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A fundamental lemma for metaplectic correspondence

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Abstract. We state a relative trace formula which will lead to the classification of the image of metaplectic correspondence as a set of distinguished representations. We prove the fundamental lemma for the relative trace formula in the case of $GL(3)$.

1. Introduction

In this paper, we will denote by $\tilde{G}L_n$ a twofold cover of the group GL_n over a local field or an adele ring, as defined in [KP]. Our purpose is to state a form of metaplectic correspondence and provide some evidences to it.

Let F be a number field; $\mathbb{A}$ its adele ring. We will write an element in $\tilde{G}L_n(\mathbb{A})$ as (g,z) with $g \in GL_n(\mathbb{A})$ and $z \in \{\pm 1\}$. The group $GL_n(F)$ of F-rational points embeds in $\tilde{G}L_n(\mathbb{A})$ by the map $g \to (g,1)$. We say a function f on $\tilde{G}L_n(\mathbb{A})$ is *genuine* if it satisfies $f(g,z) = f(g,1)z$. Denote by $\tilde{L}^2$ the subspace of $L^2\big(GL_n(F)\backslash\tilde{G}L_n(\mathbb{A})\big)$ consisting of genuine functions. A constituent of the $\tilde{G}L_n$-module $\tilde{L}^2$ is called a *genuine* automorphic representation. The metaplectic correspondence is a lifting of the genuine automorphic representations of $\tilde{G}L_n$ to the automorphic representations of GL_n. The $n = 2$ case of the metaplectic correspondence has been studied by many authors [F], [G-PS], [J], [J2], [W]. In [J], Jacquet suggested the following characterization of the image of the lifting:

> A cuspidal automorphic representation π of GL_n with trivial central character is a lifting from $\tilde{G}L_n$ if and only if π is (H_ε, χ)-distinguished for some $\varepsilon = {}^t\varepsilon$ and some quadratic character χ of $\mathbb{A}^\times/F^\times$.

Here H_ε is the similitude group

$$\{g \in GL_n \,|\, {}^tg\varepsilon g = \lambda(g)\varepsilon, \lambda(g) \text{ is a scalar}\}\,.$$

Recall that a cuspidal automorphic representation π is (H_ε, χ)-distinguished if for some ϕ lying in the space of π, we have (Z being the center of GL_n)

$$(1) \qquad \int\limits_{Z \cap H_\varepsilon(\mathbb{A}) H_\varepsilon(F) \backslash H_\varepsilon(\mathbb{A})} \phi(hg)\,\chi(\lambda(h))\,dh \not\equiv 0 .$$

The above characterization of the metaplectic correspondence follows from a relative trace formula which we now formulate. Fix a quadratic character χ on $\mathbb{A}^\times/F^\times$. Let S_n be the variety

$$\{g \in GL_n \mid {}^t g = g\} .$$

Let Φ be a function in $C_c^\infty(S_n(\mathbb{A}))$; we define a kernel function K_Φ on $GL_n(\mathbb{A})$ by:

$$(2) \qquad K_\Phi(g) = \int\limits_{\mathbb{A}^\times/F^\times} \sum_{s \in S_n(F)} \Phi({}^t gasg)\,\chi(a)\,d^\times a .$$

Let f be a genuine function in $C_c^\infty(\tilde{G}L_n(\mathbb{A}))$; for $x, y \in \tilde{G}L_n(\mathbb{A})$, let

$$(3) \qquad K_f(x,y) = \int\limits_{\mathbb{A}^\times/F^\times} \sum_{\xi \in GL(n,F)} f(x^{-1} a \xi y)\,\chi(a)\,d^\times a .$$

The unit upper triangular subgroup $N(\mathbb{A})$ of $GL_n(\mathbb{A})$ embeds in $\tilde{G}L_n(\mathbb{A})$ by the homomorphism $n \to (n,1)$. We will identify $N(\mathbb{A})$ with its embedding in $\tilde{G}L_n(\mathbb{A})$. Fix a nontrivial additive character ψ of $\mathbb{A}/F$; we define characters θ, θ' on $N(\mathbb{A})$: if $n = [x_{i,j}] \in N(\mathbb{A})$, then

$$(4) \qquad \theta'(n) = \psi\left(\frac{x_{1,2} + x_{2,3} + \cdots + x_{n-1,n}}{2}\right), \quad \theta(n) = \theta'(n)^2 .$$

The trace formula we have in mind is the following: for sufficiently many pairs of functions Φ and f

$$(5) \qquad \int\limits_{N(F)\backslash N(\mathbb{A})} K_\Phi(n)\,\theta(n)\,dn = \int\limits_{(N(F)\backslash N(\mathbb{A}))^2} K_f(n_1, n_2)\,\theta'(n_1^{-1} n_2)\,dn_1\,dn_2 .$$

In [J], the derivation of metaplectic correspondence from the above trace formula is carried out for the GL_2 case. We refer to [J], [J2], [J3], [JR], [JY] for the discussions on the trace formulas in general and their application to the lifting of automorphic representations. Our interest here is to study the related local orbital integrals. We will show some identities between these orbital integrals. At the moment, these identities are the strongest evidences that the above conjecture on metaplectic correspondence holds.

We now define the local orbital integrals. From now on, let F be a local non-Archimedean field. Let R be its integer ring, P the prime ideal, ϖ a uniformizer in P. Assume $q = \# R/P$ is odd. We will again denote by ψ an additive character on F and by θ, θ' the corresponding characters on N. The group N acts on S_n by $\delta(n): s \to {}^t nsn$, and $N \times N$ acts on GL_n by $\delta'(n_1, n_2): g \to n_1^{-1} g n_2$. Denote by N_s (or $(N \times N)_g$) the fixator of s (or g). We say an N-orbit $\{s\}$ (or $N \times N$-orbit $\{g\}$) on S_n (or GL_n) is *relevant* if $\theta(n) \equiv 1$ (or $\theta'(n_1^{-1} n_2) \equiv 1$) for $n \in N_s$ (or $n_1, n_2 \in (N \times N)_g$). For the relevant orbits $\{s\}$, $\{g\}$, for $\Phi \in C_c^\infty(S_n(F))$ and f a genuine function in $C_c^\infty(\tilde{G}L_n(F))$, define the orbital integrals:

$$(6) \qquad I(s, \Phi) = \int\limits_{N/N_s} \Phi({}^t nsn)\,\theta(n)\,dn ,$$

$$J'(g,f) = \int_{(N\times N)/(N\times N)_g} f(n_1^{-1} g n_2)\,\theta'(n_1^{-1} n_2)\,dn_1 dn_2\,. \tag{7}$$

In § 2, we classify the sets of relevant orbits. One has the following result:

Theorem 1. *The relevant N-orbits on S_n have the representatives of the form $w\mathbf{a}$ where w is the longest Weyl element of a standard parabolic subgroup in GL_n and $\mathbf{a}$ lies in the center of the corresponding Levi subgroup. The relevant $N\times N$-orbits on GL_n have the representatives of the form $w_0 w\mathbf{a}$ with w, $\mathbf{a}$ being as above and w_0 being the longest Weyl element of GL_n.*

The second part of the above theorem is well known, see [Fr], [G], [JR] and [St].

From the theorem, there is a matching between the relevant N-orbits on S_n and relevant $N\times N$-orbits on GL_n, given by $\{w\mathbf{a}\} \leftrightarrow \{w_0 w\mathbf{a}\}$. We will let $J(w\mathbf{a},f)$ be $J'(w_0 w\mathbf{a},f)$. To establish the identity (5), one needs to compare the local orbital integrals $I(w\mathbf{a},\Phi)$ and $J(w\mathbf{a},f)$.

Let K be the maximal compact subgroup $GL_n(R)$. There is a map $\kappa : K \to \{\pm 1\}$ such that $g \to (g,\kappa(g))$ is a homomorphism from K to $\tilde{G}L_n$ (see § 3). Let

$$\Phi_0(g) = \begin{cases} 1, & g\in K\cap S, \\ 0, & g\notin K\cap S, \end{cases} \qquad f_0(g,z) = \begin{cases} \kappa(g)z, & g\in K, \\ 0, & g\notin K. \end{cases}$$

Our main result is the following identity between the orbital integrals (with suitably fixed measure):

Theorem 2. *When $n=3$, ψ is of order 0, we have*

$$J(w\mathbf{a},f_0) = \mu(w\mathbf{a})\,I(w\mathbf{a},\Phi_0) \tag{8}$$

with $\mu(w\mathbf{a})$ being

1. $w\mathbf{a} = \begin{bmatrix} & & a \\ & a & \\ a & & \end{bmatrix}$, $\mu(w\mathbf{a}) = 1$,

2. $w\mathbf{a} = \begin{bmatrix} & a & \\ a & & \\ & & b \end{bmatrix}$, $\mu(w\mathbf{a}) = |a|^{-3/2}\gamma(a,\psi)$,

3. $w\mathbf{a} = \begin{bmatrix} a & & \\ & & b \\ & b & \end{bmatrix}$, $\mu(w\mathbf{a}) = |b|^{3/2}\gamma(b,\psi)$,

$$4.\quad w\mathbf{a}=\mathbf{a}=\begin{bmatrix} & & a\\ & b & \\ c & & \end{bmatrix},\quad \mu(\mathbf{a})=|a|^{-1}|b|^{-1/2}\gamma(a,\psi)\gamma(-c,\psi)\,.$$

Here $\gamma(a,\psi)$ is the Weil constant defined by the formula:

$$(9)\qquad \int \Phi^\wedge(x)\,\psi\left(\frac{1}{2}ax^2\right)dx=|a|^{-1/2}\gamma(a,\psi)\int \Phi(x)\,\psi\left(-\frac{1}{2}a^{-1}x^2\right)dx$$

where Φ is any Schwartz function on F and $\Phi^\wedge$ is its Fourier transform:

$$\Phi^\wedge(x)=\int \Phi(y)\,\psi(xy)\,dy\,.$$

Theorem 2 is the *fundamental lemma for the relative trace formula* (5) in the case of GL_3. See [J2], [JY], [JY2] for some other cases of fundamental lemmas.

We will show a proof of the theorem for case 4 which is most difficult. Here we fix the measure on N so that the volume of $K\cap N$ equals 1. Denote $J(w\mathbf{a},f_0)$ and $I(w\mathbf{a},\Phi_0)$ in this case by $J(\mathbf{a})$ and $I(\mathbf{a})$ respectively. Our result follows from the observation that the main parts of the integrals $I(\mathbf{a})$, $J(\mathbf{a})$ can be written as convolutions of orbital integrals over GL_2 (see equations (31) and (50)). One can then apply the fundamental lemma for the GL_2 case, which is proven in [J], [J2]. Our approach here is a new method in proving fundamental lemmas; it can be used to provide an easier proof for the results in [JY2].

There are other characterizations for the image of metaplectic correspondence. In [FK], strong evidence is provided to the following statement:

> A cuspidal automorphic representation π of GL_n lies in the image of the metaplectic correspondence if and only if all its local components are metic.

A representation is said to be *metic* if it is equivalent to a representation unitarily induced from an $M=\prod_i M_i$-module $\prod_i \sigma_i\nu^{s_i}$ where $M_i=GL_{r_i}$ with $\sum_i r_i=n$, σ_i are square-integrable M_i-modules whose central characters are trivial on $\{\pm 1\}$, and s_i are real numbers. Taking into account of both characterizations of the image of metaplectic correspondence, we get the implication that if π is (H_ε,χ)-distinguished, then all its local components are metic.

In § 2, we characterize the relevant orbits. In § 3, necessary information on the metaplectic group is provided. In § 4, we prove Theorem 2 for some simple cases. In § 5 and § 6, we compute $J(\mathbf{a})$ and $I(\mathbf{a})$. The comparison between $I(\mathbf{a})$ and $J(\mathbf{a})$ is done in § 7.

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2. Relevant orbits

In this section, we classify the relevant orbits and prove Theorem 1. Let T be the group of diagonal matrices and W the Weyl group of permutation matrices in GL_n. Let Φ^+ be the set of positive roots of T identified with the set $\{(i,j) \mid 1 \leqq i < j \leqq n\}$. Let Δ be the set of simple roots $\{(i,i+1)\}$. For each root α, we denote by X_α the corresponding root vector.

By Theorem 4.2 in [S], an element $s \in S_n$ can be written as $s = {}^t n w \mathbf{a} n$ with $w\mathbf{a} \in S_n$, $w \in W$, $\mathbf{a} \in T$, $n \in N$. Each N-orbit in S_n has a representative of the form $w\mathbf{a}$ with $w \in W$, $\mathbf{a} \in T$. Moreover $w\mathbf{a} \in S_n$ implies that $w^2 = e$ and $w\mathbf{a}w = \mathbf{a}$. Assume the N-orbit $\{w\mathbf{a}\}$ in S_n is relevant, then w satisfies the following property:

Lemma 1. *If* $\alpha \in \Delta$ *and* $w\alpha < 0$, *then* $\beta = -w\alpha \in \Delta$.

Proof. We consider

$$n = 1 + xX_\alpha + yX_\beta + zX_{\alpha+\beta} \in N$$

where $X_{\alpha+\beta} = 0$ if $\alpha + \beta$ is not a root. Then

$$\begin{aligned} {}^t n w \mathbf{a} n \mathbf{a}^{-1} w^{-1} = 1 &+ \big(x + y\beta(\mathbf{a})\big) X_{-\alpha} + \big(y + x\alpha(\mathbf{a})\big) X_{-\beta} \\ &+ \big(z\big(1 + \alpha(\mathbf{a})\beta(\mathbf{a})\big) + u\big) X_{-\alpha-\beta} \end{aligned}$$

where u depends only on (x,y) but not on z. As $w^2 = e$ and $w\mathbf{a}w = \mathbf{a}$, we have $\beta(\mathbf{a}) = \alpha(\mathbf{a})^{-1}$. Thus if $y = -\alpha(\mathbf{a})x$ and $z = -u/2$, then n is in the fixator of $w\mathbf{a}$. If β is not a simple root, then for this n, $\theta(n) = \psi(x)$ may be not 1, a contradiction to the definition of relevant orbit. □

Let Θ_w be the set of simple roots α such that $w\alpha < 0$. Since $w^2 = 1$, by above lemma, we have $w(\Theta_w) = -\Theta_w$. According to Lemma 1 in [JR], Θ_w then determines a standard parabolic subgroup; and w is the longest Weyl element in its Levi component L. If $\alpha \in \Theta_w$, let n be as in the proof of the lemma such that $n \in N_{w\mathbf{a}}$. Then $\theta(n) = \psi\big(x - \alpha(\mathbf{a})x\big)$. The fact that $\{w\mathbf{a}\}$ is relevant implies $\alpha(\mathbf{a}) = 1$. Thus $\mathbf{a}$ must lie in the center of L. Therefore if $\{w\mathbf{a}\}$ is a relevant orbit, w and $\mathbf{a}$ must be as described in Theorem 1. The converse is easily checked.

The relevant $N \times N$-orbits on GL_n is classified in [Fr], [G], [JR] and [St]. The corresponding statement in Theorem 1 follows immediately from the Proposition 1 of [JR].

3. The metaplectic group $\tilde{G}L_n$

Recall that the Hilbert symbol is a nondegenerate bilinear form on $F^\times/F^{\times 2}$ taking values in $\{\pm 1\}$. We will denote it by $[a,b]$.

We say an element in $F^\times$ is *even* (or *odd*) if its valuation is even (or odd). The following properties of the Hilbert symbol are well known.

Proposition 1. *For a, b, $c \in F^\times$:*

1. $[a,b]=[b,a]$.
2. $[a,bc]=[a,b][a,c]$.
3. $[a^2,b]=1$, *in particular*, $[a,b]=1$ *if* $1-a\in P$.
4. $[a,-a]=[a,1-a]=1$.
5. *If a, b are even, then* $[a,b]=1$.
6. *If a is even, b is odd, then* $[a,b]=[a,\varpi]$.
7. *If a, b are odd, then* $[a,b]=[ab,\varpi]$.
8. *If a is even,* $[a,\varpi]=1$ *if and only if a is a square.* □

The definition of the metaplectic group is given in [KP]. For $g\in GL_n$, write $R(g)=m$ if $g=n_1mn_2$ with $n_1,n_2\in N$ and $m\in W\times T$. The group $\tilde{G}L_n$ is a set of elements (g,z) with $g\in GL_n$, $z\in\{\pm 1\}$ with multiplication:

$$(g_1,z_1)(g_2,z_2)=\big(g_1g_2,z_1z_2\sigma(g_1,g_2)\big)$$

where $\sigma: GL_n\times GL_n\to\{\pm 1\}$ is the unique 2-cocycle satisfying:

1. $\sigma(\mathbf{a},\mathbf{a}')=\prod\limits_{i<j}[a_i,a_j']$ if $\mathbf{a}=\mathrm{diag}[a_i]$ and $\mathbf{a}'=\mathrm{diag}[a_i']$.
2. $\sigma(\mathbf{a},w)=\sigma(w,w')=1$ if $w,w'\in W$ and $\mathbf{a}\in T$.
3. $\sigma(w,\mathbf{a})=\prod\limits_{(i,j)\in\Theta(w)}[a_i,a_j][-1,a_i/a_j][\det(w),\det(\mathbf{a})]$ if $w\in W$, $\mathbf{a}=\mathrm{diag}[a_i]$. Here $\Theta(w)$ is the set of positive roots α such that $w\alpha<0$.
4. $\sigma(ng,g'n')=\sigma(g,g')$.
5. $\sigma(\mathbf{a},g)=\sigma\big(\mathbf{a},R(g)\big)$ when $\mathbf{a}\in T$.
6. $\sigma(s_\alpha,g)=\sigma\big(R(s_\alpha g)\,R(g)\big)$ if s_α is the reflection associated with the root $\alpha\in\Delta$.

There is a homomorphism from $K=GL_n(R)$ to $\tilde{G}L_n$ given by $g\to\big(g,\kappa(g)\big)$, where κ is a map $K\to\{\pm 1\}$ satisfying

$$\kappa|T\cap K=\kappa|W=\kappa|N\cap K=1\,.$$

Such a map κ is clearly unique. We will use the following properties of σ and κ which are easily verified from their definitions.

Lemma 2. 1. $\kappa(ng)=\kappa(gn)=\kappa(g)$ *if* $n\in N\cap K$, $g\in K$.

2. $\kappa\left(\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix}\right) = \kappa\left(\begin{bmatrix} & g_1 \\ g_2 & \end{bmatrix}\right) = \kappa(g_1)\kappa(g_2)$ *if* $\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix} \in K$.

3. $\kappa(g_1 g_2) = \kappa(g_1)\kappa(g_2)\sigma(g_1, g_2)$, *if* $g_1, g_2 \in K$.

4. $\sigma(g, g^{-1}) = \sigma(g^{-1}, g)$.

5. $\sigma(\mathbf{a}_1 w_1, \mathbf{a}_2 w_2) = \sigma(\mathbf{a}_1, w_1 \mathbf{a}_2 w_1^{-1})\sigma(w_1, \mathbf{a}_2)$ *if* $\mathbf{a}_i \in T$, $w_i \in W$, $i = 1, 2$.

6. $\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3)$. □

In the following sections, we will also need some properties of the Weil constant $\gamma(a, \psi)$:

Proposition 2. *When* ψ *is of order* 0:

1. $\gamma(a, \psi) = 1$ *if* a *is even.*

2. $\gamma(a, \psi) = q^{-1/2} \sum \psi(a\varpi^{2r} b/2)[b, \varpi]$, $b \in R/P$, *if* $|a| = q^{2r+1}$.

3. $\gamma(a, \psi)^{-1} = \gamma(-a, \psi)$.

4. $\gamma(ab, \psi) = \gamma(a, \psi)\gamma(b, \psi)[a, b]$.

5. $\gamma(a, \psi)[-a, b] = \gamma(b, \psi)$ *if* ab *is even.* □

In the rest of the paper, we assume ψ is an additive character of order 0.

4. A reduction step

In this section, we show some properties of the orbitals $J(\mathbf{a})$ and $I(\mathbf{a})$. We will use them to prove Theorem 2 for some simple cases. The proof of other cases of Theorem 2 requires different methods.

For $g \in GL_n$, denote by $\|g\|$ the maximal norm of the entries in g. Observe that $g \in K$ if and only if $|\det(g)| = 1$ and $\|g\| = 1$. From the definition in the introduction, $I(\mathbf{a}) = J(\mathbf{a}) = 0$ if $|\det(\mathbf{a})| \neq 1$. When $\mathbf{a} \in T$ and $|\det(\mathbf{a})| = 1$, the orbital integrals of concern are

$$I(\mathbf{a}) = \int_{\|{}^t n \mathbf{a} n\| = 1,\, n \in N} \theta(n)\, dn, \tag{10}$$

$$J(\mathbf{a}) = \int_{\|n_1 w_0 \mathbf{a} n_2\| = 1,\, n_1, n_2 \in N} \kappa(n_1 w_0 \mathbf{a} n_2)\, \theta'(n_1 n_2)\, dn_1\, dn_2 \tag{11}$$

where θ and θ' are defined by (4). For convenience, we set $I(\mathbf{a}) = J(\mathbf{a}) = 1$ for $\mathbf{a} \in GL_1(R)$. We have a reduction formula for the orbital integrals:

Proposition 3. *If* $\mathbf{a} \in GL_n$ *is of the form* $\begin{bmatrix} \mathbf{a}_1 & \\ & \mathbf{a}_2 \end{bmatrix}$ *with* $\mathbf{a}_1 \in GL_r$, $\mathbf{a}_2 \in GL_s$ *diagonal matrices, and* $|\det(\mathbf{a}_1)| = |\det(\mathbf{a}_2)| = 1$, *then*

$$J(\mathbf{a}) = J(\mathbf{a}_1)\,J(\mathbf{a}_2),$$

$$I(\mathbf{a}) = I(\mathbf{a}_1)\,I(\mathbf{a}_2).$$

Similar reduction formulas are shown for Kloosterman integrals in various papers, e.g. [Fr], [G], [JY 2] and [St]. What is new in our case is that $J(\mathbf{a})$ involves the splitting map κ. The proposition follows from Lemma 2 part 2 and the standard argument used in the above mentioned papers.

We apply this proposition to the case $n = 3$. From now on, let $\mathbf{a} = \operatorname{diag}[a, b, c] \in GL_3$. We will assume $|a| \leqq 1$, $|ab| \leqq 1$, $|abc| = 1$ since

Proposition 4. *When either* $|a| > 1$ *or* $|ab| > 1$ *or* $|abc| \neq 1$, *we have*

$$I(\mathbf{a}) = J(\mathbf{a}) = 0\,.$$

Proof. In all cases, either $|\det(\mathbf{a})| \neq 1$ or the domains of integration for both (10) and (11) are empty. □

Proposition 5. *When* $|a| \leqq 1$, $|ab| = |c| = 1$,

$$J(\mathbf{a}) = |a|^{-1/2}\gamma(a, \psi)\,I(\mathbf{a})\,.$$

Proof. By Prop. 3, with the assumptions, $J(\mathbf{a}) = J(\mathbf{a}_1)$ and $I(\mathbf{a}) = I(\mathbf{a}_1)$ with $\mathbf{a}_1 = \operatorname{diag}[a, b]$. If moreover $|a| = |b| = 1$, then Prop. 3 shows $J(\mathbf{a}) = I(\mathbf{a}) = 1$ and Prop. 5 holds. If $|a| < 1$, we can write down explicitly the integrals $I(\mathbf{a}_1)$ and $J(\mathbf{a}_1)$. Let E be the set of functions on $F^\times$ such that for $e(x) \in E$ one has $e\big(x(1+v)\big) = e(x)$ for all $v \in P$. For $e \in E$, m integer, $a, b \in F^\times$, set

$$J(a, b, e, q^m) = \int\limits_{|x| = |y|,\, |axy + b| \leqq q^m} \psi\left(\frac{x+y}{2}\right) e(x)\,dx\,dy\,, \tag{12}$$

$$I(a, b, q^m) = \int\limits_{|ax^2 + b| \leqq q^m} \psi(x)\,dx\,. \tag{13}$$

Then $J(\mathbf{a}_1) = J(a, b, e, 1)$ with $e(x) = [-x, a]$; and $I(\mathbf{a}_1) = I(a, b, 1)$.

The integrals (12) and (13) are studied in [J 2]. To state the results in [J 2], it is convenient to fix a square root map on $F^\times$, such that $u = \sqrt{v}$ implies $v = u^2$, $\sqrt{u\varpi^2} = \sqrt{u}\varpi$ and for any $v \in P$, there exists a $v' \in P$ with $\sqrt{u(1+v)} = \sqrt{u}(1+v')$.

Lemma 3. *For* $a, b \in F^\times$ *with* $\left|\dfrac{b}{a}\right| = q^{2l}$, *we have*: *when* $[-ab, \varpi] = 1$,

(1) *when* $l = 1$, $|a| = q^{-1}$, $e(x) = [x, \varpi]$, $J(a, b, e, 1)$ *equals*

$$(14)\quad \int_{|x|=q} \psi\left(\frac{x}{2}-\frac{b}{2ax}\right)[x,\varpi]\,dx = \sum_{i=0,1} q^{1/2}\gamma(-\varpi,\psi)\,\psi\left((-1)^i\sqrt{-\frac{b}{a}}\right),$$

(2) *when* $l>1$, $|a| \geqq q^{-(m+l)}$:

$$(15)\quad J(a,b,e,q^{-m}) = q^{-l/2}q^{-m}|a|^{-1}\sum_{i=0,1}\psi\left((-1)^i\sqrt{-\frac{b}{a}}\right)e'\left((-1)^i\sqrt{-\frac{b}{a}}\right)$$

with $e'(x)=e(x)\gamma(x,\psi)$,

(3) *when* $l \geqq 1$, $|a| \geqq q^{-(m+l)}$:

$$(16)\qquad I(a,b,q^{-m}) = q^{-m}|ab|^{-1/2}\sum_{i=0,1}\psi\left((-1)^i\sqrt{-\frac{b}{a}}\right).$$

All the above integrals equal 0 *when* $[-ab,\varpi]=-1$. □

For the calculations concerning this lemma, see [J 2].

Apply the lemma to the integrals $I(\mathbf{a}_1)$ and $J(\mathbf{a}_1)$. Notice with the notations in Lemma 3, $|a|^{-1}=|b|=q^l$; observe also when $e(x)=[-x,a]$, from Prop. 2:

$$e'\left((-1)^i\sqrt{-\frac{b}{a}}\right) = \left[-(-1)^i\sqrt{-\frac{b}{a}},a\right]\gamma\left((-1)^i\sqrt{-\frac{b}{a}},\psi\right) = \gamma(a,\psi).$$

Thus when $|a|<q^{-1}$, from part (2) and (3) of the lemma, we get

$$(17)\qquad J(\mathbf{a}_1) = |a|^{-1/2}\gamma(a,\psi)I(\mathbf{a}_1).$$

We now show Prop. 5 when $|a|=q^{-1}$. Here over the domain of (12), $|x|=q$. Using Prop. 1 we have $e(x)=[-x,a]=[ax,\varpi]$. Thus $J(\mathbf{a}_1)$ equals $[a,\varpi]$ times the expression (14). From part (1) and (3) of the Lemma 3,

$$J(\mathbf{a}_1) = [a,\varpi]\,q^{1/2}\gamma(-\varpi,\psi)I(\mathbf{a}_1).$$

From Prop. 2, $\gamma(-\varpi,\psi)[a,\varpi]=\gamma(a,\psi)$, thus we get the proposition. □

Notice that when $|c|=1$, $\gamma(-c,\psi)=1$. Thus Prop. 5 gives Theorem 2 in the case $|a|\leqq 1, |ab|=|c|=1$.

We will use another property of $I(\mathbf{a})$ and $J(\mathbf{a})$.

Proposition 6. *Let* $\mathbf{a}'=\operatorname{diag}[c^{-1},b^{-1},a^{-1}]$, *then*

$$(18)\qquad I(\mathbf{a}) = I(\mathbf{a}'),$$

$$(19)\qquad J(\mathbf{a}) = [-1,ac]\,J(\mathbf{a}').$$

Proof. We first show the identity (19). Let $g = n_1 w_0 \mathbf{a} n_2$.

Lemma 4. *Let g be as above. If* $g \in K$, *then* $\kappa(g) = \kappa(g^{-1})$.

Proof. Since $\kappa(gg^{-1}) = 1$, from Lemma 2 we need to show $\sigma(g, g^{-1}) = 1$. From Lemma 2 and the definition of σ, we have

$$\begin{aligned}\sigma(g, g^{-1}) &= \sigma(w_0 \mathbf{a} n_2, n_2^{-1} \mathbf{a}^{-1} w_0) = \sigma(n_2^{-1} \mathbf{a}^{-1} w_0, w_0 \mathbf{a} n_2) \\ &= \sigma(\mathbf{a}^{-1} w_0, w_0 \mathbf{a}) = \sigma(\mathbf{a}^{-1}, \mathbf{a})\, \sigma(w_0, w_0 \mathbf{a} w_0)\,.\end{aligned}$$

Using the definition of σ, one can compute the last expression which equals 1. □

Note that $g^{-1} = n_2^{-1} w_0 \mathbf{a}' n_1^{-1}$. As $|\det(g)| = 1$, the condition $\|g\| = 1$ is equivalent to $\|g^{-1}\| = 1$. From Lemma 4, the integral $J(\mathbf{a})$ equals

$$\begin{aligned}(20) \qquad &\int_{\|n_1 w_0 \mathbf{a} n_2\| = 1} \theta'(n_1 n_2)\, \kappa(n_1 w_0 \mathbf{a} n_2)\, dn_1\, dn_2 \\ &= \int_{\|n_2^{-1} w_0 \mathbf{a}' n_1^{-1}\| = 1} \theta'(n_1 n_2)\, \kappa(n_2^{-1} w_0 \mathbf{a}' n_1^{-1})\, dn_1\, dn_2\,.\end{aligned}$$

Make a change of variables $n_i \to \tau n_i^{-1} \tau$, $i = 1, 2$ and $\tau = \operatorname{diag}[1, -1, 1]$; then (20) becomes:

$$(21) \qquad \int_{\|n_2 w_0 \mathbf{a}' n_1\| = 1} \theta'(n_1 n_2)\, \kappa(\tau n_2 w_0 \mathbf{a}' n_1 \tau)\, dn_1\, dn_2\,.$$

A simple computation using Lemma 2 shows that

$$\begin{aligned}\kappa(\tau n_2 w_0 \mathbf{a}' n_1 \tau) &= \sigma(\tau, n_2 w_0 \mathbf{a}' n_1 \tau)\, \sigma(n_2 w_0 \mathbf{a}' n_1, \tau)\, \kappa(n_2 w_0 \mathbf{a}' n_1) \\ &= [-1, ac]\, \kappa(n_2 w_0 \mathbf{a}' n_1)\,.\end{aligned}$$

Thus $J(\mathbf{a}) = [-1, ac]\, J(\mathbf{a}')$ by the definition of $J(\mathbf{a}')$. The proof for the equality of $I(\mathbf{a})$ is similar and easier. □

As an immediate consequence, we have:

Proposition 7. *Let* $\mathbf{a}' = \operatorname{diag}[c^{-1}, b^{-1}, a^{-1}]$. *If* $J(\mathbf{a}) = \mu(\mathbf{a})\, I(\mathbf{a})$, *then*

$$J(\mathbf{a}') = \mu(\mathbf{a}')\, I(\mathbf{a}')\,.$$

Proof. We only need to show that $[-1, ac]\, \mu(\mathbf{a}) = \mu(\mathbf{a}')$ when $|abc| = 1$. This follows from the facts (using Prop. 1 and 2):

$$|a|^{-1} |b|^{-1/2} = |c^{-1}|^{-1} |b^{-1}|^{-1/2},$$

$$\gamma(a, \psi)\, \gamma(-c, \psi)\, [-1, ac] = \gamma(c^{-1}, \psi)\, \gamma(-a^{-1}, \psi)\,. \quad □$$

In particular, since Theorem 2 is true when $|ab| = |c| = 1$ as shown in Prop. 5, it is also true for the case $|a| = |bc| = 1$.

5. Computation of $J(\mathbf{a})$

From Prop. 4, 5, 7, it only remains to prove (part 4 of) Theorem 2 in the case where $|a| < 1$, $|ab| < 1$, $|abc| = 1$ and $|b| \geqq 1$. From now on, assume **a** satisfies these conditions. In this section, we compute $J(\mathbf{a})$.

(5.1) We first consider the domain for the integral $J(\mathbf{a})$ given by equation (11). Let

$$n_1 = \begin{bmatrix} 1 & x_1 & z_1 \\ & 1 & y_1 \\ & & 1 \end{bmatrix}, \quad n_2 = \begin{bmatrix} 1 & y_2 & z_2 \\ & 1 & x_2 \\ & & 1 \end{bmatrix}$$

then the matrix $n_1 w_0 \mathbf{a} n_2$ is of the form:

$$\begin{bmatrix} XS & XSY + c \\ S & SY \end{bmatrix} \tag{22}$$

with

$$S = \begin{bmatrix} ax_1 & ax_1x_2 + b \\ a & ax_2 \end{bmatrix}, \quad X = [y_1, z_1], \quad Y = \begin{bmatrix} z_2 \\ y_2 \end{bmatrix}.$$

Over the domain in (11), the matrix S has a specific Cartan Decomposition.

Lemma 5. *Let c be a number with $|c| > 1$, $S \in GL_{n-1}$. If for some $1 \times (n-1)$ matrices W, V, we have:*

$$g = \begin{bmatrix} WS & WS^tV + c \\ S & S^tV \end{bmatrix} \in GL_n(R)$$

then $S = k_1 \operatorname{diag}[c^{-1}, I]\, k_2$, for some $k_1, k_2 \in GL_{n-1}(R)$.

Proof. The condition in the lemma implies that $\|WS\| \leqq 1$, thus $\|W\| \leqq \|S^{-1}\|$. Since $WS^tV + c \in R$ and $|c| > 1$, we have $\|WS^tV\| = |c|$. Since $\|S^tV\| \leqq 1$, we get $\|W\| \geqq |c|$. Thus $\|S^{-1}\| \geqq |c|$.

Let $S = k_1 t k_2$ with $k_1, k_2 \in GL_{n-1}(R)$ and $t = \operatorname{diag}[t_i]$. Then $\min\{|t_i|\} \leqq |c|^{-1}$. On the other hand, we have also $|\det(g)| = |\det(S)c| = 1$ and $\|S\| \leqq 1$; thus $\prod |t_i| = |c|^{-1}$ and $|t_i| \leqq 1$. These conditions imply that there is a j such that $|t_j| = |c|^{-1}$ and $|t_i| = 1$ if $i \neq j$. Thus S has Cartan Decomposition as stated in the lemma. □

Over the domain of integration for $J(\mathbf{a})$ in (11), the matrix (22) lies in $GL_3(R)$. By the above lemma, we have $\|S\| = 1$. This leads to the following three cases:

1. $|ax_1| = 1$, $|ax_2| < 1$.
2. $|ax_2| = 1$, $|ax_1| < 1$.
3. $|ax_1| < 1$, $|ax_2| < 1$, $|b + ax_1x_2| = 1$.

We will denote the contribution from each of the above set to (11) by $J_i(\mathbf{a})$ with $i = 1, 2, 3$ resp. In (5.2)–(5.7), we will compute $J_3(\mathbf{a})$.

(5.2) We write down explicitly the integral $J_3(\mathbf{a})$.

Over the set with $|ax_1| < 1$, $|ax_2| < 1$, $|b + ax_1x_2| = 1$, the matrix S has the Cartan Decomposition

$$S = \begin{bmatrix} 1 & \\ \dfrac{ax_2}{b + ax_1x_2} & 1 \end{bmatrix} \begin{bmatrix} & b + ax_1x_2 \\ \dfrac{ab}{b + ax_1x_2} & \end{bmatrix} \begin{bmatrix} 1 & \\ \dfrac{ax_1}{b + ax_1x_2} & 1 \end{bmatrix}.$$

Let

$$[y_1, z_1] = [v', v] \begin{bmatrix} 1 & \\ -\dfrac{ax_2}{b + ax_1x_2} & 1 \end{bmatrix}, \quad \begin{bmatrix} z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & \\ -\dfrac{ax_1}{b + ax_1x_2} & 1 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}.$$

With these notations, the domain for integral $J_3(\mathbf{a})$ is

$$|ax_1| < 1, \quad |ax_2| < 1, \quad |b + ax_1x_2| = 1, \quad |abu| = |abv| = 1, \tag{23}$$

$$|u'| \leqq 1, \quad |v'| \leqq 1, \quad \left| \frac{ab}{b + ax_1x_2} uv + c \right| \leqq 1. \tag{24}$$

Over the domain, we have

$$\theta'(n_1n_2) = \psi\left(\frac{x_1 + x_2}{2}\right) \psi\left(\frac{-ax_1u}{2(b + ax_1x_2)} + \frac{-ax_2v}{2(b + ax_1x_2)}\right).$$

Using the properties of the κ map described in § 3, one can show:

$$\kappa(n_1 \mathrm{w}_0 \mathbf{a} n_2) = \left[\frac{-ax_1uc}{b + ax_1x_2}, \frac{ab}{b + ax_1x_2}\right] [-x_1, a] [-1, c].$$

Thus $J_3(\mathbf{a})$ is

$$\int \psi\left(\frac{x_1 + x_2}{2}\right) \psi\left(\frac{-ax_1u}{2(b + ax_1x_2)} + \frac{-ax_2v}{2(b + ax_1x_2)}\right) \tag{25}$$

$$\times \int \left[\frac{-ax_1uc}{b + ax_1x_2}, \frac{ab}{b + ax_1x_2}\right] [-x_1, a] [-1, c] \, dx_1 \, dx_2 \, du \, dv \, du' \, dv'$$

with the domain of integration given by (23) and (24). Make a change of variable $(u,v) \to \left(-\dfrac{b+ax_1x_2}{ax_1}u, -\dfrac{b+ax_1x_2}{ax_2}v\right)$; integrating over u', v' we get the following expression for $J_3(\mathbf{a})$:

$$\int \psi\left(\frac{x_1+x_2}{2}\right)\psi\left(\frac{u+v}{2}\right)\left[uc, \frac{ab}{b+ax_1x_2}\right] [-x_1,a][-1,c]|a^2x_1x_2|^{-1}dx_1dx_2dudv \tag{26}$$

with

$$|ax_1|<1, \quad |ax_2|<1, \quad |b+ax_1x_2|=1, \quad |bu|=|x_1|, \quad |bv|=|x_2|,$$
$$\left|\frac{b(b+ax_1x_2)}{ax_1x_2}uv+c\right| \leqq 1\,.$$

(5.3) We observe that over the domain of (26), the restrictions on the variables u, v depend on the values of $|x_1|$ and x_1x_2 only. There is a function $\varphi(q^z, y)$ with $z \in \mathbb{Z}$ and $y \in F$ supported on the set $|b+ay|=1$, with

$$\varphi(|x_1|, x_1x_2) = \int \psi\left(\frac{u+v}{2}\right)\left[uc, \frac{ab}{b+ax_1x_2}\right][-1,c]|a^2x_1x_2|^{-1}dudv$$

where the integration is over the domain of (26). Integrating (26) over u, v, we see $J_3(\mathbf{a})$ equals:

$$\int \psi\left(\frac{x_1+x_2}{2}\right)[-x_1,a]\,\varphi(|x_1|, x_1x_2)\,dx_1dx_2 \tag{27}$$

where the domain of integration is $|ax_1|<1, |ax_2|<1$.

The following lemma allows us to restrict the domain for (27).

Lemma 6. *Let* $m \neq n$ *be two integers with* $\max(m,n)>1$. *Then the integration of* (27) *over the set* $|x_1|=q^m$, $|x_2|=q^n$ *equals* 0.

Proof. Let $F(x_1,x_2)$ be the integrand in (27). Assume $m>n$. Make a change of variable $x_1 \to x_1(1+v\varpi^{m-1})$, $x_2 \to x_2(1+v\varpi^{m-1})^{-1}$ for $v \in R$. Then the domain remains as $|x_1|=q^m$, $|x_2|=q^n$ while $F(x_1,x_2)$ changes to $F(x_1,x_2)\psi(x_1v\varpi^{m-1}/2)$. Therefore:

$$\int_{|x_1|=q^m, |x_2|=q^n} F(x_1,x_2)\,dx_1dx_2$$
$$= \int_{v\in R} \int_{|x_1|=q^m, |x_2|=q^2} F(x_1,x_2)\,\psi(x_1v\varpi^{m-1}/2)\,dx_1dx_2dv\,.$$

Now the integration over v equals 0.

When $n>m$, we can write $\phi(|x_1|, x_1x_2)$ in the form of $\varphi'(|x_2|, x_1x_2)$ for some function $\varphi'(q^z, y)$. The above argument again shows that the integral (27) over the set $|x_1|=q^m, |x_2|=q^n$ equals 0. □

From Lemma 6, we see the contribution from the set $|x_1| \neq |x_2|$ to (26) is nonzero only when $|x_1| \leqq q$ and $|x_2| \leqq q$. Thus $J_3(\mathbf{a})$ is the sum of contributions from the following five sets:

(I) $|x_1| = |x_2| > q$.

(II) $|x_1| \leqq 1, |x_2| \leqq 1$.

(III) $|x_1| \leqq 1, |x_2| = q$.

(IV) $|x_1| = q, |x_2| \leqq 1$.

(V) $|x_1| = |x_2| = q$.

Denote the contribution from each set by $J_{\mathrm{I}}(\mathbf{a}), \ldots, J_{\mathrm{V}}(\mathbf{a})$ respectively.

(5.4) We will assume $|a| < q^{-2}$ during the computation for $J_{\mathrm{I}}(\mathbf{a})$, since the domain for $J_{\mathrm{I}}(\mathbf{a})$ is otherwise empty. Let R'' be the set of $w \in F$ with

$$|w| = 1, \quad |w-b| > |a|q^2, \quad a(w-b) \text{ is even}, \tag{28}$$

$$[a(b-w), \varpi] = 1, \quad [bcw(b-w), \varpi] = 1\,. \tag{29}$$

In (5.5)–(5.6), we prove:

Proposition 8. *When $|a| \leqq q^{-2}$, $J_{\mathrm{I}}(\mathbf{a}) = 0$. When $|a| < q^{-2}$, the integral $J_{\mathrm{I}}(\mathbf{a})$ equals*

$$\mu(\mathbf{a}) \sum_{i,j=0,1} \int_{w \in R''} \psi\left((-1)^i \sqrt{\frac{w-b}{a}} + (-1)^j \sqrt{\frac{c(b-w)}{bw}}\right) |a(w-b)|^{-1/2}\, dw\,. \tag{30}$$

For our purpose of comparing $I(\mathbf{a})$ and $J(\mathbf{a})$, there is no need in evaluating the above integral, as the same integral appears as part of $I(\mathbf{a})$.

(5.5) In proving the proposition, we use a convolution formula for $J_{\mathrm{I}}(\mathbf{a})$. The convolution formula will allow us to use Lemma 3 in computing $J_{\mathrm{I}}(\mathbf{a})$. Denote by R' the set of $w \in R^\times$ satisfying the conditions (28).

Proposition 9. *There is a suitably large r such that*

$$J_{\mathrm{I}}(\mathbf{a}) = \int_{w \in R'} q^r J(a, b-w, e_2, q^{-r})\, J\left(\frac{bw}{w-b}, c, e_1, 1\right) |a(w-b)|^{-1}\, dw \tag{31}$$

with $e_1(x) = [xc, abw]$ and $e_2(x) = [-x, a][-1, c]$. □

Proof. From equation (12), we can write down the right hand side of (31):

$$q^r \int \psi\left(\frac{x_1 + x_2 + u + v)}{2}\right) [uc, abw][-x_1, a][-1, c] \tag{32}$$

$$|a(w-b)|^{-1}dx_1dx_2dudvdw$$

with the domain of integration:

$$(33)\qquad |x_1|=|x_2|,\quad |ax_1x_2+b-w|\leqq q^{-r},$$

$$(34)\qquad |u|=|v|,\quad \left|\frac{bw}{w-b}uv+c\right|\leqq 1,$$

$$(35)\qquad |w|=1,\quad |w-b|=|a|q^{2m},\quad m>1.$$

We compare this domain with the domain of integration (26). Assume r is large and x_1, x_2, w satisfy the conditions (33). Then $|w-b|=|ax_1x_2|$. Thus $|w-b|=|a|q^{2m}, m>1$ is equivalent to $|x_1|>q$. Clearly also $|w|=1$ is equivalent to $|ax_1x_2+b|=1$. We now show the conditions (34) on u, v are equivalent to the following conditions:

$$(36)\qquad |bu|=|bv|=|x_1|,\quad \left|\frac{b(b+ax_1x_2)}{ax_1x_2}uv+c\right|\leqq 1.$$

From (34), we see $|uv|=\left|\frac{c(b-w)}{bw}\right|=|acx_1x_2b^{-1}|=|x_1^2b^{-2}|$. Thus (34) implies $|bu|=|bv|=|x_1|$. Meanwhile

$$\left|\frac{bw}{w-b}uv-\frac{b(b+ax_1x_2)}{ax_1x_2}uv\right|=\left|\frac{b^2(ax_1x_2+b-w)}{(ax_1x_2)(w-b)}uv\right|$$

which is not larger than $q^{-r}|ax_1|^{-2}$. As $|x_1|>q$, for r large enough, the above difference is an integer, thus (34) implies (36). From the same argument, (36) implies (34); they are equivalent conditions. Thus the domain of (32) is equivalent to those of (26) with extra conditions (I) and $|b+ax_1x_2-w|\leqq q^{-r}$.

Over this domain, $w=(b+ax_1x_2)(1+v)$ with $v\in P$. Using Prop. 1 and the observation $|w-b|=|ax_1x_2|$, we see the integrand of (32) is exactly that of (26). Denote by D the domain of (26), Φ the integrand of (26), then $J_{\mathrm{I}}(\mathbf{a})$ equals the integration of Φ over $D\cap\{|x_1|=|x_2|>q\}$, while (32) equals:

$$\int\limits_{D\cap\{|x_1|=|x_2|>q\}}\Phi\int\limits_{|ax_1x_2+b-w|\leqq q^{-r}}dwdx_1dx_2dudv.$$

Integrating over w introduces a factor q^{-r}; we get the identity (31). □

(5.6) In this subsection, we assume $|c|>q^2$. We will omit the proof of Prop. 8 in the case $|c|=q^2$. Though that case is a bit different, the ideas used are the same as explained below.

Given $w\in R'$, we apply Lemma 3 to find $J(a,b-w,e_2,q^{-r})$ and $J\left(\frac{bw}{w-b},c,e_1,1\right)$.

Over the set R', we have $\left|\dfrac{b-w}{a}\right| > q^2$. Thus from part (2) of Lemma 3, $J(a, b-w, e_2, q^{-r})$ equals

$$(37)\quad q^{-r}\left|\frac{a}{b-w}\right|^{1/4}|a|^{-1}\sum_{i=0,1}\psi\left((-1)^i\sqrt{\frac{w-b}{a}}\right)e_3\left((-1)^i\sqrt{\frac{w-b}{a}}\right)$$

where $e_3(x) = e_2(x)\gamma(x,\psi)$ if $[a(w-b),\varpi] = 1$ and 0 otherwise.

When $w \in R'$, since $|c| > q^2$, $\left|\dfrac{c(w-b)}{bw}\right| > q^2$. From part (2) of Lemma 3, $J\left(\dfrac{bw}{w-b}, c, e_1, 1\right)$ equals

$$(38)\quad \left|\frac{bw}{c(b-w)}\right|^{1/4}\left|\frac{bw}{w-b}\right|^{-1}\sum_{i=0,1}\psi\left((-1)^i\sqrt{\frac{c(b-w)}{bw}}\right)e_4\left((-1)^i\sqrt{\frac{c(b-w)}{bw}}\right)$$

where $e_4(x) = e_1(x)\gamma(x,\psi)$ if $\left[\dfrac{c(b-w)}{bw},\varpi\right] = 1$ or 0 otherwise.

From equation (31), using the fact $|w| = 1$ and $|abc| = 1$, we get:

$$(39)\quad J_1(\mathbf{a}) = \sum_{i,j=0,1}\int_{w\in R''}|a|^{-3/2}|b|^{-1/2}|w-b|^{-1/2}e_3\left((-1)^i\sqrt{\frac{w-b}{a}}\right)$$
$$\times \int e_4\left((-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)\psi\left((-1)^i\sqrt{\frac{w-b}{a}} + (-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)dw$$

where R'' is the set defined in (5.4).

Our next goal is to simplify the expressions in (39) involving e_3 and e_4. We first observe that one may restrict the integral (39) to a smaller set. Define $R''_{i,j}$ to be the set of elements $w \in R''$ with

$$(40)\qquad \left[(-1)^{i+j+1}bw\sqrt{\frac{w-b}{a}}\sqrt{\frac{c(b-w)}{bw}},\varpi\right] = 1\,.$$

We show that the integration in (39) over $R'' - R''_{i,j}$ is 0.

The set $R'' - R''_{i,j}$ is a disjoint union of sets $R''(w_0)$ consisting of elements $w = w_0(1+v)$ with $|v| \leqq q|a(w_0-b)|^{1/2}$, w_0 some representative $R'' - R''_{i,j}$. The following lemma shows that all the sets $R''(w_0)$ contribute 0 to the integral in (39).

Lemma 7. *For* $w_0 \in R'' - R''_{i,j}$, *when* $|a| < q^{-2}, |ab| < q^{-2}, |abc| = 1$,

$$\int_{R''(w_0)}\psi\left((-1)^i\sqrt{\frac{w-b}{a}} + (-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)dw = 0\,.$$

Proof. We will use $O(s)$ to denote any number t with $|t| \leqq |s|$. Assume

$$|a(w_0 - b)| = q^{-2m}.$$

From the assumptions, $m > 1$. The set $R''(w_0)$ is the collection of elements

$$w = w_0(1 + v\varpi^{m-1})$$

with $v \in R$. From the Taylor expansion, and the assumptions on w_0 and ψ, we have

$$\psi\left((-1)^i \sqrt{\frac{w-b}{a}}\right) = \psi\left((-1)^i \sqrt{\frac{w_0-b}{a}}\left(1 + \frac{w_0 v\varpi^{m-1}}{2(w_0-b)}\right)\right),$$

$$\psi\left((-1)^j \sqrt{\frac{c(b-w)}{bw}}\right) = \psi\left((-1)^j \sqrt{\frac{c(b-w_0)}{bw_0}}\left(1 - \frac{b}{2(b-w_0)} v\varpi^{m-1}\right)\right).$$

The integral in the lemma becomes an integral over v which is nonzero only when

$$\left|(-1)^i \sqrt{\frac{w_0-b}{a}}\,\frac{w_0}{w_0-b} + (-1)^j \sqrt{\frac{c(b-w_0)}{bw_0}}\,\frac{b}{w_0-b}\right| < q^m. \tag{41}$$

This inequality implies that w_0 satisfies (40), a contradiction to $w_0 \in R'' - R''_{i,j}$. □

Note that $|w-b|$, $e_3\left((-1)^i \sqrt{\frac{c(b-w)}{bw}}\right)$ and $e_4\left((-1)^j \sqrt{\frac{w-b}{a}}\right)$ are constant over any such set $R''(w_0)$. Thus from the lemma, the integral (39) over $R'' - R''_{i,j}$ is zero.

Using the properties of Hilbert symbol and Weil constant (see Prop. 1 and 2), taking into account of the condition (40), one can show:

Lemma 8. *Over the set $R''_{i,j}$, the product*

$$e_3\left((-1)^i \sqrt{\frac{w-b}{a}}\right) e_4\left((-1)^j \sqrt{\frac{c(b-w)}{bw}}\right)$$

is constant; it equals $\gamma(a,\psi)\gamma(-c,\psi)$.

Proof. From (40), the expression in Lemma 8 is of the form $e_3(x)e_4(y)$ with $|x/y| = |b|$, and $-xybw$ being a square. From the definition and Prop. 1,

$$e_4(y) = [yc, abw]\gamma(y,\psi) = [yc, -axy]\gamma(y,\psi),$$

$$e_3(x) = [-x, a][-1, c]\gamma(x,\psi).$$

Thus from Prop. 1, $e_3(x)e_4(y)$ equals

$$\gamma(x,\psi)\gamma(y,\psi)[-xy, ac][x, y][-a, c].$$

By Prop. 1, 2, this expression equals

$$\gamma(xy,\psi)[-xy,ac][-a,c]=\gamma(ac,\psi)[-a,c]=\gamma(a,\psi)\gamma(c,\psi)[-1,c]$$

which equals $\gamma(a,\psi)\gamma(-c,\psi)$. □

Apply the lemma to integrand in (39); we see when $|a|<q^{-2}$, $|c|\neq q^2$, $J_{\mathrm{I}}(\mathbf{a})$ equals:

$$\text{(42)}\quad \mu(\mathbf{a})\sum_{i,j=0,1}\int_{R''_{i,j}}\psi\left((-1)^i\sqrt{\frac{w-b}{a}}+(-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)|a(w-b)|^{-1/2}\,dw\,.$$

Apply Lemma 7 again, we see the domain in (42) can be replaced by R''. We proved Prop. 8 in the case $|c|\neq q^2$. □

(5.7) The computations for the integrals J_{II}, J_{III}, J_{IV}, J_{V} are straightforward. We denote their sum by $J_4(\mathbf{a})$. Below we describe the values of $J_4(\mathbf{a})$ for the various cases.

Proposition 10. *When* $|a|=q^{-1}$ *or* $|b|>1$, $J_4(\mathbf{a})=0$.

When $|a|<q^{-2}$, $|b|=1$, $J_4(\mathbf{a})$ *is* 0 *when* $|b^2+ac|=1$ *and* $q|a|^{-1}[-1,c]$ *when* $|b^2+ac|<1$.

When $|a|=q^{-2}$, $|b|=1$, $J_4(\mathbf{a})$ *is the sum of* $q^2(1+[-ac,\varpi])$ *and*

$$\text{(43)}\quad \mu(\mathbf{a})\sum_{i,j=0,1}\int\psi\left((-1)^i\sqrt{\frac{w-b}{a}}+(-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)|a(w-b)|^{-1/2}\,dw$$

with

$$\text{(44)}\qquad |w|=|w-b|=1\,,\quad [a(w-b),\varpi]=[bcw(b-w),\varpi]=1\,.$$

(5.8) We now compute $J_1(\mathbf{a})$ and $J_2(\mathbf{a})$. Following a similar discussion as in (5.2), we can get an explicit form for the integral $J_1(\mathbf{a})$. It equals:

$$\text{(45)}\quad \int\psi\left(\frac{x_1+x_2+u+v}{2}\right)[-x_1,a][-bx_1,-bx_1,-uc]|x_1|\,dx_1\,dx_2\,du\,dv$$

with

$$|ax_1|=1\,,\quad |ax_2|<1\,,\quad |b+ax_1x_2|\leqq 1\,,\quad |abvx_1|=|abu|=1\,,\quad |c-buv|\leqq 1\,.$$

For fixed u, v, the expression (45) is of the form (27). As over the domain, $|x_1|\neq|x_2|$, from Lemma 6, the integral over the subset with $|x_1|>q$ is 0. We impose the condition $|x_1|\leqq q$ on the domain. Then the domain is empty unless $|a|=q^{-1}$ and $|b|=1$, in which case it is:

$$|x_1|=|u|=q\,,\quad |v|=1\,,\quad |x_2|\leqq 1\,,\quad |c-buv|\leqq 1\,.$$

Over the domain $\psi\left(\dfrac{x_2+v}{2}\right)=1$ and from Prop. 1,

$$[-x_1, a][-bx_1, -uc] = [ac, \varpi][-uc_1, \varpi] \,.$$

Integrating over v, x_2, the integral (45) becomes

$$\int_{|x_1| = |u| = q} [ac, \varpi] \, \psi\left(\frac{x_1 + u}{2}\right) [-ux_1, \varpi] \, du dx_1$$

which by Prop. 2 equals

$$q[ac, \varpi] \, \gamma(\varpi, \psi) \, \gamma(-\varpi, \psi) = q[ac, \varpi] \,.$$

Thus $J_1(\mathbf{a}) = q[ac, \varpi]$ when $|a| = q^{-1}, |b| = 1$; it equals 0 otherwise.

We can compute $J_2(\mathbf{a})$ similarly. We get $J_2(\mathbf{a}) = q[-1, \varpi]$ when $|a| = q^{-1}$ and $|b|^{-1}$; it equals 0 otherwise.

We summarize the result in this section:

Proposition 11. *When* $|a| = q^{-1}, |b| = 1,$

$$J_1(\mathbf{a}) + J_2(\mathbf{a}) = q([-1, \varpi] + [ac, \varpi]) \,.$$

The sum equals 0 *otherwise.* □

6. Computation of $I(\mathbf{a})$

In this section, we compute $I(\mathbf{a})$ when $|abc| = 1$, $|a| < 1$, $|b| \geqq 1$ and $|ab| < 1$. Let

$$n = \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{bmatrix} .$$

From (10), $I(\mathbf{a})$ equals:

$$\int \psi(x + y) \, dx dy dz$$

over the domain

$$\begin{bmatrix} S' & S'^t W \\ WS' & WS'^t W + c \end{bmatrix} \in GL(3, R)$$

where

$$S' = \begin{bmatrix} a & ax \\ ax & b + ax^2 \end{bmatrix}, \quad W = [z, y] \,.$$

Apply Lemma 5, we see that $\|S'\| = 1$ over the domain. Since $|ab| < 1$, we deduce the conditions $|ax| < 1$ and $|b + ax^2| = 1$. Thus S' has the following Cartan Decomposition:

$$S = \begin{bmatrix} 1 & \frac{ax}{b+ax^2} \\ & 1 \end{bmatrix} \begin{bmatrix} \frac{ab}{b+ax^2} & \\ & b+ax^2 \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{ax}{b+ax^2} & 1 \end{bmatrix}.$$

Let

$$W = [u, v] \begin{bmatrix} 1 & -\frac{ax}{b+ax^2} \\ & 1 \end{bmatrix}.$$

Then the domain becomes

$$|ax| < 1, \quad |b+ax^2| = 1, \quad |v| \leqq 1, \quad \left| \frac{ab}{b+ax^2} u^2 + c \right| \leqq 1\,.$$

Since over the domain

$$\psi(y) = \psi\left(-\frac{ax}{b+ax^2} u + v \right) = \psi\left(-\frac{ax}{b+ax^2} u \right)$$

we find that $I(\mathbf{a})$ is the integration

$$\int \psi\left(x - \frac{axu}{b+ax^2} \right) dx du dv$$

over

$$|ax| < 1, \quad |b+ax^2| = 1, \quad |v| \leqq 1, \quad \left| \frac{ab}{b+ax^2} u^2 + c \right| \leqq 1\,.$$

Integrating over v, we get

$$I(\mathbf{a}) = \int \psi\left(x - \frac{axu}{b+ax^2} \right) dx du \tag{46}$$

with

$$|ax| < 1, \quad |b+ax^2| = 1, \quad \left| \frac{ab}{b+ax^2} u^2 + c \right| \leqq 1\,.$$

Denote by $I_1(\mathbf{a})$ the contribution to (46) from the set with $|x| > 1$ and $I_2(\mathbf{a})$ from the set with $|x| \leqq 1$.

Proposition 12. *When* $|b| = 1$, $I_2(\mathbf{a})$ *equals* $1 + [-ac, \varpi]$. *It is* 0 *when* $|b| \neq 1$. □

Proof. Since $|b+ax^2| = 1$ and $|x| \leqq 1$ over the domain, the contribution is nonzero only when $|b| = 1$. When $|b| = 1$, we have $|u| = |a|^{-1}$, thus

$$\left| \frac{axu}{b+ax^2} \right| = |x| \leqq 1\,.$$

Thus the integrand in (46) is 1; $I_2(\mathbf{a})$ is the volume of the set

$$\left\{ (x, u) \mid x \in R, \left| \frac{ab}{b+ax^2} u^2 + c \right| \leqq 1 \right\}. \tag{47}$$

Under the current assumptions, $\frac{b}{b+ax^2}$ is a square. When $[-ac,\varpi]=-1$, the set (47) will be empty. When $[-ac,\varpi]=1$, the set has volume 2.

Proposition 13. *When* $|a|=q^{-1}$, $I_1(\mathbf{a})=0$. *When* $|a|\leqq q^{-2}$, $I_1(\mathbf{a})$ *equals*

$$(48)\qquad \sum_{i=0,1}\sum_{j=0,1}\int |a(w-b)|^{-1/2}\psi\left((-1)^i\sqrt{\frac{w-b}{a}}+(-1)^j\sqrt{\frac{c(b-w)}{bw}}\right)dw$$

with

$$a(w-b)\ \textit{even},\quad [a(w-b),\varpi]=1,\quad [bcw(b-w),\varpi]=1,$$
$$|w|=1,\quad |w-b|\geqq|a|q^2.$$

The domain for (48) *is the same as* R'' *introduced in* (5.4) *if* $|b|>1$.

Proof. The integral (46) over $|x|>1$ is clearly 0 when $|a|=q^{-1}$, since the domain will then be empty. When $|a|\leqq q^{-2}$, we make a change of variable $u\to-\frac{b+ax^2}{ax}u$. The integral $I_1(\mathbf{a})$ is

$$(49)\qquad \int\psi(x+u)|ax|^{-1}dx\,du$$

with

$$|x|>1,\quad |ax|<1,\quad |ax^2+b|=1,\quad \left|\frac{b(b+ax^2)}{ax^2}u^2+c\right|\leqq1.$$

The domain for u is a function of x^2, not merely a function of x. The argument used in the proof of Prop. 9 leads to a convolution formula for $I_1(\mathbf{a})$:

$$(50)\qquad I_1(\mathbf{a})=\int_{R_I'}q^rI(a,b-w,q^{-r})\,I\left(\frac{bw}{w-b},c,1\right)|a(w-b)|^{-1/2}dx$$

where r is a large number and R_I' is the set of w with

$$|w|=1,\quad |w-b|\geqq|a|q^2,\quad a(w-b)\ \text{even}.$$

The identity (50) allows us to apply part (3) of Lemma 3 in computing $I_1(\mathbf{a})$; what we get is the equality (48). The last assertion is clear. □

For the comparison with $J(\mathbf{a})$, we still need the following result on $I_1(\mathbf{a})$ in the case $|a|<q^{-2}$ and $|b|=1$. In this case, we write $I_1(\mathbf{a})=I_{\mathrm{I}}(\mathbf{a})+I_{\mathrm{II}}(\mathbf{a})$, where $I_{\mathrm{II}}(\mathbf{a})$ is the contribution to (48) from the subset with $|w-b|=|a|q^2$ and $I_{\mathrm{I}}(\mathbf{a})$ that from the subset R''.

Proposition 14. *When* $|b|=1$, $|a|<q^{-2}$, $J_{\mathrm{I}}(\mathbf{a})=\mu(\mathbf{a})I_{\mathrm{I}}(\mathbf{a})$. *The contribution* $I_{\mathrm{II}}(\mathbf{a})$ *equals* $q-2$ *if* $|b^2+ac|<1$, *it equals* -2 *if* $|b^2+ac|=1$ *and* $[-ac,\varpi]=1$. *It equals* 0 *if* $[-ac,\varpi]=-1$.

Proof. The first assertion is clear from the expressions (30) and (48). We now compute $I_{\mathrm{II}}(\mathbf{a})$. With $|w-b| = |a|q^2$ and $[a(w-b), \varpi] = 1$, we can write $w = b + ax^2$ with $|x| = q$. Using Taylor expansion, we can show that $[bcw(b-w), \varpi] = 1$ is equivalent to $[-ac, \varpi] = 1$. Also, over the domain of (48):

$$\psi\left(\sqrt{\frac{w-b}{a}}\right) = \psi(\pm x), \quad \psi\left(\sqrt{\frac{c(b-w)}{bw}}\right) = \psi(\pm x\sqrt{-acb^{-2}})$$

where the sign is determined by the choice of x. With this change of variable in (48), $I_{\mathrm{II}}(\mathbf{a})$ becomes (when $[-ac, \varpi] = 1$)

$$\sum_{i=0,1} \int_{|x|=q} \psi\left(x\left(1 + (-1)^i\sqrt{-acb^{-2}}\right)\right) dx\,. \tag{51}$$

When $|ac + b^2| = 1$, $1 + (-1)^i\sqrt{-acb^{-2}}$ is a unit, thus the above sum gives -2. When $|ac + b^2| < 1$, for one choice of i, $1 + (-1)^i\sqrt{-acb^{-2}}$ is a unit, for the other, it lies in P, thus the sum gives $q - 2$. □

7. The comparison of $I(\mathbf{a})$ and $J(\mathbf{a})$

With the assumptions $|a| < 1$, $|b| \geqq 1$, $|ab| < 1$ and $|abc| = 1$, we show that

$$J(\mathbf{a}) = \mu(\mathbf{a}) I(\mathbf{a})\,. \tag{52}$$

Recall that $J(\mathbf{a}) = J_1(\mathbf{a}) + J_2(\mathbf{a}) + J_{\mathrm{I}}(\mathbf{a}) + J_4(\mathbf{a})$, $I(\mathbf{a}) = I_1(\mathbf{a}) + I_2(\mathbf{a})$ and

$$\mu(\mathbf{a}) = |a|^{-1}|b|^{-1/2}\gamma(a,\psi)\gamma(-c,\psi)$$

Case 1. When $|a| = q^{-1}$ and $|b| = 1$.

From Prop. 12 and 13, $I(\mathbf{a})$ equals $1 + [-ac, \varpi]$. From Prop. 8, 10 and 11, we see $J(\mathbf{a})$ is $q([ac, \varpi] + [-1, \varpi])$. Therefore $J(\mathbf{a}) = q[ac, \varpi] I(\mathbf{a})$. The above equation holds since $[ac, \varpi] = \gamma(a,\psi)\gamma(-c,\psi)$ when $|a|^{-1} = |c| = q$.

Case 2. When $|a| = q^{-2}, |b| = q$.

From Prop. 8, 10, 11, $J(\mathbf{a}) = 0$. From Prop. 12, $I_2(\mathbf{a}) = 0$. As the domain for (48) is empty, $I_1(\mathbf{a}) = 0$. Thus $I(\mathbf{a}) = 0$ also.

Case 3. When $|a| = q^{-2}$, $|b| = 1$.

From Prop. 8, 10, 11, $J(\mathbf{a})$ equals $q^2(1 + [-ac, \varpi]) + (43)$. Meanwhile, from Prop. 12, 13, $I(\mathbf{a}) = 1 + [-ac, \varpi] + (48)$. Note (43) and (48) has the same domains of integration since $|w - b| = 1$ over the domain of (48). Comparing the integrand, we see that the expression (43) is $\mu(\mathbf{a})$ times (48). The equation (52) follows since $q^2 = \mu(\mathbf{a})$ in this case.

Case 4. When $|a| < q^{-2}$, $|b| > 1$.

The equality (52) follows immediately from Prop. 8, 10, 11, 12, 13.

Case 5. When $|a| < q^{-2}$, $|b| = 1$.

From Prop. 11, we see $J(\mathbf{a})$ is the sum of $J_{\mathrm{I}}(\mathbf{a})$ and $J_4(\mathbf{a})$. From Prop. 14, we have $J_{\mathrm{I}}(\mathbf{a}) = \mu(\mathbf{a}) I_{\mathrm{I}}(\mathbf{a})$. From Prop. 12, 14, we have $I_{\mathrm{II}}(\mathbf{a}) + I_2(\mathbf{a}) = q$ when $|b^2 + ac| < 1$ and 0 otherwise. From Prop. 10, $J_4(\mathbf{a})$ is $[-1, c]\, q |a|^{-1}$ when $|b^2 + ac| < 1$ and 0 otherwise. Since $\gamma(a, \psi)\gamma(-c, \psi) = [-1, c]$ when $-ac$ is a square, we have $J_4(\mathbf{a}) = \mu(\mathbf{a})\big(I_2(\mathbf{a}) + I_{\mathrm{II}}(\mathbf{a})\big)$. The equation (52) again holds.

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