiplicity theorem ([W3, Theorem 3]) gives a beautiful description of a near equivalence class $\tilde{\Pi}$ of irreducible cuspidal representations. At a local place v , either $\tilde{\pi}_v$ equals a fixed principal series representation for all $\tilde{\pi} \in \tilde{\Pi}$, or $\tilde{\pi}_v$ belong to a packet consisting of two distinct representations that are not principal series representation. In the latter case, the two representations have different central character. In particular, since our choice of χ determines the central character of $\tilde{\pi}_i$, we see $\tilde{\pi}_i$ is actually the same representation $\tilde{\pi}$ for all i , being the unique element in the near equivalence class of $\Theta(\pi, \psi)$ with central character determined by χ .

We now describe the local components of $\tilde{\pi}$ (where $\tilde{\varphi}$ lies). At places $v = p \mid N$, $v = p = 2$ and $v = \infty$, the local components of π are principal series representations, thus so is that of $\Theta(\pi, \psi)_v$, which implies $\tilde{\pi}_v = \Theta(\pi, \psi)_v$. In particular, at ∞ , $\tilde{\pi}_\infty$ is of the form $\tilde{\pi}_{s,\eta}$ with $\eta = \eta_f$. Since $V(\tilde{\pi}_{s,\pm})$ does not contain vectors of weight $\mp\frac{1}{2}$, we get $S_\epsilon(f, 4M, \chi) = \{0\}$ when $\epsilon = -\frac{1}{2}\eta_f$.

At a place $v = l \mid N$, π_v is a special representation, and $\tilde{\pi}_v$ is not a principal series representation. The two possibilities for $\tilde{\pi}_v$ are either a special representation or an odd Weil representation, determined by the value of $\chi_{(l)}(-1)$. At $l \in T'_\chi$, $\tilde{\pi}_l$ is an odd Weil representation (see [BM1, §8.3]). At $l \in T - T'_\chi$, $\tilde{\pi}_l$ is a special representation. The lowest level of

vectors in $V(\tilde{\pi}_l)$ is l^2 when $\tilde{\pi}_l$ is an odd Weil representation. When $\tilde{\pi}_l$ is a special representation, the lowest level of vectors is l . Thus the level for $g(z)$ is divisible by $N(\chi)$. We get $S_\epsilon(f, 4M, \chi) = \{0\}$ when $N(\chi) \nmid M$.

Next we show $S^+(f, \chi)$ is non-empty. Let $\tilde{\pi}$ be the cuspidal representation determined by χ as above, let $\tilde{\varphi}_0 = \otimes \tilde{\varphi}_{0,v}$ be a vector in $V(\tilde{\pi})$ as follows: $\tilde{\varphi}_{0,\infty}$ lies in weight $\frac{1}{2}\eta_f$ space of $\tilde{\pi}_{0,\infty}$ which is one dimensional; for $p \nmid 2$, $\tilde{\varphi}_{0,p}$ lies in the space of lowest level vectors which is one dimensional (see [BM1, §8] for description of these vectors); for $p = 2$, $\tilde{\varphi}_{0,2}$ is the Kohnen vector described in [BM1, §9.4] (with $k = 0$), which is also uniquely determined. Then $t^{-1}(\tilde{\varphi}_0)$ lies in $S^+(f, \chi)$.

If ϵ and M satisfy the conditions in Theorem 1.1, then $S^+(f, \chi) \subset S_\epsilon(f, 4M, \chi)$; thus $S_\epsilon(f, 4M, \chi)$ is nonzero. We have shown Theorem 1.1.

Now let $g \in S^+(f, \chi)$ and $\tilde{\varphi} = t(g) = \sum_i \otimes \tilde{\varphi}_{i,v}$ a finite sum. The restriction on the level of $g(z)$ implies that $\tilde{\varphi}_{i,p}$ is a multiple of $\tilde{\varphi}_{0,p}$ when $p \neq 2$. When $p = 2$, the vanishing condition (C2) of Fourier coefficients of $g(z)$ implies $\tilde{\varphi}_{i,2}$ is a multiple of $\tilde{\varphi}_{0,2}$ ([BM1]). Thus $\tilde{\varphi}$ must be a multiple of $\tilde{\varphi}_0$ and the space $S^+(f, \chi)$ is one dimensional. \square

PROOF OF PROPOSITION 1.3 AND THEOREM 1.4: Let $g(z) = g_{f,\chi}(z)$ be the generator of $S^+(f, \chi)$ and $\tilde{\varphi} = t(g)$. If $\tilde{\varphi} \in \tilde{\pi}$, the description of the near equivalence class implies there is a $D_0 \in \mathbb{Q}^*$ so that $\tilde{\pi} = \Theta(\pi \otimes \chi_{D_0}, \psi^{D_0})$. We can proceed as in [BM1, §10.4] to find such a D_0 . [BM1, Corollary 10.7] states that we can take D_0 to be an integer such that $(D_0, N) = 1$ and for $p \in T$, $\left(\frac{D_0}{p}\right) = w_p \chi_{(p)}(-1)$. We will fix a choice of such a D_0 . We remark that unlike the situation for holomorphic modular forms, here $\tilde{\pi}_\infty$ is a principal series representation, thus independent of choice of D_0 . In particular we no longer have the sign restriction on D_0 described in [BM1, Corollary 10.7].

Proposition 1.3 follows from local vanishing of Whittaker functionals. When $\tilde{\pi}_p$ is a special representation, and $\tilde{\varphi}$ is the lowest level vector in $V(\tilde{\pi}_p)$, the local Whittaker functional \tilde{L}_p^n satisfies $\tilde{L}_p^n(\tilde{\varphi}_p) = 0$ for n in a particular square class of \mathbb{Q}_p^* , (see [BM1, §8.3.2]). In our case, when $\tilde{\pi}_p = \Theta(\pi_p \otimes \chi_{D_0}, \psi^{D_0})$, $\tilde{L}_p^n(\tilde{\varphi}_p) = 0$ when n is a unit in \mathbb{Z}_p which does not lie in the same square class of D_0 in $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$. Thus when $\left(\frac{n}{p}\right) = -w_p \chi_{(p)}(-1)$ for some $p \in T$, $\tilde{W}^n(\tilde{\varphi}) = 0$ and by (2.8) we get $c(n) = 0$. On the other hand, if $\tilde{\pi}_p$ is an odd Weil representation, and $\tilde{\varphi}_p$ is a lowest level vector in $V(\tilde{\pi}_p)$, then $\tilde{L}_p^n(\tilde{\varphi}_p) \neq 0$ for n in only one square class of \mathbb{Q}_p^* (see

[BM1, §8.3.3]). In particular $\tilde{L}_p^n(\tilde{\varphi}) = 0$ if $p|n$. Thus if $l|n$ for some $l \in T'_\chi$, we have the local Whittaker functional \tilde{L}_l^n vanishes at $\tilde{\varphi}_l$, and thus $c(n) = 0$. Proposition 1.3 follows.

Now assume D is a fundamental discriminant such that $\left(\frac{D}{p}\right) \neq -w_p\chi(p)(-1)$ for all $p \in T$ and $l \nmid D$ when $l \in T'_\chi$. Then $\Theta(\pi \otimes \chi_D, \psi^D) = \Theta(\pi \otimes \chi_{D_0}, \psi^{D_0}) = \tilde{\pi}$ by [BM1, Corollary 10.7]. Now we can apply (3.1). Let $\varphi = s(f)$. Along with (2.7), (2.8) and (3.7) we get:

$$(5.1) \quad \frac{|c(D)|^2 |W_{\frac{\eta}{4} \operatorname{sgn}(D), \frac{s}{2}}(4\pi|D)|^2}{\|\tilde{\varphi}\|^2} = \frac{L(f, D, \frac{1}{2}) |W_{0,s}(4\pi)|^2}{\|\varphi\|^2} \kappa$$

with

$$\kappa = L(\pi_{f,\infty} \otimes \chi_D, \frac{1}{2}) \prod_{p|2N, p \neq \infty} E_p(\varphi_p, \tilde{\varphi}_p, \psi, D).$$

Proposition 4.1 shows that $L(\pi_{f,\infty} \otimes \chi_D, \frac{1}{2}) E_\infty(\varphi_\infty, \tilde{\varphi}_\infty, \psi, D)$ equals

$$(2\pi|D|)^{-1} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} - \frac{s}{2}\right) \left| \frac{W_{\frac{\eta}{4} \operatorname{sgn}(D), \frac{s}{2}}(4\pi|D|)}{W_{0,s}(4\pi)} \right|^2.$$

Thus (5.1) becomes:

$$(5.2) \quad \frac{|c(D)|^2}{\|\tilde{\varphi}\|^2} = \frac{L(f, D, \frac{1}{2})}{\|\varphi\|^2} (2\pi|D|)^{-1} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(D)\eta}{4} - \frac{s}{2}\right) \prod_{p|2N} E_p(\varphi_p, \tilde{\varphi}_p, \psi, D)$$

From [BM1, Lemma 9.1], we have $\frac{\|\varphi\|^2}{\langle f, f \rangle} = \frac{\|\tilde{\varphi}\|^2}{\langle g, g \rangle}$. The values of $E_p(\varphi_p, \tilde{\varphi}_p, \psi, D)$ for $p|2N$ are given in [BM1, (10.4)]. We have if $p = 2$, $E_p(\varphi_p, \tilde{\varphi}_p, \psi, D) = 2$; if $p \in T - T'_\chi$, $E_p(\varphi_p, \tilde{\varphi}_p, \psi, D) = 2$ if $p \nmid D$ and 1 if $p|D$; if $p \in T'_\chi$, $E_p(\varphi_p, \tilde{\varphi}_p, \psi, D) = 2\frac{p}{p+1}$. Putting these data together we get (1.2) from (5.2). \square

PROOF OF THEOREM 1.5: First assume $g(z)$ in the theorem is of the form $t(\tilde{\varphi})$ where $\tilde{\varphi}$ lies in an irreducible representation $\tilde{\pi}$ of $\overline{\mathrm{SL}}_2$. Assume $c(n) \neq 0$, then $\tilde{\pi} = \Theta(\pi \otimes \chi_n, \psi^n)$ for some irreducible representation π . Let $f(z)$ be the newform corresponding to φ the lowest level vector in π . As in the proof of Theorem 1.4, From (3.1), (2.7), (2.8) and Proposition 4.1 we get equation (5.2), which we restate here:

$$(5.3) \quad \frac{|c(n)|^2}{\langle g, g \rangle} = \frac{L^f(\pi \otimes \chi_n, \frac{1}{2})}{\langle f, f \rangle} (2\pi|n|)^{-1} \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(n)\eta}{4} + t\right) \Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn}(n)\eta}{4} - t\right) \prod_{p|2M} E_p(\varphi_p, \tilde{\varphi}_p, \psi, n)$$

Here we have $t = \frac{s}{2}$ as in the situation of Theorem 1.4, $\Delta_{\pm\frac{1}{2}}g = -(\frac{1-s^2}{4})g$. The L -function $L^f(\pi \otimes \chi_n, s)$ is defined to be $L(\pi \otimes \chi_n, s)/L_\infty(\pi_\infty \otimes \chi_n, s)$. Given n , there is a fundamental

discriminant D with $n = D\delta^2/4$ for some integer δ . Then $\chi_n = \chi_D$ and by (3.7)

$$L^f(\pi \otimes \chi_n, s) = L(\pi \otimes \chi_D, s)/L_\infty(\pi_\infty \otimes \chi_D, s) = L(f, D, s).$$

Thus from (1.3) we get

$$(5.4) \quad L^f(\pi \otimes \chi_n, \frac{1}{2}) \ll_\epsilon ((1 + |t|)N|n|)^\epsilon (1 + |t|)^B N^C |n|^{\frac{1}{2} - \frac{1}{8}(1-2\theta)}.$$

Since $|\Gamma(x + iy)|$ is asymptotically $\sqrt{2\pi}|y|^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}$ as $|y| \mapsto \infty$, we have:

$$(5.5) \quad \Gamma(\frac{1}{2} - \frac{\operatorname{sgn}(n)\eta}{4} + t)\Gamma(\frac{1}{2} - \frac{\operatorname{sgn}(n)\eta}{4} - t) \ll (1 + |t|)^{-\frac{1}{2}\operatorname{sgn}(n)\eta}.$$

In [M1], the bounds for $E_p(\varphi_p, \tilde{\varphi}_p, \psi, n)$ are given. One has (see [M1, §7.4]):

$$(5.6) \quad \prod_{p|2M} E_p(\varphi_p, \tilde{\varphi}_p, \psi, n) \ll_\epsilon M^{2+\epsilon}.$$

Since $\langle g, g \rangle = 1$, we get from (5.3), (5.4), (5.5) and (5.6):

$$\begin{aligned} \langle f, f \rangle |c(n)|^2 &\ll_\epsilon |n|^{-1} M^{2+\epsilon} (1 + |t|)^{-\frac{1}{2}\operatorname{sgn}(n)\eta} e^{-\pi|t|} \\ &\quad ((1 + |t|)N|n|)^\epsilon (1 + |t|)^B N^C |n|^{\frac{1}{2} - \frac{1}{8}(1-2\theta)}. \end{aligned}$$

The level N of $f(z)$ is bounded by M : from [M1, Lemma 7.1], $N|(2M)^\infty$ and $N \leq 4M^2$. Thus:

$$(5.7) \quad \langle f, f \rangle |c(n)|^2 \ll_\epsilon ((1 + |t|)M|n|)^\epsilon (1 + |t|)^{-\frac{1}{2}\operatorname{sgn}(n)\eta+B} e^{-\pi|t|} M^{2C+2} |n|^{-\frac{1}{2} - \frac{1}{8}(1-2\theta)}.$$

[HL, Corollary 0.3] states that $\langle f, f \rangle^{-1} \ll_\epsilon ((1 + |t|)M)^\epsilon e^{2\pi|t|}$. Thus (5.7) gives:

$$(5.8) \quad |c(n)| \ll_\epsilon ((1 + |t|)M|n|)^\epsilon e^{\frac{\pi|t|}{2}} (1 + |t|)^{-\frac{1}{4}\operatorname{sgn}(n)\eta + \frac{B}{2}} M^{C+1} |n|^{-\frac{1}{4} - \frac{1}{16}(1-2\theta)}.$$

Now an arbitrary $g(z)$ with $\langle g, g \rangle = 1$ is a linear combination $\sum_i b_i g_i(z)$ where $\langle g_i, g_j \rangle = \delta_{i,j}$ (thus $\sum_i |b_i|^2 = 1$) and the Fourier coefficients of g_i satisfy the bound of (5.8). From the Cauchy-Schwartz inequality,

$$(5.9) \quad |c(n)|^2 \leq \sum_i |c_i(n)|^2,$$

where $g_i(z) = \sum c_i(n)e(nz)$. From Selberg's work on Weyl's law, we have dimension of the space of level N weight 0 Maass forms f with $\Delta_0 f = -(\frac{1}{4} + (2t)^2)f$ is $\ll_\epsilon (N(1 + |t|))^{1+\epsilon}$. Through theta correspondence, dimension of $S_{\pm\frac{1}{2}}(4M)$ has the same bound with $N = M^2$.

We get the Theorem from the Cauchy-Schwartz inequality (5.9) and the bound (5.8). \square

6. APPENDIX

We prove equation (4.8) here.

Proof. We make a change of variable in (4.7), $\tan(\theta) = x$. Then we have:

$$\begin{aligned} F(t, k) &= \int_{-\infty}^{\infty} (1+x^2)^{-t+\frac{1}{2}} \left((1+x^2)^{-1} e^{ik \tan^{-1}(x)} \right) dx \\ &= \int_{-\pi/2}^{\pi/2} \cos(\theta)^{2t-1} e^{ik\theta} d\theta \\ &= 2 \int_0^{\pi/2} \cos(\theta)^{2t-1} \cos(k\theta) d\theta. \end{aligned}$$

By [GR, p.391, formula 9],

$$F(t, k) = \frac{2\pi\Gamma(2t+1)}{2^{2t}2t\Gamma(t+\frac{k}{2}+\frac{1}{2})\Gamma(t-\frac{k}{2}+\frac{1}{2})}.$$

Thus from (4.9),

$$F(t, k) = \frac{2^{1-2t}\pi\Gamma(2t)}{\Gamma(t+\frac{k}{2}+\frac{1}{2})\Gamma(t-\frac{k}{2}+\frac{1}{2})} = \frac{\sqrt{\pi}\Gamma(t)\Gamma(t+\frac{1}{2})}{\Gamma(t+\frac{k}{2}+\frac{1}{2})\Gamma(t-\frac{k}{2}+\frac{1}{2})}.$$

□

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