BESSEL IDENTITIES IN THE WALDSPURGER CORRESPONDENCE OVER A $p-$ADIC FIELD

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Abstract. In this paper we develop the local theory for a Jacquet's relative trace formula. The local theory is essential to the application of the trace formula: it is an identity of Bessel and relative Bessel distributions. We show that the relative Bessel distribution attached to a distinguished (that is, self contragradient) unitary representation of $GL_2(k)$ where k is a p-adic field is given by a locally integrable function, namely the relative Bessel function. We compute the relative Bessel function for principal series, complementary series and special representations. We also show that the Bessel distributions associated to unitary irreducible admissible representations of the double covers of $GL_2(k)$ and $SL_2(k)$ are given by a locally integrable Bessel functions. We compute these Bessel functions for principal series, complementary series and special representations. Finally, we obtain the Waldspurger correspondence via Bessel identities between relative Bessel functions on $GL_2(k)$ and Bessel functions on the double cover of $SL_2(k)$. The Bessel identity also implies the identity between Bessel and relative Bessel distributions.

1. Introduction. The Waldspurger correspondence between representations of PGL_2 and representations of the double cover of SL_2 over local and global fields was developed by Waldspurger [W1] using theta correspondence methods. Jacquet [J] gave a new proof of the global Waldspurger correspondence using the relative trace formula. In this paper we develop the p-adic theory of this relative trace formula. This is the first such treatment of the p-adic theory of the relative trace formula and it puts the relative trace formula on the same footing as the more established trace formulas and theta correspondences which come with a p-adic and archimedean theory. Moreover, as noticed by Jacquet, the relative trace formula is more adequate for investigating special values of L-functions. This observation arises very clearly from this paper, the corresponding archimedean theory and our future work where we apply these results to study the central value of automorphic L-functions. The setting for this paper is the following.

Let F be a number field, $\mathbf{A}_F$ its adele ring. We denote the local places of F by v. Let $\overline{SL}_2$ be the double cover of SL_2. From Jacquet's relative trace formula, when π a cuspidal automorphic representation of PGL_2 corresponds to π' a cuspidal automorphic representation of $\overline{SL}_2$, there is an identity of distributions:

$$I_\pi(f) = J_{\pi'}(\bar{f}). \tag{1.1}$$

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Here f and $\bar{f}$ are Schwartz functions of $PGL_2(\mathbf{A}_F)$ and $\overline{SL}_2(\mathbf{A}_F)$ respectively. The distributions I_π and $J_{\pi'}$ factor into local distributions. From [B-M], we have: (when $f = \otimes f_v, \bar{f} = \otimes \bar{f}_v$)

$$\begin{aligned} I_\pi(f) &= L\left(\pi, \frac{1}{2}\right) |d(\pi)|^2 \prod I_{\pi,v}(f_v), \\ J_{\pi'}(\bar{f}) &= |d(\pi')|^2 \prod J_{\pi',v}(\bar{f}_v). \end{aligned}$$

Here $d(\pi)$ and $d(\pi')$ are some constant associated to the representations π and π', roughly they are the “Fourier coefficients” of π and π'. From the above identities, an identity between the local distributions $I_{\pi,v}(f_v)$ and $J_{\pi',v}(\bar{f}_v)$ would give a relation between $L(\pi, \frac{1}{2})$ and the Fourier coefficient of π'. Such a relation is studied in [W2] when F is the field of rational numbers.

The local theory of the relative trace formula would establish the identity of local distributions. The local distributions $I_{\pi,v}(f_v)$ and $J_{\pi',v}(\bar{f}_v)$ are called (relative) Bessel distributions, as they are related to the Bessel functions when v is archimedean [C-PS], and the p-adic Bessel functions [Av], [S] when v is p-adic.

This local theory is in complete analog with the local theory of the regular or twisted trace formula. Here the character distributions are replaced with Bessel distributions and the character identities are replaced with Bessel identities. However, Bessel distributions are much more difficult to study and we first need to establish basic facts about their behavior. The Bessel identities are difficult to establish and reflect very interesting identities of special functions, in our case, p-adic Bessel functions.

Our main results are Theorem 6.3 and Theorem 11.1 stating that the Bessel distributions and relative Bessel distributions are given by locally integrable functions; Theorem 13.3, which states that there is a Bessel identity (13.2) between the relative Bessel function for π_v and the Bessel function for π'_v, where π_v and π'_v are matching local representations under the Waldspurger correspondence; and Theorem 13.6 which is the identity between relative Bessel distributions and Bessel distributions. We note that Theorem 13.3 gives another way to define the Waldspurger correspondence, namely via the Bessel identities.

Our definition of the Bessel distributions follows that of [B]. We will show the equivalence of our definition and the definition of the local distributions which appear in the relative trace formula. This paper is the first paper which studies the identities of the Bessel functions associated to Jacquet's relative trace formula . We hope that this paper will serve as a model for other cases of local theory for the relative trace formula. Lapid and Rogawski [La-Ro] have studied another case of equalities between Bessel distributions without resorting to Bessel functions. A finite field version of another Bessel function identity is studied in [M]. Some other types of local distributions of relative trace formulas have been studied before, in the papers of J. Hakim [H1], [H2] and Rader and Rallis [Ra-R].

A crucial tool in studying the local Bessel distributions is an inner product formula using Kirillov models. As the representations of $\overline{SL}_2$ do not have a Kirillov model, we study instead the Bessel distributions over $\overline{GL}_2$, the double cover of GL_2, then use the results of [G-PS, 2] to relate the results to $\overline{SL}_2$.

The paper is organized as follows. We begin by studying certain orbital integrals on GL_2. In Section 3 we show that these orbital integrals give rise to locally integrable functions on GL_2. In Section 4 we define the relative Bessel function of a self dual irreducible representation of GL_2. We show that this function can be obtained via the orbital integrals which were studied in Section 2 hence it is locally integrable. In Section 5 we compute the relative Bessel functions of induced representations. In Section 6 we define the relative Bessel distribution associated to a self dual representation and show that it is given by the relative Bessel functions which were defined in Section 4. Our proof here relies on a G invariant bilinear form on the Whittaker model of π which is given in [Go].

Next we need to repeat this process for Bessel functions and Bessel distributions on the double cover of GL_2 and SL_2. In Section 7 we study orbital integrals on the double cover of GL_2 and show that they give rise to locally integrable functions. In Section 8 we study Bessel functions of irreducible admissible genuine representations of the double cover of GL_2 and show that they are given by the orbital integrals of Section 7, hence are locally integrable. We again would like to show that the Bessel distributions are given by Bessel functions. However, we do not have at hand the inner product formula in the Whittaker model in this case. In Section 9 we prove the existence of an inner product formula in the Whittaker model of unitary representations and give an explicit form for induced representations. In Section 10 we define the Bessel distributions of the double cover and show using the inner product formula that they are given by the Bessel functions of Section 8. In Section 11 we derive the results for $\overline{SL}_2$ from the results for $\overline{GL}_2$, in particular we show that the Bessel distributions on $\overline{SL}_2$ are given by Bessel functions. In Section 12 we compute these Bessel functions for induced representations.

All these results have independent interest and lead us to the main theorems of this paper which appear in Section 13. These are the Bessel identities. The Bessel identities for principal series and special representations follow from works in previous sections, while we use the global identity (1.1) to derive the Bessel identity for supercuspidal representations. The identity between the distributions follows immediately from the Bessel identity.

We encourage the reader to start with Section 13 and move backward if needed.

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2. Preliminaries and notations. Let k be a p-adic field, $O = O_k$ be the ring of integers in k, P be the maximal ideal in O and $\varpi \in P$ be a uniformizer. Let $q = |O/P|$ be the order of the residue field of k. Let $|x|$ be the normalized absolute value for $x \in k$. Let ψ be a nontrivial additive character of k. For $\Phi \in C_c^\infty(k)$, we define its Fourier transform to be

$$\Phi^\wedge(x) = \int_k \Phi(y)\psi(-xy)\,dy.$$

We set the Haar measure on k so that it is self dual with respect to this Fourier transform. Let $d^*x = dx/|x|$.

Let $G = GL_2(k)$ and $S = SL_2(k)$; let $\bar{G} = \overline{GL}_2(k)$ and $\bar{S} = \overline{SL}_2(k)$. Let $B = B_G$ be the Borel subgroup of upper triangular matrices in G and B_S the Borel subgroup of upper triangular matrices in S. Let $A = A_G$ be the subgroup of diagonal matrices. Let

$$N = \left\{ n(y) = \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} : y \in k \right\}.$$

We will also use ψ to denote a character of N given by

$$\psi(n(y)) = \psi(y).$$

Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$s(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \quad t(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \quad \text{and} \quad \bar{n}(z) = \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix}.$$

We fix some more notation. For a function $\phi\colon\ k \mapsto \mathbb{C}$ we let

$$\begin{aligned} \int_k^{+} \phi(x)\,dx &= \lim_{m\to\infty} \int_{|x|\le q^m} \phi(x)\,dx, \\ \int_k^{-} \phi(x)\,dx &= \lim_{m\to\infty} \int_{|x|\ge q^{-m}} \phi(x)\,dx, \\ \int_k^{+,-} \phi(x)\,dx &= \lim_{m\to\infty} \int_{q^{-m}\le|x|\le q^m} \phi(x)\,dx. \end{aligned} \tag{2.1}$$

if the limits exist. Let $D \in k^*$; we will use ψ^D to denote the character of k such that $\psi^D(x) = \psi(Dx)$.

We will use (a,b) to denote the Hilbert symbol on $k^* \times k^*$. We recall here some facts about the Weil constant: The Weil constant $\gamma(a, \psi^D)$ is defined by the formula

$$\int \hat{\Phi}^{2D}(x)\psi^D(ax^2)\,dx = |a|^{-\frac{1}{2}}\gamma(a,\psi^D)\int \Phi(x)\psi^D(-a^{-1}x^2)\,dx. \tag{2.2}$$

Here $\Phi \in C_c^\infty(k)$ and

$$\hat{\Phi}^D(x) = \int \Phi(y)\psi^D(-xy)\,dy.$$

The following properties of $\gamma(a, \psi^D)$ are well known:
(1) $\gamma(ab^2, \psi^D) = \gamma(a, \psi^D)$.
(2) $\gamma(ab, \psi^D) = \gamma(a, \psi^D)\gamma(b, \psi^D)\gamma(1, \psi^D)^{-1}(a,b)$.
(3) $\gamma(a, \psi^D)^2 = \gamma(1, \psi^D)^2(-1, a)$.
If we further assume that $|2| = 1$ and that ψ is unramified then:
(4) $\gamma(a, \psi) = 1$ if the valuation of a is even and in particular $\gamma(1, \psi) = 1$.
(5) If the valuation of a is odd, $a = b\varpi^{2r}$ with $|b| = q$, then

$$\gamma(a,\psi) = q^{-\frac{1}{2}} \sum_{x \in O/P} \psi(bx)(b, \varpi).$$

We will also use the following properties:

$$\gamma\left(\frac{a}{D}, \psi\right) = |D|^{\frac{1}{2}}\gamma(a,\psi^D) \quad \gamma(a,\psi^D)\gamma(-a,\psi^D) = |2D|^{-1}.$$

As in [W1], we will let $\chi_{\psi^D}(a)$ to be $(-1,a)\gamma(a,\psi^D)\gamma(1,\psi^D)^{-1}$, then

$$\chi_{\psi^D}(ab) = \chi_{\psi^D}(a)\chi_{\psi^D}(b)(a,b).$$

3. Spherical functions and orbital integrals on GL_2. In this section we consider orbital integrals of certain spherical functions as in [J]. We show that these integrals give rise to locally integrable functions on $G = GL_2(k)$ where k is a p-adic field.

Let $\mathcal{H}$ be the space of functions $H\colon G \to \mathbb{C}$ which are smooth on the right and satisfy

$$H(ag) = H(g), \quad a \in A, g \in G.$$

We let G act on $\mathcal{H}$ by right translations, i.e., For $g_0 \in G$ we define $(g_0H)(g) = H(gg_0)$. For each $H \in \mathcal{H}$ we let $\phi_H\colon k \to \mathbb{C}$ be the restriction of H to N, i.e.,

$$\phi_H(x) = H(n(x)), \quad x \in k. \tag{3.1}$$

Let

$$\mathcal{H}^0 = \{H \in \mathcal{H}\colon\ \phi_H \text{ is compactly supported}\}. \tag{3.2}$$
$$\mathcal{H}^c = \{H \in \mathcal{H}\colon\ H \text{ is compactly supported mod } A\}.$$

It follows from the Iwasawa decomposition that $\mathcal{H}^c \subset \mathcal{H}^0$.

LEMMA 3.1. *Let $H \in \mathcal{H}$. Let m be a positive integer such that the additive character ψ is nontrivial on P^{-m}. If*

$$H(gn(x)) = \psi(x)H(g) \text{ for all } g \in G \text{ and all } x \text{ such that } |x| \leq q^m, \tag{3.3}$$

then $H \in \mathcal{H}^0$.

Proof. Let $a \in k$ be of the form $a = 1 + b$, where $b \in P^n$ and $n > 0$ is large enough so that $H(gt(a)) = H(g)$ for all $g \in G$ and $b \in P^n$. Then

$$\begin{aligned} H(n(x)) &= H(n(x)t(a^{-1})) = H(t(a^{-1})t(a)n(x)t(a^{-1})) = H(n(x+bx)) \\ &= H(n(x)n(bx)). \end{aligned}$$

If x is large enough then we choose $b \in P^n$ such that $|bx| \leq q^m$ and such that $\psi(bx) \neq 1$. It follows from (3.3) that $H(n(x)n(bx)) = \psi(bx)H(n(x))$ and consequently that $H(n(x)) = 0$. □

For $H \in \mathcal{H}$ we define the functions $H_m \in \mathcal{H}$ to be

$$H_m(g) = \int_{|x|\leq q^m} H(gn(x))\psi^{-1}(x)\,dx. \tag{3.4}$$

COROLLARY 3.2. *If m is large enough then $H_m \in \mathcal{H}^0$.*

Proof. It is clear that H_m satisfy (3.3). Now choose m large enough so that ψ is nontrivial on P^{-m}. □

LEMMA 3.3. *Assume ψ is trivial on O and nontrivial on P^{-1}. Let $H \in \mathcal{H}$. Then for large m*

$$H_m(n(x)) = \psi(x)H_m(e)\Phi_m(x) \tag{3.5}$$

where Φ_m is the characteristic function of P^{-m}.

Proof. Let $M > 0$ be such that $H(gt(a)) = H(g)$ whenever $a \in 1 + P^M$. Let $m \geq M$. Using our assumption on ψ, it is easy to see that $t(a)$ fixes H_m for all $a \in 1 + P^m$. It follows from the proof of Lemma 3.1 that $H_m(n(x))$ is supported on P^{-m} and from the invariance of H_m (see (3.3)) that it is given by (3.5). □

LEMMA 3.4. *If $H \in \mathcal{H}^0$ and $g \in G$ then the function*

$$y \mapsto H(gn(y))$$

is compactly supported.

Proof. If $g \in B$ then the lemma follows from the definition of H_0. Thus it is enough to consider $g \in Bw_0B$ of the form $g = n(x)w_0$. We shall need the following matrix equation. Let $y \neq 0$. Then

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -\frac{1}{y} & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{y} & \\ & y \end{pmatrix} \begin{pmatrix} 1 & -y^2x - y \\ & 1 \end{pmatrix}. \tag{3.6}$$

If $|y|$ is large enough then $H(g\bar{n}(-\frac{1}{y})) = H(g)$ for all $g \in G$. Hence by (3.6) we have

$$H(n(x)w_0n(y)) = H\left(n(x)w_0n(y)\bar{n}\left(-\frac{1}{y}\right)\right) = H(n(-y^2x - y)).$$

If $|y|$ is large then $|-y^2x - y|$ will be large and since $H \in \mathcal{H}^0$, $H(n(-y^2x - y))$ will vanish. □

We define the following orbital integrals (See [J] (8)):

$$I(H,g) = \int H(gn(x))\,dx, \quad I_\psi(H,g) = \int^+ H(gn(x))\psi^{-1}(x)\,dx. \tag{3.7}$$

From [J], the function $I_\psi(H,g)$ is determined by its value at $g = e$ and $g = n(z)w_0$, $z \in k$.

PROPOSITION 3.5. *The integral defining $I(H,g)$ converges absolutely for every $H \in \mathcal{H}^0$ and $g \in G$. The integral defining I_ψ converges for every $H \in \mathcal{H}$ and $g \in G$. Moreover, for a fixed $H \in \mathcal{H}$ and $g \in G$ there exist an integer M_0 such that for all $m \geq M_0$*

$$\int_{|x|=q^m} H(g(n(x))\psi^{-1}(x)\,dx = 0.$$

Proof. The first statement follows from Lemma 3.4. To show that the integral in I_ψ converges we notice that for $n \geq m$

$$\int_{|y|\leq q^n} H(gn(y))\psi^{-1}(y) = q^{-m}\int_{|y|\leq q^n} H_m(gn(y))\psi^{-1}(y)\,dy. \tag{3.8}$$

By Corollary 3.2, if m is large enough then $H_m \in \mathcal{H}^0$ and by Lemma 3.4, $H_m(gn(y))$ is compactly supported in y. □

A closer look at the proof of Lemma 3.4 will give the following:

LEMMA 3.6. *Let H be as above. Then $I(H,g)$ and $I_\psi(H,g)$ are locally constant on the set $Bw_0B - w_0B$.*

We would like to show that $I(H,g)$ and consequently $I_\psi(H,g)$ give rise to locally integrable functions on G. This will follow from the next Proposition.

PROPOSITION 3.7. *Let $H_1 \in \mathcal{H}^c$ and $H_2 \in \mathcal{H}^0$. There exist positive constants $C = C_{H_1}, D = D_{H_2}$ and $E = E_{H_2}$ such that*

(a) $I(H_1, n(x)w_0) = 0$ *if* $|x| > C$.

(b) $|I(H_2, n(x)w_0)| < D$ *and* $|I_\psi(H_2, n(x)w_0)| < D$ *if* $|x| < E$.

Proof. We will prove both (a) and (b) simultaneously. Let $H \in \mathcal{H}^0$. let $D' = D'_H$ be a positive constant such that $H(g\bar{n}(y)) = H(g)$ for all $g \in G$ and $|y| < D_H'^{-1}$. We have

$$(3.9) \quad I(H, n(x)w_0) = \int_{|y|\le D'} H(n(x)w_0n(y))\,dy + \int_{|y|>D'} H(n(x)w_0n(y))\,dy$$
$$= H_{D'}(n(x)w_0) + \int_{|y|>D'} H(n(-y^2x - y))\,dy.$$

Here $H_{D'}(g) = \int_{|y|\le D'} H(gn(y))\,dy$. (See the proof of Lemma 3.4 for the second equality.) If $H \in \mathcal{H}^c$ then $H_{D'} \in \mathcal{H}^c$ hence $H_{D'}(n(x)w_0) = 0$ when $|x|$ is large. Since $|-y^2x - y| = |yx||y + \frac{1}{x}|$ and since $|y + \frac{1}{x}| = |y|$ for all $|y| > D'$ and $|x|$ large enough it follows that $|y^2x + y|$ is large for all $|y| > D'$ and $|x|$ large, hence the right integral vanishes for $|x|$ large and we get (a).

Since $|H| \in \mathcal{H}^0$ whenever $H \in \mathcal{H}^0$ and since $|I_\psi(H,g)| \le I(|H|,g)$, it is enough to prove (b) for $I(H,g)$ where $H \in \mathcal{H}^0$ is a nonnegative function.

Let $H_{D'}(w_0) = C_1$. If x is small then $H_{D'}(n(x)w_0) = H_{D'}(w_0)$ and by (3.9),

$$I(H, n(x)w_0) \le C_1 + C_2 \operatorname{vol}\{y \in k\colon\ |y^2x + y| < E'\}$$

where $C_2 = \max\{H(n(x)), x \in k\}$ and E' is some positive constant depending on ϕ_H. We write $|y^2x + y| = |yx||y + \frac{1}{x}|$ and divide our set above into three subsets:

(1) $|y| < |\frac{1}{x}|$. Then $|y^2x + y| < E'$ implies $|y| < E'$.

(2) $|y| = |\frac{1}{x}|$. Then $|y^2x + y| < E'$ implies $|y + \frac{1}{x}| < E'$. Hence $y \in -\frac{1}{x} + P^k$ where k depends only on E'.

(3) $|y| > |\frac{1}{x}|$. Then $|y^2x + y| < E'$ implies $|y^2x| < E'$. Also $|y| < |y^2x| < E'$.

Thus the set above is contained in a union of two sets whose volumes are bounded by a constant depending on E' and independent of x. □

THEOREM 3.8.

(a) *Let $H \in \mathcal{H}^0$. Then $I(H,g)$ defines a locally integrable function on G.*

(b) *Let $H \in \mathcal{H}$. Then $I_\psi(H,g)$ defines a locally integrable function on G.*

Proof. Let $f \in S(G)$ and define

$$H_f(g) = \int_A f(ag)\, da. \tag{3.10}$$

It is easy to see that $H_f \in \mathcal{H}^c$. In order to prove that $I(H, g)$ is locally integrable it is enough to show that

$$\int_G |I(H, g) f(g)|\, dg < \infty$$

for all $f \in S(G)$. Choosing a Haar measure $dg = da\, dx\, dy$ on the set of elements of the form $an(x)w_0n(y)$, $a \in A, x, y \in k$, we get

$$\int_G |I(H, g) f(g)|\, dg = \int_k |I(H, n(x)w_0)| I(H_{|f|}, n(x)w_0))\, dx.$$

By Proposition 3.7 (a), $I(H_{|f|}, n(x)w_0) = 0$ if $|x|$ is large and by Proposition 3.7 (b), $|I(H, n(x)w_0)|$ and $I(H_{|f|}, n(x)w_0)$ are bounded when $|x|$ is small. Since both functions are locally constant in $x \neq 0$, part (a) follows. For part (b) we notice that $I_\psi(H, g) = I_\psi(H', g)$ for some $H' \in \mathcal{H}^0$ and all $g \in G$. (By (3.8) and Corollary 3.2 we can choose $H' = q^{-m}H_m$ for m large enough.) Now the proof is the same as for part (a). (Alternatively, $|I_\psi(H', g)| \leq I(|H'|, g)$ and part (b) follows from (a)). □

We end this section by providing the exact asymptotics for $I_\psi(H, n(x)w_0)$ when $|x|$ is small.

Proposition 3.9. *Let H be in $\mathcal{H}$. Let ψ be as in Lemma 3.3. There exists an integer $M = M_H$ such that for every $m \geq M$ there exists a positive constant $C = C_{m,H}$ so that if $|x| < C$ and $x \neq 0$ then*

$$I_\psi(H, n(x)w_0) = I_\psi(H, w_0) + q^{-m} I_\psi(H, e) \int_{y+\frac{1}{x} \in P^{-m}} \psi^{-1}(y^2x + 2y)\, dy. \tag{3.11}$$

Proof. Since $I_\psi(H, g) = I_\psi(q^{-m}H_m, g)$ we can replace H with $q^{-m}H_m$. By Lemma 3.3 we can choose M' large enough so that $H_m(n(x)) = H_m(e)\psi(x)\Phi_m(x)$ for all $m \geq M'$. By Proposition 3.5 we can choose $M \geq M'$ large enough so that $H_m(e) = I_\psi(H, e)$ for all $m \geq M$. Fix $m \geq M$ Let $k \geq m$ be large enough so that $\bar{n}(y)H_m = H_m$ for all $|y| \leq q^{-k}$. By an argument similar to that in (3.9) we have

$$\begin{aligned} I_\psi(H, n(x)w_0) \;=\; & q^{-m} \int_{|y| \leq q^k} H_m(n(x)w_0n(y))\psi^{-1}(y)\, dy + q^{-m} \\ & \times \int_{|y| > q^k} H_m(n(x)w_0n(y))\psi^{-1}(y)\, dy \end{aligned}$$

$$= H_k(n(x)w_0) + q^{-m} \int_{|y|>q^k} H_m(n(-y^2x - y))\psi^{-1}(y)\, dy$$
$$= H_k(n(x)w_0) + I_\psi(H, e)q^{-m} \int_{|y|>q^k, |y^2x+y|\leq q^m} \psi(-y^2x - 2y)\, dy.$$

Fix k large enough so that $H_k(w_0) = I_\psi(H, w_0)$. If x is small (depending on k) then $H_k(n(x)w_0) = H_k(w_0)$ and we get

$$I_\psi(H, n(x)w_0) = I_\psi(H, w_0) + I_\psi(H, e)q^{-m} \int_{|y|>q^k, |y^2x+y|<q^m} \psi(-y^2x - 2y))\, dy.$$

If $|y| > |x|^{-1}$ then $|y^2x + y| > |y|$. If $|y| < |x|^{-1}$ then $|y^2x + y| = |y|$. In both cases $|y^2x + y| \geq |y|$. Since $k > m$ it follows that

$$\{y\colon\ |y| > q^k, |y^2x + y| \leq q^m\} = \left\{y\colon\ |y| > q^k, \left|y + \frac{1}{x}\right| \leq q^m\right\}.$$

If $|x| < q^{-k}$ then

$$\left\{y\colon\ |y| > q^k, \left|y + \frac{1}{x}\right| \leq q^m\right\} = \left\{y\colon\ \left|y + \frac{1}{x}\right| \leq q^m\right\}$$

and we get our formula. □

COROLLARY 3.10. *If $|x| \neq 0$ is small then*

$$I_\psi(H, n(x)w_0) = I_\psi(H, w_0) + I_\psi(H, e)\psi\left(\frac{1}{x}\right).$$

Proof. Assume for the moment ψ is as in Lemma 3.3. Since $xy^2 + 2y = x(y + \frac{1}{x})^2 - \frac{1}{x}$ we can write (3.11) as

$$I_\psi(H, n(x)w_0) = I_\psi(H, w_0) + I_\psi(H, e)\psi\left(\frac{1}{x}\right) q^{-m} \int_{|y+\frac{1}{x}|\leq q^m} \psi^{-1}\left(x\left(y + \frac{1}{x}\right)^2\right) dy$$
$$= q^{-m} I_\psi(H, w_0) + I_\psi(H, e)\psi\left(\frac{1}{x}\right) q^{-m} \int_k \Phi_m(y)\psi^{-1}(xy^2)\, dy.$$

where Φ_m is the characteristic function of P^{-m}. Since $|x|$ is very small we have $\psi^{-1}(xy^2) = 1$ for all $|y| \leq q^m$ Hence

$$q^{-m} \int_F \Phi_m(y)\psi^{-1}(xy^2)\, dy = 1.$$

and our proof is complete in this case.

In general, there is a character $\psi^D(x) = \psi(Dx)$ satisfying Lemma 3.3. From a change of variable, we see the relation

$$I_{\psi^D}(H, g) = |D|^{-1} I_\psi(H^D, gt(D^{-1})) \tag{3.12}$$

where $H^D(g) = H(gt(D))$ and $t(D)$ is defined in Section 2. The general case follows easily from this relation and the special case above. □

The corollary is also proved in [J] in the case $H \in \mathcal{H}^c$.

4. Relative Bessel functions. In this section we define the relative Bessel function i_π which is attached to a distinguished representation of $G = GL_2(k)$ where k is a p-adic field.

Let (π, V) be an infinite dimensional irreducible admissible representation of G. There exists a nontrivial functional L: $V \to C$ such that:

$$L(\pi(n)v) = \psi(n)L(v) \quad n \in N, v \in V.$$

It is well known that such a functional is unique up to scalar multiples. We call this functional the ψ-Whittaker functional. The Whittaker model $\mathcal{W}(\pi, \psi)$ is the space of functions

$$W_v(g) = L(\pi(g)v), \quad g \in G, v \in V. \tag{4.1}$$

The representation π is called A-distinguished if there exists a nontrivial linear functional R: $V \to \mathbf{C}$ such that

$$R(\pi(a)v) = R(v) \quad a \in A, v \in V. \tag{4.2}$$

This functional is also unique up to scalar multiples [J-L], [J-R]. The representation π is A-distinguished if and only if the central character of π is trivial [J-L]. The spherical model $\mathcal{H}(\pi)$ is the space of functions

$$H_v(g) = R(\pi(g)v), \quad g \in G, v \in V.$$

G acts on these models via right translations.

From now on, fix an A-distinguished representation (π, V) of G, and nonzero Whittaker functional L: $V \mapsto \mathbb{C}$. Following [J-L], [J-R] we construct a nontrivial spherical functional R: $V \mapsto \mathbb{C}$ satisfying (4.2). For each $W \in \mathcal{W}(\pi, \psi)$. We define

$$R(W) = \left. \frac{\int W(t(a))|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right|_{s=\frac{1}{2}} \tag{4.3}$$

where $L(\pi, s)$ is the local L-factor attached to π in [J-L], [Go]. Since the integral converges in some half plane $\mathrm{Re}(s) > C$ and can be meromorphically continued to the whole complex plane, and since the quotient is an analytic function of s [Go], R is well defined. Since π has trivial central character it follows that R satisfies (4.2). For each $W \in \mathcal{W}(\pi, \psi)$ we define $H_W \in \mathcal{H}(\pi)$ by

$$H_W(g) = R(\pi(g)W) = \left. \frac{\int W(t(a)g)|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right|_{s=\frac{1}{2}} .$$

It is easy to see that the mapping $W \mapsto H_W$ is a G isomorphism between the Whittaker model $\mathcal{W}(\pi, \psi)$ and the spherical model $\mathcal{H}(\pi)$.

The space $\mathcal{H}(\pi)$ gives a unique realization of π on a space of smooth functions H on G satisfying $H(ag) = H(g)$ for all $a \in A$, $g \in G$, hence $\mathcal{H}(\pi) \subset \mathcal{H}$ (see Section 3). For each $H \in \mathcal{H}(\pi)$ we define the integral

$$I_\psi(H, g) = \int_k^+ H(gn(y))\psi^{-1}(y)\, dy$$

where $\int_k^+$ is defined in (2.1). It follows from Proposition 3.5 that the integral converges. For a fixed $g \in G$ it defines a Whittaker functional on π. We write $H = H_W$ as in the isomorphism above. From the uniqueness of Whittaker functionals it follows that there exists a complex scalar $i_{\pi,\psi}(g)$ such that

$$I_\psi(H_W, g) = i_{\pi,\psi}(g)W(e), \quad \text{for all } W \in \mathcal{W}(\pi, \psi). \tag{4.4}$$

It follows from Proposition 3.5 that $i_{\pi,\psi}$ is locally constant on $Bw_0B - w_0B$ and from Theorem 3.8 that $i_{\pi,\psi}$ is locally integrable.

The function $i_{\pi,\psi}$ is the relative Bessel function associated to π. It is determined by the choice of character ψ and the Haar measure dx on k. Recall that the measure dx is chosen to be self dual for ψ.

LEMMA 4.1. *There exists a positive constant C_π such that if $|x| \leq C_\pi$, and $x \neq 0$ then*

$$i_{\pi,\psi}(n(x)w_0) = \frac{\epsilon(\pi, \frac{1}{2}, \psi) + \psi(\frac{1}{x})}{L(\pi, \frac{1}{2})}$$

where $\epsilon(\pi, s, \psi)$ and $L(\pi, s)$ are the ϵ-factor and L-factor respectively, attached to π in [Go]. Notice that $L(\pi, s)^{-1}$ is a polynomial in q^s hence we put $L(\pi, \frac{1}{2}) = L(\pi, s)|_{s=\frac{1}{2}}$.

Proof. Let $W \in \mathcal{W}(\pi, \psi)$ be such that $W(t(a)) = \Phi_{1+P}(a)$. where Φ_{1+P} is the characteristic function of $1 + P$. By Corollary 3.10 we have that for $|x|$ small

$$i_{\pi,\psi}(n(x)w) = I_\psi(H_W, w_0) + I_\psi(H_W, e)\psi\left(\frac{1}{x}\right).$$

Now

$$\begin{aligned} I_\psi(H_W, e) &= \int_k \left(\frac{\int_{k^*} W(t(a)n(y))|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right)\Bigg|_{s=\frac{1}{2}} \psi(-y)\, dy \\ &= L\left(\pi, \frac{1}{2}\right)^{-1} \int_k \int_{k^*} \Phi_{1+P}(a)\psi((a-1)y)\, d^*a\, dy \\ &= L\left(\pi, \frac{1}{2}\right)^{-1} \int_k \int_{k^*} \Phi_{1+P}(a)\psi((a-1)y)\, da\, dy \\ &= L\left(\pi, \frac{1}{2}\right)^{-1} \int_k \int_k \Phi_P(a)\psi(ay)\, da\, dy \\ &= L\left(\pi, \frac{1}{2}\right)^{-1}, \end{aligned}$$

the last identity being the Fourier inversion formula.

To compute $I_\psi(H, w_0)$ we recall the Jacquet-Langlands functional equation for irreducible admissible representations with trivial central character [J-L].

$$\gamma(\pi, s, \psi) \int_{k^*} W(t(a))|a|^{s-\frac{1}{2}}\, d^*a = \int_{k^*} W(w_0 t(a))|a|^{\frac{1}{2}-s}\, d^*a, \tag{4.5}$$

where each integral converges in some half plane (possibly disjoint) and the equality is in the sense of analytic continuation. Now

$$I_\psi(H_W, w_0) = \int_k \left(\frac{\int_{k^*} W(t(a)w_0 n(y))|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right)\Bigg|_{s=\frac{1}{2}} \psi(-y)\, dy.$$

Using (4.5) we get

$$I_\psi(H_W, w_0) = \int_k \left(\gamma(\pi, s, \psi) \frac{\int_{k^*} W(t(a)n(y))|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right)\Bigg|_{s=\frac{1}{2}} \psi(-y)\, dy,$$

and as in our first integral, this is exactly $\frac{\gamma(\pi, \frac{1}{2}, \psi)}{L(\pi, \frac{1}{2})}$. Since π is self dual, we have $\epsilon(\pi, \frac{1}{2}, \psi) = \gamma(\pi, \frac{1}{2}, \psi)$. □

Let X be the set of functions $W \in \mathcal{W}(\pi, \psi)$ such that

$$\mu_W(a) = W(t(a))$$

is compactly supported in k^*. Let $\xi_W(a) = |a|^{-1}\mu_W(a)$. For such W we have

$$\begin{aligned}
\phi_{H_W}(z) = H_W(n(z)) &= L\left(\pi, \frac{1}{2}\right)^{-1} \int W(t(a)n(z))\, d^*a \\
&= L\left(\pi, \frac{1}{2}\right)^{-1} \int W(t(a))\psi(az)\, d^*a \\
&= L\left(\pi, \frac{1}{2}\right)^{-1} \hat{\xi}_W(-z),
\end{aligned}$$

where $\hat{\xi}_W$ is the Fourier transform of ξ_W. In particular, ϕ_{H_W} is compactly supported in k. Since we know, [Go], that every function in $S(k^*)$ can be realized as μ_W for some $W \in \mathcal{W}(\pi, \psi)$ it follows that the set Z of functions

$$\phi_{H_W}(z) = H_W(n(z)) \tag{4.6}$$

where $W \in X$, is exactly the set of functions $\phi \in S(k)$ such that $\hat{\phi}(0) = 0$.

Lemma 4.2. *Let $W \in \mathcal{W}(\pi, \psi)$ be such that the function*

$$a \to W\begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

belongs to $S(k^*)$. Then

$$H_W(g) = \int_{k^*} i_{\pi,\psi}(gt(a)^{-1})W(t(a))\, d^*a$$

for all $g \in Bw_0B - w_0B$.

Proof. Let

$$\phi_{W,g}(y) = H_W(gn(y)).$$

Since $a \mapsto W(t(a))$ is compactly supported on k^* it follows that the function $x \mapsto \phi_{H_W}(x) = \phi_{W,e}(n(x))$ is compactly supported on k, hence by Lemma 3.4 the function $\phi_{W,g}$ defined above is compactly supported on k. We have

$$I_\psi(H_W, g) = \hat{\phi}_{W,g}(1) = i_{\pi,\psi}(g)W(e)$$

where $\hat{\phi}(z) = \int \phi(y)\psi^{-1}(zy)\,dy$ and similarly

$$\hat{\phi}_{W,g}(z) = i_{\pi,\psi}(gt(z)^{-1})W(t(z))|z|^{-1}.$$

Applying Fourier inversion for $\phi_{W,g}(0) = H_W(g)$ we obtain the equation above. □

Finally we would like to obtain a relation between the Bessel function $j_\pi = j_{\pi,\psi}$ and the relative Bessel function $i_\pi = i_{\pi,\psi}$. Recall (see [S]) that $j_{\pi,\psi}$ is defined by the equation

$$\int^+ W(gn(x))\psi^{-1}(x)\,dx = j_{\pi,\psi}(g)W(e)$$

where $g \in Bw_0B$ and $W \in \mathcal{W}(\pi,\psi)$.

PROPOSITION 4.3. *We have*

$$i_{\pi,\psi}(n(x)w_0) = \left(\frac{\int^+ j_{\pi,\psi}(t(a)w_0)\psi(ax)|a|^{s-\frac{1}{2}}\,d^*a}{L(\pi,s)}\right)\Bigg|_{s=\frac{1}{2}}$$

where the integral above converges for Re(s) *large and has analytic continuation to the whole complex plane. Moreover, if* π *is unitary then the above integral converges for Re*$(s) \geq \frac{1}{2}$, *hence for such* π *we have*

$$i_{\pi,\psi}(n(x)w_0) = L\left(\pi,\frac{1}{2}\right)^{-1}\int^+ j_{\pi,\psi}(t(a)w_0)\psi(ax)\,d^*a.$$

Proof. We shall analyze the right-hand side of the above formula. It follows from ([S], Lemma 4.2 (3)) that the above integral vanishes on the set $\{t(a)\colon\ |a| = q^r\}$ when r is a large integer (depending on x). We fix an $x \in k^*$ and $W \in \mathcal{W}(\pi,\psi)$ such that $W(e) = 1$. By ([S], Lemma 4.2 (4)) we have that if m is large enough then

$$W_m(t(a)w_0) = \begin{cases} j_\pi(t(a)w_0) & \text{if } |a| \leq q^{2m};\\ 0 & \text{if } |a| > q^{2m}.\end{cases}$$

Hence, if m is large enough then we have

$$\left(\frac{\int^+ j_{\pi,\psi}(t(a)w_0)\psi(ax)|a|^{s-\frac{1}{2}}\,d^*a}{L(\pi,s)}\right)\Bigg|_{s=\frac{1}{2}}$$
$$= \left(\frac{\int^+ W_m(t(a)w_0)\psi(ax)|a|^{s-\frac{1}{2}}\,d^*a}{L(\pi,s)}\right)\Bigg|_{s=\frac{1}{2}}$$

$$= \left(\frac{\int^+ W_m(t(a)n(x)w_0)|a|^{s-\frac{1}{2}}\, d^*a}{L(\pi, s)} \right)\Bigg|_{s=\frac{1}{2}}$$

$$= H_{W_m}(n(x)w_0).$$

Now $H_{W_m} = H_m$ for $H = H_W$ (see (3.4)) and we have that for m large (and x fixed) $H_m(n(x)w_0) = i_{\pi,\psi}(n(x)w_0)$. If π is unitary then the integral converges for $\mathrm{Re}\,(s) \geq \frac{1}{2}$ and our second formula follows. □

5. Computations of Bessel and relative Bessel functions for the principal series of $GL(2)$, nonarchimedean case. We begin by computing the Bessel function $j_{\pi,\psi}$ defined in [B] (see also [S]) for the principal series.

Let χ_1, χ_2 be multiplicative quasi-characters on k^*. We let $B(\chi_1, \chi_2)$ be the space of smooth functions $f\colon\ G \mapsto \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right|^{\frac{1}{2}} \chi_1(a_1)\chi_2(a_2) f(g), \quad a_1, a_2 \in k^*, x \in k. \tag{5.1}$$

We let π denote the G action on this space by right translations. We define the quasi character χ to be such that $f(bg) = \chi(b)f(g)$ whenever $f \in B(\chi_1, \chi_2)$ and $b \in B, g \in G$, i.e, $\chi(b) = |\frac{a_1}{a_2}|^{\frac{1}{2}}\chi_1(a_1)\chi_2(a_2)$, for $b \in B$ written as in (5.1).

We now define functions f_m, $m \geq 0$ in $B(\chi_1, \chi_2)$ by

$$f_m(g) = \begin{cases} \chi(b)\psi(x) & \text{if } g = bw_0n(x), \text{ and } |x| \leq q^m; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$f_m(g) = \int_{|x| \leq q^m} f_0(gn(y))\psi^{-1}(y)\, dy. \tag{5.2}$$

We now describe the Whittaker model associated to $B(\chi_1, \chi_2)$. Let $f(g) \in B(\chi_1, \chi_2)$.

We define

$$W_f(g) = \int_k^+ f(w_0n(y)g)\psi^{-1}(y)\, dy$$

where $\int_k^+$ is defined in (2.1). The integral converges ([Sh], Proposition 3.1). Since

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{a}{z} & 1 \\ & z \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \frac{a}{z} \\ & 1 \end{pmatrix},$$

we have

$$W_{f_m}(t(a)w_0) = \lim_{n\to\infty} \int_{|\frac{a}{z}|\le q^m}^{|z|\le q^n} \chi_1\left(-\frac{a}{z}\right)\chi_2(z)\left|\frac{a}{z^2}\right|^{\frac{1}{2}} \psi\left(\frac{a}{z}-z\right)\, dz.$$

Let $\pi = \pi(\chi_1, \chi_2)$ be the infinite dimensional irreducible component of $B(\chi_1, \chi_2)$ (see [J-L]). By ([S] (4.7)) and (5.2) we have that if m is large enough then $j_{\pi,\psi}(t(a)w_0) = W_{f_m}(t(a)w_0)$ for all $|a| \le q^{2m}$, hence

$$j_{\pi,\psi}(t(a)w_0) = \int^{+,-} \chi_1\left(-\frac{a}{z}\right)\chi_2(z)\left|\frac{a}{z^2}\right|^{\frac{1}{2}} \psi\left(\frac{a}{z}-z\right)\, dz \tag{5.3}$$

where $\int^{+,-}$ is as in (2.1).

Let $\pi_\chi = \pi(\chi, \chi^{-1})$ be the infinite dimensional component of $B(\chi, \chi^{-1})$ and $i_{\pi_\chi,\psi}$ be the relative Bessel function of π_χ as defined in (4.4). It is clear that the function $i_{\pi_\chi,\psi}$ is determined by its value at $n(x)w_0$ with $x \ne 0$. We now compute $i_{\pi_\chi,\psi}(n(x)w_0)$, $x \ne 0$.

Define

$$I_{\pi_\chi,\psi}(n(x)w_0, s) = \int^{+} j_{\pi_{\chi,s}}(t(a)w_0)|a|^{s-\frac{1}{2}}\psi(ax)\, d^*a.$$

By Proposition 4.3, the above integral converges for $Re(s)$ large and has meromorphic continuation to $\mathbf{C}$. The function $I_{\pi_\chi,\psi}(n(x)w_0, s)/L(\pi_\chi, s)$ is entire and we have that

$$i_{\pi_\chi,\psi}(n(x)w_0) = \left.\frac{I_{\pi_\chi,\psi}(n(x)w_0, s)}{L(\pi_\chi, s)}\right|_{s=\frac{1}{2}}.$$

If π_χ is unitary then $I_{\pi_\chi,\psi}(n(x)w_0, s)$ converges for $\mathrm{Re}\,(s) \ge \frac{1}{2}$ (see the table in [Go] p. 1.36) and is entire at $s = \frac{1}{2}$. For such π we have

$$i_{\pi_\chi,\psi}(n(x)w_0) = \frac{I_{\pi_\chi,\psi}(n(x)w_0, \frac{1}{2})}{L(\pi_\chi, \frac{1}{2})}.$$

By (5.3) we have

$$I_{\pi_\chi,\psi}\left(n(x)w_0, \frac{1}{2}\right) = \int^{+}\int^{+,-} \chi\left(-\frac{a}{z^2}\right)|a|^{1/2}|z|^{-1}\psi\left(\frac{a}{z}-z+ax\right)\, dz\, d^*a. \tag{5.4}$$

Let $r = r(\chi) \in \mathbf{R}$ be such that $|\chi(x)| = |x|^r$ for all $x \in k^*$ and write $\chi(x) = \chi_0(x)|x|^r$ where χ_0 is a unitary character. By [J-L] we have that if π_χ is unitary then $|r(\chi)| \le 1/2$. If $|r(\chi)| = 1/2$ then π_χ is unitary if and only if $\chi_0^2 = 1$.

THEOREM 5.1. *Let π_χ be the representation defined above and assume that π_χ is unitary. Assume further that $\chi \neq |\ |^{1/2}$ and $\chi \neq -|\ |^{-1/2}$, (that is, $|r(\chi)| \neq 1/2$ or $|r(\chi)| = 1/2$ and $\chi_0^2 \neq 1$.) Then for $x \neq 0$,*

$$i_{\pi_\chi,\psi}(n(x)w_0) = L\left(\pi_\chi, \frac{1}{2}\right)^{-1} \psi\left(\frac{1}{2x}\right) |x|^{-\frac{1}{2}} \times \int^{+,-} \chi(-a)\gamma(ax,\psi)\psi\left(-\frac{a}{4x} - \frac{1}{4ax}\right) d^*a$$

where $\gamma(\cdot,\psi)$ is the Weil constant (see (2.2)).

Proof. We will assume that $0 < r(\chi) < 1/2$ and prove the equality above. For the case where $r = 0$ or the case where $r = 1/2$, $\chi_0^2 = 1$ and $\chi_0 \neq 1$ we shall use a limit process to establish the above equality. Since every unitary representation π_χ with $r(\chi) < 0$ is equivalent to one with $r(\chi) \geq 0$ (namely $\pi_{\chi^{-1}}$) and since the right-hand side of the above equality is invariant under $\chi \to \chi^{-1}$ we do not lose any generality with this restriction.

We shall manipulate the integral in (5.4) to the form of the theorem. We postpone the justification of the mainupulation to the Appendix.

We start by switching the order of integration in (5.4)

$$I_{\pi_\chi,\psi}(n(x)w_0, 1/2) = \int^{+}\int^{+} \chi\left(-\frac{a}{z^2}\right) |a|^{1/2}|z|^{-1}\psi\left(\frac{a}{z} - z + ax\right) d^*a\, dz. \tag{5.5}$$

Making a change of variable $a \mapsto az^2$ (no need for justification) we get

$$I_{\pi_\chi,\psi}(n(x)w_0, 1/2) = \int^{+}\int^{+} \chi(-a)|a|^{1/2}\psi(az - z + az^2x)\, d^*a\, dz. \tag{5.6}$$

Switching again order of integration we get

$$I_{\pi_\chi,\psi}(n(x)w_0, 1/2) = \int^{+,-} \chi(-a)|a|^{1/2} \int^{+} \psi(az - z + az^2x)\, dz\, d^*a. \tag{5.7}$$

Completing the square

$$axz^2 + z(a-1) = ax\left(z + \frac{a-1}{2ax}\right)^2 - \frac{(a-1)^2}{4ax}$$

we get

$$i_{\pi_\chi,\psi}(n(x)w_0) = \int^{+,-} \chi(-a)|a|^{\frac{1}{2}}\psi\left(-\frac{(a-1)^2}{4ax}\right) \times \left(\int^{+} \psi\left(ax\left(z + \frac{a-1}{2ax}\right)^2\right) dz\right) d^*a. \tag{5.8}$$

We now analyze the inner integral. Let Φ_m be the characteristic function of P^{-m}. For m large (depending on a and x) we can write the dz integral in (5.8) in the form

$$\lim_{m\to\infty}\int \Phi_m(z)\psi(axz^2)\,dz$$
$$= \left|\frac{ax}{4}\right|^{-\frac{1}{2}} \gamma(ax,\psi) \lim_{m\to\infty}\int \hat{\Phi}_m^2(z)\psi\left(-\frac{z^2}{ax}\right)\,dz,$$

where the equality follows from Weil formula (2.2). When m is large, over the support of $\hat{\Phi}_m^2$, we have $\psi(-\frac{z^2}{ax}) \equiv 1$; thus the integral above equals $|2|^{-1}$. From this and (5.8) we get the desired formula. Now it follows from Lemma 14.3 that this formula extends to the case where $r = 0$ or $r = 1/2$, $\chi_0^2 = 1$ and $\chi_0 \neq 1$. □

Remark 5.2. It follows from the theorem that if we define i_{π_χ,ψ^D} by replacing ψ with ψ^D (in particular, we change the definition of Haar measure and thus of $\gamma(*,\psi^D)$) then

$$i_{\pi_\chi,\psi^D}(n(x)w_0) = i_{\pi_\chi,\psi}\left(n\left(\frac{x}{D}\right)w_0\right).$$

THEOREM 5.3. *Let $\chi = ||^{1/2}$ and ψ be unramified. Then*

$$i_{\pi_\chi,\psi}(n(x)w_0) = L\left(\pi_\chi,\frac{1}{2}\right)^{-1}(\psi(x^{-1})-1) = (1-q^{-1})^{-1}(\psi(x^{-1})-1).$$

Proof. Fix $x \in k^*$ and let $\chi_r(y) = |y|^r$. Let $I(r,x) = I_{\pi_{\chi_r},\psi}(n(x)w_0,\frac{1}{2})$. (See (5.4)). We need to compute $I(-1/2,x)$ and use the equality

$$i_{\pi_{\chi_{-1/2}},\psi}(n(x)w_0) = L\left(\pi_{\chi_{-1/2}},\frac{1}{2}\right)^{-1} I(-1/2,x)$$

to finish the proof. By Lemma 14.4 we have that $\lim_{r\to -1/2} I(r,x) = \psi(x^{-1}) + 1$. We will show that $\lim_{r\to -1/2} I(r,x) - I(-1/2,x) = 2$ for all x hence $I(-1/2,x) = \psi(x^{-1}) - 1$ and we are done. To do that we separate the d^*a integral in (5.4) that divides $I(r,x)$ into two parts: The first where $|a| > 1$ which we denote by $I^+(r,x)$ and the second where $|a| \leq 1$ we denote by $I^-(r,x)$. Hence $I(r,x) = I^+(r,x) + I^-(r,x)$ for every r such that $-1/2 \leq r < 0$. Using the Dominated Convergence Theorem it is possible to show that $\lim_{r\to -1/2} I^+(r,x) - I^+(-1/2,x) = 0$. By explicit calculation of both terms we can show that $\lim_{r\to -1/2} I^-(r,x) - I^-(-1/2,x) = 2$. □

We remark here that we were surprised to discover that $I(r,x)$ is discontinuous in r at $r = 1/2$, that is, at the special representation with $\chi(x) = |x|^{1/2}$. Theorem 5.3

can be translated into the following exponential sums identity:

COROLLARY 5.4. *Assume that ψ is unramified then*

$$\int^{+}\int^{+} |a|^{-1}\psi\left(\frac{a}{z} - z + ax\right) dz\, da = \psi(x^{-1}) - 1.$$

6. Relative Bessel distributions. In this section, we define the relative Bessel distribution and relate it to the relative Bessel function $i_{\pi,\psi}$.

Let $S(G)$ be the space of compactly supported locally constant functions on G. Let (π, V) be an A-distinguished representation of G with a ψ-Whittaker functional L as in (4.1) and a spherical functional R as in (4.3). Let $(\hat{\pi}, \hat{V})$ be the representation contragredient to π where $\hat{V}$ is the space of smooth linear functionals on V. Let $\hat{L}$ be a ψ^{-1} Whittaker functional on $\hat{V}$. It follows from [J-L] or ([Go], Theorem 2, p. 1.18) that we can (and will) normalize $\hat{L}$ so that if $v \in V$, and $\hat{v} \in \hat{V}$ are such that either $a \mapsto W_v(t(a))$ or $a \mapsto \hat{W}_{\hat{v}}(t(a))$ have compact support in k^*, then

$$\hat{v}(v) = \langle \hat{v}, v\rangle = \int_{k^*} W_v(t(a))\hat{W}_{\hat{v}}(t(a))\, d^*x. \tag{6.1}$$

Here $W_v \in \mathcal{W}(\pi, \psi)$ and $\hat{W}_{\hat{v}} \in \mathcal{W}(\hat{\pi}, \psi^{-1})$ are defined as in (4.1). The assumption on the support is made to ensure that the integral converges. If π is unitary then this integral converges for all $v \in V$, $\hat{v} \in \hat{V}$, and defines a nondegenerate G invariant bilinear form on $V \times \hat{V}$.

It is well known that $\hat{\hat{\pi}} \cong \pi$. We identify V and $\hat{\hat{V}}$. For a linear functional (smooth or not) T: $\hat{V} \mapsto \mathbb{C}$ and $f \in S(G)$ we define the linear functional $\rho(f)T$: $\hat{V} \mapsto \mathbb{C}$ by

$$(\rho(f)T)(\hat{v}) = \int_G f(g)T(\hat{\pi}(g^{-1})\hat{v})\, dg, \qquad \hat{v} \in \hat{V}. \tag{6.2}$$

It is clear that $\rho(f)T$ is a smooth linear functional hence we can identify $\rho(f)T$ with a vector $v_{f,T} \in V$. We now define the Relative Bessel distribution $I_{\pi,\psi}$: $S(G) \mapsto \mathbb{C}$ to be

$$I_{\pi,\psi}(f) = \langle R, \rho(f)\hat{L}\rangle = R(v_{f,\hat{L}}).$$

It is easy to check that the distribution $I_{\pi,\psi}(f)$ is independent of our choice of Whittaker model L, and the choice of Haar measure dx. It is determined by the choice of Haar measure dg over G.

Let T be a linear functional on V. Let $f \in S(G)$. we define the linear functional $\rho^*(f)T$ to be

$$\rho^*(f)T(v) = \int_G f(g)T(\pi(g)v)\, dg, \quad v \in V.$$

Notice that $\rho^*(f)T = \rho(f^*)T$ where $f^*(g) = f(g^{-1})$. We will use the notation $\hat{v}_{f^*,R} = \rho(f^*)R$. The following Lemma is immediate.

LEMMA 6.1.

$$I_{\pi,\psi}(f) = \langle \rho^*(f)R, \hat{L}\rangle.$$

LEMMA 6.2. *Let $v \in V$ be such that $a \mapsto W_v(t(a))$ has compact support in k^*. Then*

$$\int_{k^*} I_{\pi,\psi}(\rho_r(t(a))f)W_v(t(a))\, d^*a = \int_G f(g)H_v(g)\, dg \tag{6.3}$$

where $H_v(g) = R(\pi(g)v)$.

Proof. It follows from Lemma (6.1) that $I_{\pi,\psi}(\rho_r(t(a))f) = W_{\hat{v}_{f^*,R}}(t(a))$. Hence, by (6.1) the left-hand side of (6.3) is the same as $\langle \hat{v}_{f^*,R}, v\rangle$ which equals $\langle \rho(f^*)R, v\rangle$. By (6.2) this is exactly the right-hand side of (6.3). □

THEOREM 6.3.

$$I_{\pi,\psi}(f) = \int_G i_{\pi,\psi}(g)f(g)\, dg.$$

Proof. We define the distribution $I'_{\pi,\psi}$ on $S(G)$ to be

$$I'_{\pi,\psi}(f) = \int_G i_{\pi,\psi}(g)f(g)\, dg, \quad f \in S(G). \tag{6.4}$$

Since $i_{\pi,\psi}$ is locally integrable, $I'_{\pi,\psi}$ is well defined. We shall prove that $I'_{\pi,\psi} = I_{\pi,\psi}$. Let $f \in S(G)$. There exist an integer m such that $\rho_r(t(a))f = f$ for all $a \in 1 + P^m$. Let χ be the characteristic function of $1 + P^m$ and let $W \in \mathcal{W}(\hat{\pi}, \psi)$ be such that $\hat{W}(t(a)) = q^m\chi(a)$ for all $a \in k^*$. We have

$$\begin{aligned} I'_{\pi,\psi}(f) &= \int_{k^*} I'_{\pi,\psi}(\rho_r(t(a))f)W(t(a))\, d^*a \\ &= \int_{k^*} W(t(a)) \left(\int_G f(gt(a))i_{\pi,\psi}(g)\, dg\right) d^*a \\ &= \int_G f(g) \left(\int_{k^*} i_{\pi,\psi}(gt(a)^{-1})W(t(a))\, d^*a\right) dg \\ &= \int_G f(g)H_W(g)\, dg, \end{aligned}$$

where the last equality follows from Lemma 4.2. By (6.3) this last integral equals

$$\int_{k^*} I_{\pi,\psi}(\rho_r(t(a))f)W(t(a))\, d^*a$$

which is just $I_{\pi,\psi}(f)$. □

COROLLARY 6.4. *We fix a Haar measure dg on G such that $dg = d^*a\,dx\,dy$ on the set of elements of the form $an(x)w_0n(y)$ where $a \in A$ and $x, y \in k$. Then*

$$I_{\pi,\psi}(f) = \int_k i_{\pi,\psi}(n(x)w_0)O_\psi(f, n(x)w_0)\,dx$$

where $O_\psi(f,g) = \int_A \int_k f(agn(y))\psi(y)\,d^*a\,dy$ is the orbital integral of f as defined in [J].

Proof. Using the invariance properties of $i_{\pi,\psi}$ the corollary is clear. □

We finish this section by relating our definition of I_π to the local distributions appearing in Jacquet's relative trace formula [J]. Let $\{v_i\colon\ i = 1, 2, \ldots\}$ be a linear basis of V obtained by putting together bases of the various K-types of π. (Here K is a compact open subgroup of G such as $K = GL_2(O)$.) Let $\{v_i^*\colon\ i = 1, 2, \ldots\}$ denote the dual basis. We let $H_{v_i} \in \mathcal{H}(\pi)$ be the spherical function corresponding to v_i and $\hat{W}_{v_i^*} \in \mathcal{W}(\hat{\pi}, \psi^{-1})$ be the Whittaker functions corresponding to v_i^*.

LEMMA 6.5.

$$I_{\pi,\psi}(f) = \sum_i \left(\int_G f(g)H_{v_i}(g)\,dg\right)\hat{W}_{v_i^*}(e) = \sum_i H_{\pi(f)(v_i)}(e)\hat{W}_{v_i^*}(e),$$

where the integral vanishes for all but finite number of indices i.

Proof. Let $\hat{v}_{f^*,R} \in V^*$ defined as above (see (6.2)). By Lemma 6.1 we have $I_{\pi,\psi}(f) = \hat{L}(\hat{v}_{f^*,R})$. Since $\hat{v}_{f^*,R} \in \hat{V}$ we have

$$\hat{v}_{f^*,R} = \sum_i \lambda_i v_i^*, \tag{6.5}$$

where all but a finite number of the λ_is are zero. The λ_is are given by

$$\lambda_i = \langle v_i, \hat{v}_{f^*,R}\rangle = \int_G f(g^{-1})R(\pi(g^{-1})v_i)\,dg = \int_G f(g)H_{v_i}(g)\,dg.$$

Applying $\hat{L}$ to (6.5) we get the desired equality. □

Recall when (π, V) is unitary, there is a G invariant Hermitian form on V given by

$$\langle v_1, v_2\rangle = \int W_{v_1}(t(a))\overline{W_{v_2}(t(a))}\,d^*a.$$

Corollary 6.6. *Assume that* (π, V) *is unitary. Let* $\{v_i\colon\ i = 1, 2, \ldots\}$ *be an orthonormal basis of* K *types with respect to the above Hermitian form. Then*

$$I_{\pi,\psi}(f) = \sum_i H_{\pi(f)v_i} \overline{W_{v_i}(e)} \tag{6.6}$$

where $\overline{W_{v_i}(e)}$ *is the complex conjugate of* $W_{v_i}(e)$.

Proof. Let $\hat{v}_i$ be the linear form $V \mapsto \mathbb{C}$ given by:

$$\hat{v}_i(v) = \int W_v(t(a))\overline{W_{v_i}(t(a))}\, d^*a.$$

Then the set $\{\hat{v}_i\}$ gives a basis of $\hat{\pi}$ which is the dual basis of $\{v_i\}$. Compare the definition of $\hat{v}_i$ with (6.1), we see $\hat{L}(\hat{v}_i) = \overline{W_{v_i}(e)}$. Thus our formula is an immediate consequence of Lemma 6.5. □

The right-hand side of (6.6) is the local distribution appearing in Jacquet's relative trace formula.

7. Whittaker functions for the double cover of GL_2**.** In the next few sections, we study the Bessel distribution on the double cover of GL_2. We first fix some notations.

Let k be a p-adic field. For $a, b \in k^*$ the Hilbert symbol (a, b) is defined by

$$(a, b) = \begin{cases} 1 & \text{if there are } x, y \in k \text{ such that } a = x^2 - by^2; \\ 0 & \text{if not.} \end{cases}$$

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and set

$$\chi(g) = \begin{cases} c & \text{if } c \neq 0\ ; \\ d & \text{if } c = 0. \end{cases}$$

For $g_1, g_2 \in Gl_2(k)$ we define the cocycle $\alpha(g_1, g_2)$ to be

$$\alpha(g_1, g_2) = \left(\det(g_1), \frac{\chi(g_1g_2)}{\chi(g_1)}\right)\left(\frac{\chi(g_1g_2)}{\chi(g_1)}, \frac{\chi(g_1g_2)}{\chi(g_2)}\right)$$

(see [K-P], p. 41). We let $\bar{G}$ be the metaplectic cover of G, i.e., $\bar{G} = \{[g, \epsilon]\colon\ g \in G, \epsilon = \pm 1\}$ with multiplication given by

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1g_2, \alpha(g_1, g_2)\epsilon_1\epsilon_2)].$$

The group $\bar{G}$ fits into the exact sequence

$$1 \to \{\pm 1\} \to \bar{G} \to G \to 1. \tag{7.1}$$

In general if H is any subgroup of G we let $\bar{H}$ denote its full inverse image in G. In particular if (7.1) splits over H, then $\bar{H}$ is the direct product of $\{\pm 1\}$ with H. We note in particular that (7.1) splits over N.

We let $S = SL_2(k)$ and $\bar{S}$ the metaplectic cover of S viewed as a subgroup of $\bar{G}$. We shall confuse an element $g \in G$ and $\epsilon = \pm 1$ with

$$g = [g, 1] \in \bar{G}, \quad \epsilon = [e, \epsilon] \in \bar{G}. \tag{7.2}$$

We let N, A and B be subgroups of G as defined in Section 2. We let

$$Z^2 = \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} : \ z \in (k^*)^2 \right\}.$$

The center of $\bar{G}$ is

$$\bar{Z}^2 = Z^2 \times \{\pm 1\}.$$

Let Z be the center of G. The group $\bar{Z}$ is abelian but not central in $\bar{G}$. Let $\bar{z} = \mathrm{diag}(z, z)$. The commutation is given by

$$g\bar{z} = \bar{z}g(\det(g), z) \tag{7.3}$$

where $\bar{z} = [\,\mathrm{diag}(z, z), 1]$ is in $\bar{Z}$, $g = [g, 1]$ is in $\bar{G}$ and (x, y) is the Hilbert symbol of $x, y \in k^*$.

Let $\bar{G}$ act on a complex vector space V and assume that $\bar{Z}^2$ acts according to the character ω, i.e., $z(v) = \omega(z)v$ for all $z \in \bar{Z}^2$ and all $v \in V$. Let $\Omega(\omega)$ be the set of characters extending ω to $\bar{Z}$. For $\mu \in \Omega(\omega)$ we define

$$V_\mu = \{v \in V \mid z(v) = \mu(z)v \text{ for all } z \in \bar{Z}. \tag{7.4}$$

Lemma 7.1.

$$V = \bigoplus_{\mu \in \Omega(\omega)} V_\mu$$

where $\bar{Z}$ acts on V_μ according to the character μ.

Proof. The proof is immediate from the fact that $\bar{Z}/\bar{Z}^2 \cong k^*/(k^*)$ is a finite abelian group. □

Notice that $\bar{S}$ stabilizes each V_μ and that V_μ might be zero.

In the following we shall analyze Whittaker functions and orbital integrals on $\bar{G}$. The proofs are standard and are in complete analog to the ones in Section 3 and in [S], [B] and [J]. We will omit them.

Let ψ be a nontrivial character on k viewed as a character on N as in Section 2. Let μ be a genuine quasi-character on $\bar{Z}$, i.e., μ is a quasi-character on $\bar{Z}$ satisfying $\mu(z\epsilon) = \epsilon\mu(z)$ for all $z \in \bar{Z}$. Note that considered as a function on k^*,

$$\mu(ab) = \mu(a)\mu(b)(a, b).$$

We define the Whittaker space $\mathcal{W}(\psi, \mu)$ to be the space of functions $W\colon\ \bar{G} \mapsto \mathbb{C}$ which are smooth on the right and satisfy

$$W(zn(y)g) = \mu(z)\psi(y)W(g) \quad z \in \bar{Z}, y \in k, g \in \bar{G}. \tag{7.5}$$

We let $\bar{G}$ act on this space by right translations. We denote by ω the restriction of μ to $\bar{Z}^2$, the center of $\bar{G}$. It is clear that $\bar{Z}^2$ acts on $\mathcal{W}(\psi, \mu)$ through the character ω. By Lemma 7.1 we can decompose

$$\mathcal{W}(\psi, \mu) = \bigoplus_{\mu' \in \Omega(\omega)} \mathcal{W}_{\mu'}(\psi, \mu)$$

where $\bar{Z}$ acts on the space $\mathcal{W}_{\mu'}(\psi, \mu)$ through the character μ'. We will denote $\mathcal{W}_\mu(\psi) = \mathcal{W}_\mu(\psi, \mu)$. Let $\mathcal{W}^0(\psi, \mu)$ be the space of functions $W \in \mathcal{W}(\psi, \mu)$ such that the function $a \mapsto W(t(a))$ is compactly supported in k^*; let $\mathcal{W}^0_\mu(\psi) = \mathcal{W}^0(\psi, \mu) \cap \mathcal{W}_\mu(\psi)$. For $W \in \mathcal{W}_\mu(\psi)$ we define

$$W_m(g) = \int_{|x| \le q^m} W(g(n(x))\psi^{-1}(x)\, dx. \tag{7.6}$$

LEMMA 7.2. *If m is large enough then $W_m \in \mathcal{W}^0_\mu(\psi)$.*

As in Lemma 3.4, the following Lemma is a consequence of the matrix equality:

$$\begin{aligned} &\begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -\frac{1}{y} & 1 \end{pmatrix} \\ &\quad = \left[\begin{pmatrix} y & \\ & y \end{pmatrix}, (-a, y) \right] \begin{pmatrix} 1 & -\frac{a}{y} \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{a}{y^2} & \\ & 1 \end{pmatrix}. \end{aligned} \tag{7.7}$$

LEMMA 7.3. *If $W \in \mathcal{W}^0(\psi, \mu)$. and $g \in \bar{B}w\bar{B}$ then the function*

$$y \mapsto W(gn(y))$$

is compactly supported.

We now define the orbital integrals [J]. Let

$$J(W,g)=\int W(gn(x))\,dx,\quad J_\psi(W,g)=\int^{+} W(gn(x))\psi^{-1}(x)\,dx. \tag{7.8}$$

The convergence of these integrals depends on g and W. The following Proposition specifies the cases that we shall consider. It follows from Lemma 7.2 and Lemma 7.3.

PROPOSITION 7.4. *The integral defining $J(W,g)$ converges absolutely for every $W\in\mathcal{W}^0(\psi,\mu)$ and $g\in\bar{B}w\bar{B}$. The integral defining $J_\psi(W,g)$ converges for every $W\in\mathcal{W}(\psi,\mu)$ and $g\in\bar{B}w\bar{B}$. Moreover, for a fixed W, J and J_ψ are locally constant on the set BwB.*

LEMMA 7.5. *Let $W\in\mathcal{W}(\psi,\mu)$ and let m be a large integer. There exists a constant $D=D_{W,m}$ such that for $|a|>D$*

$$J_\psi(W,t(a)w)=W(e)\int_{|\frac{a}{y^2}-1|<q^{-m}}(-a,y)\mu(y)\psi^{-1}\left(\frac{a}{y}+y\right)\,dy.$$

Proof. The proof is identical to the one in ([S], Lemma 4.2 (2)). We use the matrix equality (7.7). See also Proposition 3.9 for a similar argument. □

COROLLARY 7.6. *Let $W\in\mathcal{W}(\psi,\mu)$. There exists a constant $C=C_W$ such that for $|a|>C$:*

(a) $J_\psi(W,t(a)w)=0$ *if a is not a square.*

(b) *If $a=b^2$ then*

$$\begin{aligned}J_\psi(W,t(a)w) \;=\; & W(e)|b|^{\frac{1}{2}}\mu(b)(-1,b)(\mu(-1)(-1,-b)\gamma(b,\psi)\psi(2b)\\ &+\gamma(-b,\psi)\psi(-2b)).\end{aligned} \tag{7.9}$$

Proof. Choose a large m and let $C=D_{W,m}$ as in the lemma. Since the integral that appears in Lemma 7.5 does not depend on W, we can choose any $W\in\mathcal{W}(\psi,\mu)$ and compute $J_\psi(W,t(a)w)$ to obtain the result above. This is done in ([G-PS, 1], Proposition 4.4.2) for a particular W, hence we are done.

We now sketch a more direct proof:

For (a) we can use Hensel's Lemma to see that if m is large enough and $\frac{a}{y^2}\in 1+P^m$ then $a\in(k^*)^2$. In particular $(-a,y)=(-1,y)$.

For (b) we assume $a=b^2$ and let m be large enough so that $m>2\,\mathrm{val}(2)$ and $\mu(z)\equiv 1$ when $z\in 1+P^m$. Let $m'=m+\mathrm{val}(2)$. The set $\{y\colon\ |(\frac{b}{y})^2-1|<q^{-m}\}$ can be separated into a union of the two disjoint sets $\{y\colon\ |\frac{b}{y}-1|<q^{-m'}\}$ and

$\{y\colon |\frac{b}{y}+1| < q^{-m'}\}$. Integrating on the first set we get

$$W(e)\int_{|\frac{b}{y}-1|<q^{-m'}} \mu(y)(-1,y)\psi^{-1}\left(\frac{b^2}{y}+y\right)\,dy$$
$$= W(e)\int_{|\frac{b}{y}-1|<q^{-m'}} \mu(y)(-1,y)\psi^{-1}\left(y\left(\frac{b}{y}-1\right)^2+2b\right)\,dy.$$

Changing variable $z = \frac{b}{y} - 1$ we get

$$W(e)|b|\psi(-2b)\mu(b)(-1,b)\int_{|z|<q^{-m'}} \psi^{-1}\left(\frac{bz^2}{z+1}\right)\,dz.$$

It is not difficult to check that if m is large enough then the map $z^2 \mapsto \frac{z^2}{z+1}$ is a well-defined isometry from $k^2 \bigcap P^{m'}$ to $k^2 \bigcap P^{m'}$. Hence we can change variables in the above integration to get

$$W(e)|b|\psi(-2b)\mu(b)(-1,b)\int \Phi_{m'}(z)\psi^{-1}(bz^2)\,dz.$$

By the Weil formula (2.2) we have that

$$\int \Phi_m(z)\psi^{-1}(bz^2)\,dz = \left|\frac{b}{4}\right|^{-\frac{1}{2}}\gamma(-b,\psi)\int \hat{\Phi}^2_{m'}(z)\psi\left(-\frac{z^2}{b}\right)\,dz.$$

Now as in the proof of Theorem 5.1, if $|b|$ is large enough then this last integral equals $|2|^{-1}$ and we see that the contribution from the set $|\frac{b}{y}-1| < q^{-m'}$ is:

$$W(e)|b|^{\frac{1}{2}}\psi(-2b)\mu(b)(-1,b)\gamma(-b,\psi).$$

Similar arguments apply for the integral over the second set. □

COROLLARY 7.7. *There exist positive constants constant $C = C_W$ and D such that for $|a| > C$*

$$|J_\psi(W, t(a)w)| \leq D|a|^{\frac{1}{4}}|\mu(a)|^{\frac{1}{2}}.$$

We now come to our main theorem in this section:

THEOREM 7.8. *The function $J_\psi(W, g)$ defined on the open cell $\bar{B}wN$ is locally integrable in $\bar{G}$.*

Proof. The proof is essentially the same as in [B]. We replace Corollary 4.2 in [B] with Corollary 7.7. (See also Theorem 3.8 for a similar situation.) □

8. Bessel functions for the double cover of GL_2. In this section we define and analyze the Bessel functions $j_\pi^{\mu,\mu'}$ which are attached to a genuine irreducible admissible representation π of $\bar{G} = \overline{GL_2(k)}$ where k is a p-adic field. We show that they are locally integrable and give a kernel function for the action of w on the μ-Whittaker space of π.

Let (π, V) be an infinite dimensional irreducible admissible representation of $\bar{G}$. We shall assume that π is genuine, that is, $\pi(\epsilon\bar{g}) = \epsilon\pi(\bar{g})$ for all $\bar{g} \in \bar{G}$ and $\epsilon = \pm 1$ as defined in (7.2). By [G-PS, 1] there exists a genuine character μ of $\bar{Z}$ and a nontrivial functional l^μ: $V \to C$ such that:

$$l^\mu(\pi(\bar{z})\pi(n)v) = \mu(\bar{z})\psi(n)l^\mu(v) \quad n \in N, \bar{z} \in \bar{Z},\ v \in V. \tag{8.1}$$

Moreover by [G-PS, 1] the space of such functionals for any fixed μ and a fixed nontrivial character ψ is at most one dimensional. Let ω_π be the central character of π and let $\Omega(\omega_\pi)$ be the set of characters μ of $\bar{Z}$ extending ω_π. For each $\mu \in \Omega(\omega_\pi)$ we fix a Whittaker functional l^μ satisfying (8.1) and nontrivial if possible. If l^μ is nontrivial then we define $\mathcal{W}(\pi, \mu, \psi)$ to be the space of functions

$$W_v^\mu(\bar{g}) = l_\pi^\mu(\pi(\bar{g})v), \quad \bar{g} \in \bar{G}, v \in V.$$

The space $\mathcal{W}(\pi, \mu, \psi)$ gives a unique realization of π on a space of smooth functions W on $\bar{G}$ satisfying $W(\bar{z}n\bar{g}) = \mu(\bar{z})\psi(n)W(\bar{g})$ for all $\bar{z} \in \bar{Z}$, $n \in N$ and $\bar{g} \in \bar{G}$, hence $\mathcal{W}(\pi, \mu, \psi) \subset \mathcal{W}(\mu, \psi)$ (see (7.5)). By Lemma 7.1 we can write the space $\mathcal{W}(\pi, \mu, \psi)$ as a direct sum: $\mathcal{W}(\pi, \mu, \psi) = \bigoplus_{\tilde{\mu} \in \Omega(\omega_\pi)} \mathcal{W}_{\tilde{\mu}}(\pi, \mu, \psi)$ where the group $\bar{Z}$ acts on $\mathcal{W}_{\tilde{\mu}}(\pi, \mu, \psi)$ through the character $\tilde{\mu}$. For each $\tilde{\mu} \in \Omega(\omega_\pi)$ there exists $x \in k^*$ defined up to its square class in $k^*/(k^*)^2$ such that

$$\mu(\bar{z}) = \tilde{\mu}(\bar{z})(x, z) \tag{8.2}$$

for all $\bar{z} = [\mathrm{diag}\,(z, z), 1] \in \bar{Z}$.

Notation. If μ and $\tilde{\mu}$ satisfy (8.2) then we write

$$(\mu, \tilde{\mu}) = x \bmod (k^*)^2.$$

LEMMA 8.1. *Let $W \in \mathcal{W}_{\tilde{\mu}}(\pi, \mu, \psi)$. Then W is supported on the set $g = [g, 1]$ such that $det(g) = (\tilde{\mu}, \mu)$ mod $(k^*)^2$.*

Proof. This follows immediately from (7.3). □

For $W \in \mathcal{W}(\pi,\mu,\psi)$ and $\tilde{\mu} \in \Omega(\omega_\pi)$ we define the integral (See [G-PS, 1] (4.2.1))

$$J_{\psi,\tilde{\mu}}(W,\bar{g}) = \int_k^+ \sum_{\bar{z}\in\bar{Z}/\bar{Z}^2} W(\bar{g}n(y)\bar{z})\psi^{-1}(y)\tilde{\mu}^{-1}(\bar{z})\,dy$$

where the sum is taken over a finite set of representatives of $k^*/(k^*)^2$. The summation over $\bar{Z}/\bar{Z}^2$ projects $\mathcal{W}(\pi,\mu,\psi)$ onto $\mathcal{W}_{\tilde{\mu}}(\pi,\mu,\psi)$ and we have

$$J_{\psi,\tilde{\mu}}(W_v,\bar{g}) = J_\psi(W_{\tilde{v}},\bar{g})$$

where $\tilde{v} = \sum_{\bar{Z}/\bar{Z}^2} \tilde{\mu}^{-1}(\bar{z})\pi(\bar{z})v$ and J_ψ is the integral defined in (7.8). Hence the next Theorem follows from Proposition 7.4 and Theorem 7.8.

THEOREM 8.2. *$J_{\psi,\tilde{\mu}}(W,\bar{g})$ is convergent for all $\bar{g} \in \bar{B}wN$ and gives a locally constant function on $\bar{B}wN$ which is locally integrable on all of $\bar{G}$.*

For a fixed $\bar{g} \in \bar{B}wN$ and a fixed $\tilde{\mu}$ in $\Omega(\omega_\pi)$, $J_{\psi,\tilde{\mu}}(W,\bar{g})$ defines a $(\psi,\tilde{\mu})$ Whittaker functional on V and from the uniqueness of such Whittaker functionals it follows that there exists a scalar $j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g})$ such that

$$J_{\psi,\tilde{\mu}}(W^\mu_v,\bar{g}) = j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g})l^{\tilde{\mu}}(v) = j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g})W^{\tilde{\mu}}_v(1). \tag{8.3}$$

The function $j^{\mu,\tilde{\mu}}_{\pi,\psi}$ is called the $(\mu,\tilde{\mu})$-Bessel function of π. If $l^{\tilde{\mu}}$ is identically zero, we define $j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g})$ to be identically zero. It is clear that $j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g})$ is dependent on the choice of Whittaker functional $l^{\tilde{\mu}}$ unless $\tilde{\mu} = \mu$, when it is only dependent on the choice of Haar measure on k.

The next proposition is the analog of ([S], Lemma 4.2 (1)).

PROPOSITION 8.3. *Let $v \in V$ such that the functions $a \mapsto W^{\tilde{\mu}}_v(t(a))$ are compactly supported in k^* for all $\tilde{\mu} \in \Omega(\omega_\pi)$. Let $g \in \bar{B}wB$. Then*

$$W^\mu_v(\bar{g}) = \sum_{\tilde{\mu}\in\Omega(\omega_\pi)} \int j^{\mu,\tilde{\mu}}_{\pi,\psi}(\bar{g}t(a)^{-1})W^{\tilde{\mu}}_v(t(a))\,d^*a. \tag{8.4}$$

Proof. The proof is the same as in [S]. Since $V = \bigoplus_{\tilde{\mu}\in\Omega(\omega)} V_{\tilde{\mu}}$, it follows from Lemma 8.1 that it is enough to prove (8.4) for $v \in V_{\tilde{\mu}}$. Let $\phi(y) = W^\mu_v(\bar{g}n(y))$. Fix $a \in k^*$. Then

$$\hat{\phi}(a) = \int W^\mu_v(\bar{g}n(y))\psi^{-1}(ay)\,dy = |a|^{-1}\int W^\mu_v(\bar{g}t(a)^{-1}n(y)t(a))\psi^{-1}(y)\,dy,$$

where we obtained the last equality by the change of variables $y \mapsto ay$. $\pi(t(a))$ sends v to a vector $v' = \pi(t(a))v \in V_{\mu'}$ where $\mu'(z) = \tilde{\mu}(z)(a,z)$. It follows that the last integral is the same as

$$J_{\psi,\mu'}(W^{\mu}_{v'}, \bar{g}t(a)^{-1}) = j^{\mu,\mu'}_{\pi,\psi}(\bar{g}t(a)^{-1})W^{\mu'}_{v'}(e) = j^{\mu,\mu'}_{\pi,\psi}(\bar{g}t(a)^{-1})W^{\mu'}_{v}(t(a)).$$

Since $W^{\mu''}_{v'}(t(a)) = 0$ unless $\mu'' = \mu'$ we can write

$$\hat{\phi}(a) = |a|^{-1} \sum_{\mu''\in\Omega(\omega_\pi)} j^{\mu,\mu''}_{\pi,\psi}(\bar{g}t(a)^{-1})W^{\mu''}_{v}(t(a)).$$

We now apply Fourier inversion to find $\phi(0) = W_v(\bar{g})$ to get the result. □

COROLLARY 8.4. [G-PS, 1] *The mapping* $v \mapsto \{W^{\tilde{\mu}}_v(t(a)) \mid \tilde{\mu} \in \Omega(\omega_\pi)\}$ *is injective.*

Proof. Assume $W^{\tilde{\mu}}_v(t(a)) = 0$ for all such $\tilde{\mu}$ and $a \in k^*$. Then by (8.4) $W^{\mu}_v(\bar{g}) = 0$ for all $\bar{g} \in \bar{G}$ and all $\mu \in \Omega(\omega_\pi)$. This is a contradictory to the existence of a nontrivial Whittaker functional. □

9. Inner product formulas in the Whittaker-Kirillov models. Our purpose in this section is to give a positive definite $\bar{G}$ invariant Hermitian form (the inner product) on the Whittaker or Kirillov models for irreducible unitary representations of $\bar{G}$. Since the complex conjugate of a unitary representation is isomorphic to the contragredient of that representation, these formulas will also give a $\bar{G}$ invariant nondegenerate bilinear form on the Whittaker models of the representation and its contragredient representation. It is possible to generalize these formulas to every irreducible admissible representation of $\bar{G}$ as in [Go]. We will leave that to the interested reader. The proofs and formulas here are almost identical to the ones given in Godement [Go] so we will omit many details.

Our main theorem of this section is the following: Let k be a p-adic field $S = SL_2(k)$ and $G = GL_2(k)$. Let $\bar{S}$ and $\bar{G}$ be double covers of S and G respectively. Let $n(x), t(a), s(a), w$ be as in Section 2. Let (π, V) be a unitary irreducible admissible genuine representation of $\bar{G}$. Let ω_π be the central character of π and $\Omega(\omega_\pi)$ the finite set of characters of $\bar{Z}$ extending ω_π. For each $\mu \in \Omega(\omega_\pi)$ we fix l^μ to be a (ψ, μ) Whittaker functional on V nontrivial if possible.

THEOREM 9.1. *Let* $\langle , \rangle$ *be a* $\bar{G}$ *invariant inner product on* V. *There exist scalars* $\lambda_{l^\mu} > 0$, $\mu \in \Omega(\omega_\pi)$ *such that*

$$\begin{aligned} (9.1) \qquad \langle v_1, v_2\rangle &= \sum_{\mu\in\Omega(\omega_\pi)} \lambda_{l^\mu} \int_{k^*} l^\mu(\pi(t(a))v_1)\overline{l^\mu(\pi(t(a))v_2)}\, d^*a \\ &= \sum_{\mu\in\Omega(\omega_\pi)} \lambda_{l^\mu} \int_{k^*} W^{\mu}_{v_1}(t(a))\overline{W^{\mu}_{v_2}(t(a))}\, d^*a. \end{aligned}$$

By choosing suitable linear forms l^μ, the constants λ_{l^μ} can be taken to be 1.

We shall prove this Theorem on a case-by-case basis starting with supercuspidal representations. For the case of unitary principal series, unitary complementary series and special representations we shall use the Jacquet integrals to define our Whittaker functionals and determine the λ_{l^μ} appearing in (9.1) explicitly.

9.1. Supercuspidal representations. Let (π, V) be a unitary supercuspidal representation of $\bar{G}$ and $\langle , \rangle$ a $\bar{G}$ invariant inner product on V. Since π is unitary, it follows that ω_π is unitary and that every character extending ω_π to $\bar{Z}$ is unitary. Let $\mu_1, \ldots, \mu_r \in \Omega(\omega_\pi)$ be the characters of $\bar{Z}$ that support a nontrivial Whittaker functional l^{μ_j}. Let $v \mapsto (W_v^{\mu_1}(t(a)), \ldots, W_v^{\mu_r}(t(a)))$ be the Kirillov model for π. By [G-PS, 1] this map is injective and the image is $S(k^*)^r = S(k^*) \times S(k^*) \times \cdots \times S(k^*)$. For each such μ we let the subspace $\mathcal{W}^\mu$ be the space of functions in the Kirillov model of the form $(0, \ldots, 0, \phi, 0, \ldots, 0)$ where $\phi \in S(k^*)$ is in the μ component. Equivalently we denote by $\mathcal{V}^\mu$ the space of vectors $v \in V$ such that $l^{\mu'}(\pi(t(a))v) = 0$ for all $\mu' \neq \mu$. (Notice that this space is different from V_μ.)

LEMMA 9.2. *Let $v \in V^\mu$ and $\tilde{v} \in V^{\tilde{\mu}}$. If $\mu \neq \tilde{\mu}$ then $\langle v, \tilde{v} \rangle = 0$.*

Proof. Let $\{\delta\}$ be a set of representatives for the square classes k^*/k^{*2}. Each function ϕ in $S(k^*)$ can be written in the form $\phi = \sum_\delta \phi_\delta$ where ϕ_δ is supported on the square class of δ. Hence it is enough to show that $\langle \phi_1, \phi_2 \rangle = 0$ for functions in $\mathcal{W}^\mu \times \mathcal{W}^{\tilde{\mu}}$ which are supported on some square classes. Let $\phi_1 \in \mathcal{W}^\mu$ and $\phi_2 \in \mathcal{W}^{\tilde{\mu}}$ be such functions. Applying $\bar{z} \in \bar{Z}$ to the inner product of ϕ_1 and ϕ_2 we get

$$\langle \phi_1, \phi_2 \rangle = \langle \pi(\bar{z})\phi_1, \pi(\bar{z})\phi_2 \rangle = \mu(\bar{z})(z, \delta_1)\overline{\tilde{\mu}(\bar{z})}(z, \delta_2)\langle \phi_1, \phi_2 \rangle$$

where δ_i is the square class support of ϕ_i, $i = 1, 2$ and $\overline{\tilde{\mu}(\bar{z})}$ is the complex conjugate of $\tilde{\mu}(z)$. If δ_1 gives the same square class as δ_2 then our assumption that $\tilde{\mu} \neq \mu$ gives $\langle \phi_1, \phi_2 \rangle = 0$. So assume that δ_1 and δ_2 are not in the same square class. It follows that ϕ_1 and ϕ_2 are supported on disjoint compact sets A and B. Let χ_X be the characteristic function of X. We will show that there exists an integer M depending on A and B such that

$$\langle \chi_{a+P^m}, \chi_{b+P^n} \rangle = 0 \quad \text{for all } n, m \geq M \text{ and all } a \in A, b \in B. \tag{9.2}$$

Acting by $n(x)$ we see that $\pi(n(x))\phi(t) = \psi(tx)\phi(t)$. Writing $t = a+p$ where $p \in P^m$ we have $\psi(tx) = \psi(xa + xp)$. If $|x| \leq q^m$ then $\psi(xp) = 1$ for all $p \in P^m$. Fix an integer $M > 0$ and let $m, n \geq M$. For $|x| \leq q^M$ we have

$$\begin{aligned}\langle \chi_{a+P^m}, \chi_{b+P^n} \rangle &= \langle \pi(n(x))\chi_{a+P^m}, \pi(n(x))\chi_{b+P^n} \rangle \\ &= \psi(ax)\psi(-bx)\langle \chi_{a+P^m}, \chi_{b+P^n} \rangle = \psi(ax - bx)\langle \chi_{a+P^m}, \chi_{b+P^n} \rangle.\end{aligned}$$

Since A and B are disjoint compact sets, it follows that there exists an integer $M_1 > 0$ such that $|a-b| > q^{-M_1}$ for all $a \in A, b \in B$. It follows that if we choose

$M = M_1$ then for a fixed $a \in A$ and $b \in B$ we can find $x \in k$ such that $|x| \geq q^M$ and $\psi((a-b)x) \neq 1$. Hence we get (9.2) and we have proved the Lemma. □

We now restrict $\langle , \rangle$ to V^μ.

LEMMA 9.3. *There exists a scalar λ_{l^μ} such that*

$$\langle v_1, v_2 \rangle = \lambda_{l^\mu} \int W^\mu_{v_1}(t(a)) \overline{W^\mu_{v_2}(t(a))}\, d^*a$$

for every $v_1, v_2 \in V^\mu$.

Proof. The restriction $\langle , \rangle$ to $\mathcal{W}^\mu$ gives an inner product $S(k^*)$ which is invariant under the action of the mirabolic P which splits in $\bar{G}$. It follows from [Go] that every P inner product on the Kirillov model $S(k^*)$ is a scalar multiple of the form $\langle , \rangle_P$ given by

$$\langle \phi_1, \phi_2 \rangle_P = \int \phi_1(a) \overline{\phi_2(a)}\, d^*a.$$

Hence we have proved our Lemma. □

It follows from the injectivity of the Kirillov model and from Lemma 9.2 and Lemma 9.3 that the inner product $\langle , \rangle$ on V is given by (9.1). Hence we have proved Theorem 9.1 for supercuspidal representations.

Remark 9.4. Since the Kirillov model of every infinite dimensional unitary representation contains the space $S(k^*)^r$ we can use the same argument to prove that the given inner product on such a representation is of the form of (9.1) when restricted to this subspace. While this is sufficient for the applications in this paper we shall need the explicit form of the inner product formula in our subsequent work. For simplicity, we will assume ψ is trivial on O, nontrivial on P^{-1}.

9.2. Induction from $\bar{G}^*$ to $\bar{G}$. We recall some facts from [G-PS, 2]. Let $\bar{G}^* = \{(g, \epsilon) \mid g \in \bar{G}, \det g \in (k^*)^2\}$. Its center is $\bar{Z}$. Let σ be an irreducible admissible genuine representation of $\bar{S}$. Let μ be a character of $\bar{Z}$ whose restriction to $\bar{Z} \cap \bar{S}$ agrees with the central character of σ. We can extend σ to a representation $\mu \times \sigma$ of $\bar{G}^* = \bar{Z}\bar{S}$ given by

$$\mu \times \sigma(\bar{z}\bar{s}) = \mu(\bar{z})\sigma(\bar{s}).$$

PROPOSITION 9.5. ([G-PS, 2] 1.1.4,1.1.6) *The representation $\pi = Ind(\mu \times \sigma, \bar{G}^*, \bar{G})$ is irreducible. Moreover, every irreducible admissible genuine representation π of $\bar{G}$ is of the form $\pi = Ind(\mu \times \sigma, \bar{G}^*, \bar{G})$ for some irreducible admissible genuine representation σ of $\bar{S}$ and some character μ of $\bar{Z}$.*

It is also easy to describe the unitary representations of $\bar{G}$ via the unitary representations of $\bar{S}$.

LEMMA 9.6. *Let σ be an irreducible admissible genuine representation of $\bar{S}$. Then $\pi = Ind(\mu \times \sigma, \bar{G}^*, \bar{G})$ is unitary if and only if σ and μ are unitary and the G invariant hermitian form on V_π is given (up to scalar) by*

$$\langle F_1, F_2\rangle_{\bar{G}} = \sum_{a\in k^*/(k^*)^2} \langle F_1(t(a)), F_2(t(a))\rangle_{\bar{S}}$$

where $\langle , \rangle_{\bar{S}}$ is the invariant inner product on V_σ and F_1, F_2 are in the induced space V_π, that is, F_i: $\bar{G} \to V_\sigma$ is smooth and satisfies $F_i(g^*\bar{g}) = \mu \times \sigma(g^*)F_i(\bar{g})$ where $g^* \in \bar{G}^*$, $\bar{g} \in \bar{G}$ and $i = 1, 2$.

Remark 9.7. If V_σ is given as a space of function $V_\sigma = \{f_\alpha\}$ where f_α: $\bar{S} \to \mathbf{C}$ and the action of $\bar{S}$ is by right translations then we can realize $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ as a space of smooth functions F: $\bar{G}^* \times \bar{G} \to \mathbf{C}$ satisfying:

(1) $\bar{s} \mapsto F(\bar{s}, \bar{g})$ is in V_σ for every fixed $\bar{g} \in \bar{G}$.

(2) $F(\bar{z}g^*, \bar{g}) = \mu(\bar{z})F(g^*, \bar{g})$ for all $\bar{z} \in \bar{Z}, g^* \in \bar{G}^*$ and $\bar{g} \in \bar{G}$.

(3) $F(g_1^* g_2^*, \bar{g}) = F(g_1^*, g_2^*\bar{g})$ for all $g_1^*, g_2^* \in \bar{G}^*$ and $\bar{g} \in \bar{G}$.

9.3. Unitary principal series. We take σ to be a unitary principal series of $\bar{S}$. That is, there exists a unitary character χ of k^* such that $\sigma = \mathrm{Ind}(\chi\chi_\psi, \bar{B_S}, \bar{S})$. (Recall the definition of χ_ψ in Section 2.) Let $\pi = \mathrm{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$. Then by the remark above V_π can be realized as the space of smooth functions F: $\bar{G}^* \times \bar{G} \to \mathbf{C}$ satisfying

(1) $F(s(a)n(x)g^*, \bar{g}) = \chi(a)\chi_\psi(a)|a|F(g^*, \bar{g})$

and satisfying (2) and (3) of the remark.

It follows from [W1] and from Lemma 9.6 that the following is a positive definite $\bar{G}$ invariant hermitian form:

$$\langle F_1, F_2\rangle = \sum_{a\in k^*/(k^*)^2} \int_k F_1(wn(x), t(a))\overline{F_2(wn(x), t(a))}\, dx. \tag{9.3}$$

For each $F \in \pi$ and $a \in k^*$ let $V_{F,a}(x) = F(wn(x), t(a))$. Then the above formula reads

$$\langle F_1, F_2\rangle = \sum_{a\in k^*/(k^*)^2} \int_k V_{F_1,a}(x)\overline{V_{F_2,a}(x)}\, dx = \sum_{a\in k^*/(k^*)^2} \int_k \hat{V}_{F_1,a}(y)\overline{\hat{V}_{F_2,a}(y)}\, dy.$$

Here $\hat{V}_{F,a}(y) = \int^+ V_{F,a}(x)\psi(-xy)\, dx$ where $\int^+$ is defined in (2.1). By ([W1], p. 7) this integral converges. Define

$$l(a, b, F) = \int^+ V_{F,a}(x)\psi(-bx)\, dx, \quad a, b \in k^*, F \in V_\pi.$$

Then

$$l(a,b,s(\alpha)^{-1}F) = (\alpha,a)\chi(\alpha)\chi_{\psi}(\alpha)|\alpha|l(a,\alpha^2 b,F)$$

and

$$l(ac,b,F) = l(a,b,t(c)F).$$

Thus we have

$$
\begin{aligned}
(9.4)\qquad \langle F_1,F_2\rangle &= \sum_{a\in k^*/(k^*)^2} \int l(a,b,F_1)\overline{l(a,b,F_2)}\,db \\
&= \frac{1}{2}\sum_{a\in k^*/(k^*)^2}\sum_{b\in k^*/(k^*)^2} |2b^{-1}| \\
&\quad \int l(a,\alpha^2 b^{-1},F_1)\overline{l(a,\alpha^2 b^{-1},F_2)}|\alpha|\,d\alpha \\
&= \frac{1}{2}\sum_{a\in k^*/(k^*)^2}\sum_{b\in k^*/(k^*)^2} |2b^{-1}| \\
&\quad \int l(a,b^{-1},s(\alpha)^{-1}F_1)\overline{l(a,b^{-1},s(\alpha)^{-1}F_2)}\,d^*\alpha \\
&= \frac{1}{2}\sum_{b\in k^*/(k^*)^2} |2b^{-1}| \sum_{a\in k^*/(k^*)^2} \\
&\quad \int l(b,b^{-1},t(b)^{-1}t(a)s(\alpha)^{-1}F_1) \\
&\qquad \times \overline{l(b,b^{-1},t(b)^{-1}t(a)s(\alpha)^{-1}F_2)}\,d^*\alpha \\
&= \sum_{b\in k^*/(k^*)^2} |b^{-1}| \int l(b,b^{-1},t(a)F_1)\overline{l(b,b,^{-1}t(a)F_2)}\,d^*a.
\end{aligned}
$$

For $b \in k^*$ we can define the Whittaker functional $l^{\mu_b}\colon \pi \to \mathbf{C}$ by

$$l^{\mu_b}(F) = l(b,b^{-1},F).$$

It is easy to see that $l^{\mu_b}(n(x)F) = \psi(x)l^{\mu_b}(F)$ and that $l^{\mu_b}(\bar{z}F) = \mu_b(z)l^{\mu_b}(F)$ where $\mu_b(z) = \mu(z)(b,z)$. Hence l^{μ_b} is a (ψ,μ_b)-Whittaker functional. If we define $W_F^{\mu_b}(\bar{g}) = l(b,b^{-1},\bar{g}F) = l^{\mu_b}(\bar{g}F)$ then equation (9.4) reads

$$(9.5)\qquad \langle F_1,F_2\rangle = \sum_{b\in k^*/(k^*)^2} |b^{-1}| \int W_{F_1}^{\mu_b}(t(a))\overline{W_{F_2}^{\mu_b}(t(a))}\,d^*a$$

which is of the form of (9.1) hence we have proved Theorem 9.1 for this case.

9.4. A useful identity. Let $\psi^D(x) = \psi(Dx)$.

PROPOSITION 9.8. *Let Φ be a smooth function on k and assume that $\Phi(x) = O(|x|^{-1-\epsilon})$ for some $\epsilon > 0$. Assume that $\hat{\Phi}$ is compactly supported on k. Let $\nu \in \mathbf{C}$ be such that $0 < Re(\nu) < 1$. Then there exists a function $\Delta(y) = \Delta_{\psi,D,\nu}(y)$ independent of Φ and constant on square classes such that*

$$\int \Phi(x)(-1,x)\gamma(x,\psi^D)|x|^{\nu-1}\,dx = \int \Delta(y)\hat{\Phi}(y)|y|^{-\nu}\,dy. \tag{9.6}$$

Assume that $|2| = |D| = 1$, ψ is unramified and τ is a nonsquare unit. Then $1, \tau, \varpi, \tau\varpi$ form a complete set of square class representatives and

$$\Delta_{\psi,D,\nu}(y) = \begin{cases} \frac{q^{-\nu}-q^{\nu-1}}{1-q^{-2\nu}} & \text{if } y = \varpi \text{ or } y = \tau\varpi; \\ \frac{1-q^{-1}}{1-q^{-2\nu}} + q^{\nu-\frac{1}{2}}(yD,\varpi) & \text{if } y = 1 \text{ or } y = \tau. \end{cases} \tag{9.7}$$

Proof. Write $\Phi(x) = \int \hat{\Phi}(y)\psi(yx)\,dx$. Then

$$\begin{aligned}
&\int \Phi(x)(-1,x)\gamma(x,\psi^D)|x|^{\nu-1}\,dx \\
&= \frac{\gamma(1,\psi^D)}{\gamma(-1,\psi^D)} \int \left(\int \hat{\Phi}(y)\psi(xy)\,dy\right) \gamma(-x,\psi^D)|x|^{\nu-1}\,dx \\
&= \frac{\gamma(1,\psi^D)}{\gamma(-1,\psi^D)} \lim_{m\to\infty} \int_{|x|\le q^m} \int \hat{\Phi}(y)\psi(xy)\gamma(-x,\psi^D)|x|^{\nu-1}\,dy\,dx \\
&= \frac{\gamma(1,\psi^D)}{\gamma(-1,\psi^D)} \lim_{m\to\infty} \int \int_{|x|\le q^m} \hat{\Phi}(y)\psi(xy)\gamma(-x,\psi^D)|x|^{\nu-1}\,dx\,dy \\
&= \frac{\gamma(1,\psi^D)}{\gamma(-1,\psi^D)} \int \hat{\Phi}(y) \left(\int^{+} \psi(xy)\gamma(-x,\psi^D)|x|^{\nu-1}\,dx\right)\,dy,
\end{aligned}$$

where the last equality follows from the Dominated Convergence Theorem. To see that we write

$$\begin{aligned}
&\int |\hat{\Phi}(y)| \left|\int_{|x|\le q^m} \psi(xy)\gamma(-x,\psi^D)|x|^{\nu-1}\,dx\right|\,dy \\
&= \int |\hat{\Phi}(y)||y|^{\mathrm{Re}(\nu)} \left|\int_{|x|\le q^m|y|} \psi(x)\gamma(-xy^{-1},\psi^D)|x|^{\nu-1}\,dx\right|\,dy. \\
&\le \int |\hat{\Phi}(y)||y|^{\mathrm{Re}(\nu)} \int_{|x|\le B} |x|^{\mathrm{Re}(\nu)-1}\,dx\,dy.
\end{aligned} \tag{9.8}$$

Here the existence of the constant B above follows from the fact that the inner integral stabilizes independently of y (see an argument below). Since the last integral does not depend on m we can use our assumptions on $\hat{\Phi}$ and ν to apply the Dominated Convergence Theorem.

Continuing with the above integral and making a change of variables we write

$$\begin{aligned}\Delta(y) &= \int^{+} \psi(xy)\gamma(-x,\psi^{D})|x|^{\nu-1}\,dx \\ &= \int^{+} \psi(x)\gamma(-xy^{-1},\psi^{D})|xy^{-1}|^{\nu-1}|y|^{-1}\,dx \\ &= |y|^{-\nu}\int^{+} \gamma(-xy,\psi^{D})|x|^{\nu-1}\psi(x)\,dx.\end{aligned}$$

Hence we have

$$\Delta(y) = \frac{\gamma(1,\psi^{D})}{\gamma(-1,\psi^{D})}\int^{+} \gamma(-xy,\psi^{D})|x|^{\nu-1}\psi(x)\,dx$$

and it is clear that Δ is constant on square classes. We now compute Δ under the assumptions described in the proposition. Our arguments will show that the integral giving Δ stabilizes for large m independent of y hence the existence of the constant B above. (The stabilization does not depend on our assumptions on ψ and D).

Assume $|2| = |D| = 1$ and ψ is unramified. Then $\gamma(x,\psi^{D}) = (x,x) = 1$ if $\operatorname{val}(x)$ is even. Write

$$\Delta(y) = \sum_{m=-\infty}^{\infty} A_m$$

where

$$A_m = \int_{|x|=q^m} \gamma(-xy,\psi^{D})|x|^{\nu-1}\psi(x)\,dx = q^{m\nu}\int_{|x|=1} \gamma(-\varpi^{-m}xy,\psi^{D})\psi(\varpi^{-m}x)\,d^{*}x.$$

Since $\gamma(-yx\varpi^{-m},\psi^{D}) = \gamma(-y\varpi^{-m},\psi^{D})(-y\varpi^{-m},x)$ and since $(-y\varpi^{-m},x)$ is a quadratic character in x which is trivial on $1+P$, the integral vanishes for $m > 1$. We now compute $\Delta(y)$ for $y = 1,\tau,\varpi,\tau\varpi$.

Case 1: *when* $y = \varpi,\tau\varpi$. When $y = \varpi$ or $y = \tau\varpi$ we have that A_m vanishes when m is even. When m is odd $\gamma(-yx\varpi^{-m},\psi^{D}) = 1$ hence we have

$$\Delta(y) = \sum_{M=1}^{\infty} q^{(-2M+1)\nu}(1-q^{-1}) - q^{\nu-1} = \frac{(1-q^{-1})q^{-\nu}}{1-q^{-2\nu}} - q^{\nu-1} = \frac{q^{-\nu}-q^{\nu-1}}{1-q^{-2\nu}}.$$

Case 2: *when* $y = 1, \tau$. When $y = 1$ or $y = \tau$ we have that the integral vanishes when m is odd $m \neq 1$. Hence

$$\Delta(y) = \left(\sum_{M=0}^{\infty} q^{-2M\nu}(1 - q^{-1})\right) + q^{\nu} \int_{|x|=1} \gamma(-\varpi^{-1}yx, \psi^D)\psi(\varpi^{-1}x)\, d^*x.$$

By [J], if $|z| = q$ then

$$\gamma(z, \psi^D) = q^{-\frac{1}{2}} \sum_{u \in R^*/P} \psi(uz\varpi^{-1}D)(u, \varpi).$$

Hence

$$\begin{aligned}\Delta(y) &= \frac{1 - q^{-1}}{1 - q^{-2\nu}} + q^{\nu-\frac{1}{2}-1} \sum_{v \in R^*/P} \sum_{u \in R^*/P} \psi(-uyv\varpi^{-1}D)(u, \varpi)\psi(v\varpi^{-1}) \\ &= \frac{1 - q^{-1}}{1 - q^{-2\nu}} + q^{\nu-\frac{1}{2}-1} \sum_{u \in R^*/P} (u, \varpi) \sum_{v \in R^*/P} \psi(-uyv\varpi^{-1}D + v\varpi^{-1}).\end{aligned}$$

The last sum is -1 when $uyD \neq 1$ and $q - 1$ otherwise. Hence

$$\begin{aligned}\Delta(y) &= \frac{1 - q^{-1}}{1 - q^{-2\nu}} + q^{\nu-\frac{1}{2}-1} \left(\left(\sum_{u \in R^*/P, uyD \neq 1} -(u, \varpi)\right) + (q - 1)(yD, \varpi)\right) \\ &= \frac{1 - q^{-1}}{1 - q^{-2\nu}} + q^{\nu-\frac{1}{2}}(yD, \varpi).\end{aligned}$$

□

9.5. Complementary series. We take σ to be a unitary complementary series of $\bar{S}$. Let $\psi^D(x) = \psi(Dx), D \in k^*$. Let χ_{ψ^D} be as in Section 2. Let ν be a positive number such that $\nu < \frac{1}{2}$. Then a unitary complementary series representation has the form $\sigma = \mathrm{Ind}\,(\chi_{\psi^D}|\,|^{\nu}, \bar{B_S}, \bar{S})$.

Let $\pi = \mathrm{Ind}\,(\mu \times \sigma, \bar{G}^*, \bar{G})$. Then by the remark above V_π can be realized as the space of smooth functions F: $\bar{G}^* \times \bar{G} \to \mathbf{C}$ satisfying:

(1) $F(s(a)n(x)g^*, \bar{g}) = \chi_{\psi^D}(a)|a|^{1+\nu}F(g^*, \bar{g})$

and satisfying (2) and (3) of the remark in subsection 9.2.

It follows from [Go] and from Lemma 9.6 that there exists a scalar λ such that the following is a positive definite G invariant hermitian form:

$$\langle F_2, F_1\rangle = \lambda \sum_{a \in k^*/(k^*)^2} \int_k \overline{F_1(w^{-1}n(x), t(a))}A(F_2)(w^{-1}n(x), t(a))\, dx \tag{9.9}$$

where $A(F)$ is the intertwining operator given by

$$A(F)(g^*, \bar{g}) = \int F(wn(y)g^*, \bar{g})\, dy.$$

This integral converges absolutely for $\nu > 0$ and we have

$$\begin{aligned} A(F)(w^{-1}n(x), t(a)) &= \int F(wn(-y)w^{-1}n(x), t(a))\, dy \\ &= \int (y, -y)F(n(y^{-1})s(y^{-1})w^{-1}n(y^{-1}+x), t(a))\, dy \\ &= \int (y, -y)\chi_{\psi^D}(y^{-1})|y|^{-\nu-1}F(w^{-1}n(y^{-1}+x), t(a))\, dy \\ &= \int (-1, y)\frac{\gamma(y, \psi^D)}{\gamma(1, \psi^D)}|y|^{\nu-1}F(w^{-1}n(y+x), t(a))\, dy. \end{aligned}$$

For each $F \in \pi$ and $a \in k^*$ let $V_{F,a}(x) = F(w^{-1}n(x), t(a))$. It follows from [W1] Lemme 5 and Lemme 6 that $V_{F,a}$ satisfies the assumptions of Proposition 9.8. Hence, by Proposition 9.8 there exists a function $\Delta_{\psi,D,\nu}(x)$ independent of $F \in V_\pi$ and constant on square classes such that

$$\int V_{F,a}(x)(-1, x)\gamma(x, \psi^D)|x|^{\nu-1}\, dx = \int \hat{V}_{F,a}(x)|x|^{-\nu}\Delta_{\psi,D,\nu}(x)\, dx$$

for all $F \in V_\pi$. Hence

$$V_{A(F),a}(x) = \int \hat{V}_{F,a}(y)|y|^{-\nu}\Delta_{\psi,D,\nu}(y)\psi(yx)\, dy.$$

Thus we have (for $\lambda' = \lambda\gamma(1, \psi^D)^{-1}$):

$$\begin{aligned} (9.10)\quad \langle F_2, F_1\rangle &= \lambda' \sum_{a\in k^*/(k^*)^2} \int \overline{V_{F_1,a}(x)}V_{A(F_2),a}(x)\, dx \\ &= \lambda' \sum_{a\in k^*/(k^*)^2} \int\int \hat{V}_{F_2,a}(y)\overline{V_{F_1,a}(x)}|y|^{-\nu}\Delta_{\psi,D,\nu}(y)\psi(yx)\, dy\, dx \\ &= \lambda' \sum_{a\in k^*/(k^*)^2} \int \overline{\hat{V}_{F_1,a}(y)}\hat{V}_{F_2,a}(y)|y|^{1-\nu}\Delta_{\psi,D,\nu}(y)\, d^*y. \end{aligned}$$

Let

$$l(a, b, F) = \int V_{F,a}(x)\psi(-bx)\, dx = \hat{V}_{F,a}(b), \quad a, b \in k^*, F \in V_\pi.$$

Since $\nu > 0$ this last integral converges absolutely. It is easy to see that

$$l(a, b, s(\alpha)^{-1}F) = (\alpha, a)\chi_{\psi^D}(\alpha)|\alpha|^{1+\nu} l(a, \alpha^2 b, F)$$

and that

$$l(a, b, t(c)F) = l(ac, b, F).$$

Now (9.10) reads ($\lambda'' = \lambda'|2|/2$)

$$\begin{aligned}
\langle F_2, F_1 \rangle &= \lambda' \sum_{a \in k^*/(k^*)^2} \int \overline{l(a, y, F_1)} l(a, y, F_2) \Delta_{\psi,D,\nu}(y)|y|^{1-\nu}\, d^*y \\
&= \lambda'' \sum_{a \in k^*/(k^*)^2} \sum_{b \in k^*/(k^*)^2} \\
&\quad \times \int \overline{l(a, b^{-1}\alpha^2, F_1)} l(a, b^{-1}\alpha^2, F_2) \Delta_{\psi,D,\nu}(b^{-1}\alpha^2)|b^{-1}\alpha^2|^{1-\nu}\, d^*\alpha \\
&= \lambda'' \sum_{a \in k^*/(k^*)^2} \sum_{b \in k^*/(k^*)^2} |b|^{\nu-1} \Delta_{\psi,D,\nu}(b) \\
&\quad \times \int \overline{l(a, b^{-1}, s(\alpha)^{-1}F_1)} l(a, b^{-1}, s(\alpha)^{-1}F_2)\, d^*\alpha \\
&= \lambda'' \sum_{b \in k^*/(k^*)^2} |b|^{\nu-1} \Delta_{\psi,D,\nu}(b) \sum_{a \in k^*/(k^*)^2} \\
&\quad \times \int \overline{l(b, b^{-1}, t(b)^{-1}t(a)s(\alpha)^{-1}F_1)} l(b, b^{-1}, t(b)^{-1}t(a)s(\alpha)^{-1}F_2)\, d^*\alpha \\
&= \lambda' \sum_{b \in k^*/(k^*)^2} |b|^{\nu-1} \Delta_{\psi,D,\nu}(b) \int \overline{l(b, b^{-1}, t(a)F_1)} l(b, b^{-1}, t(a)F_2)\, d^*a.
\end{aligned}$$

For $b \in k^*$ we can define the Whittaker functional $l^{\mu_b}\colon \pi \to \mathbf{C}$ by

$$l^{\mu_b}(F) = l(b, b^{-1}, F).$$

It is easy to see that $l^{\mu_b}(n(x)F) = \psi(x) l^{\mu_b}(F)$ and that $l^{\mu_b}(\bar{z}F) = \mu_b(z) l^{\mu_b}(F)$ where $\mu_b(z) = \mu(z)(b, z)$. Hence l^{μ_b} is a (ψ, μ_b)-Whittaker functional. If we define $W_F^{\mu_b}(\bar{g}) = l(b, b^{-1}, \bar{g}F) = l^{\mu_b}(\bar{g}F)$ then equation (9.9) reads

$$(9.11) \quad \langle F_2, F_1 \rangle = \lambda' \sum_{b \in k^*/(k^*)^2} \Delta_{\psi,D,\nu}(b)|b|^{\nu-1} \int \overline{W_{F_1}^{\mu_b}(t(a))} W_{F_2}^{\mu_b}(t(a))\, d^*a.$$

This is the inner product formula for the complementary series. Now we derive a more explicit formula when $|2| = |D| = 1$, and ψ is unramified. It is convenient to take the representatives of square classes to be $D, D\tau, \varpi$ and $\tau\varpi$.

To find a choice of a constant λ' such that this inner product is positive definite it is enough to require that $c_{\mu_D} = 1$ in a $\bar{G}$ invariant form

$$\langle F_2, F_1\rangle = \sum_{b\in k^*/(k^*)^2} c_{\mu_b} \int \overline{W_{F_1}^{\mu_b}(t(a))} W_{F_2}^{\mu_b}(t(a))\, d^*a. \tag{9.12}$$

To do that we choose $\lambda = \Delta_{\psi,D,\nu}(D)^{-1}$ and we get the following form for $c_{\mu_b,\nu}$ in (9.12):

$$c_{\mu_b} = \begin{cases} 1 & \text{if } b = D; \\ \frac{(1-q^{\nu-\frac{1}{2}})(1+q^{-\nu-\frac{1}{2}})}{(1+q^{\nu-\frac{1}{2}})(1-q^{-\nu-\frac{1}{2}})} & \text{if } b = D\tau; \\ \frac{q^{\frac{1}{2}-\nu}-1}{1-q^{-\nu-\frac{1}{2}}} & \text{if } b = \varpi \text{ or } b = \tau\varpi. \end{cases} \tag{9.13}$$

9.6. Special representations. We shall now prove the inner product formula for special representations. Our treatment follows ([Go] 1.64–1.67) We consider the case where $\chi(a) = \chi_{\psi^D}(a)|a|^{\frac{1}{2}}$ for some $D \in k^*$. Let $I(\chi) = \mathrm{Ind}\,(\chi, \bar{B_S}, \bar{S})$ be the representation consisting of smooth functions $f\colon \bar{S} \to \mathbf{C}$ satisfying $f(s(\alpha)n(x)\bar{s}) = \chi(\alpha)|\alpha| f(\bar{s})$ for all $\alpha \in k^*$, $x \in k$ and $\bar{s} \in \bar{S}$. By ([W1], Proposition 2), this representation is reducible and has a unique subrepresentation which we now describe. For each $f \in I(\chi)$ we let $V_f(x) = f(w^{-1}n(x))$. Let $V(\chi)$ be the subspace of functions $f \in I(\chi)$ such that $\hat{V}_f(Dy^2) = 0$ for all $y \in k^*$. Then $V(\chi)$ is an invariant subspace of $I(\chi)$. We denote the representation of $\bar{S}$ on $V(\chi)$ by σ_χ. These are the special representations of $\bar{S}$. The representation σ_χ is an irreducible unitary representation of $\bar{S}$. The following lemma gives another description of the vectors in $V(\chi)$; it is the analog of formula (298) in [Go]:

LEMMA 9.9. *Let χ be as above and let $f \in V(\chi)$. Then*

$$\int V_f(y+x)(-1,y)\gamma(y,\psi^D)|y|^{-1/2}\,dy = 0$$

for all $x \in k$.

Proof. By [W1] Lemme 5 and Lemme 6 we have that V_f satisfies the assumptions of Proposition 9.8. Hence, by Proposition 9.8 we have that

$$\int V_f(y+x)(-1,y)\gamma(y,\psi^D)|y|^{-1/2}\,dy = \int \Delta_{\psi,D,1/2}(y)\hat{V}_f(y)|y|^{-1/2}\psi(xy)\,dy.$$

By a direct computation (see (9.7)), we see that if $\nu = 1/2$ then $\Delta(y) = \Delta_{\psi,D,\nu}(y)$ is zero unless yD is a square in k^*. However, if yD is a square in k^* then $y = Dz^2$ for some $z \in k^*$ and by our assumptions on V_f we have that $V_f(Dz^2) = 0$. □

To get the inner product formula we view σ_χ as a limit of complementary series representations as ν goes to $\frac{1}{2}$ (see [Go] (300)). We will continue to assume that $|2| = |D| = 1$ and ψ is unramified, the general case being similar.

Let $F_1, F_2 \in \pi = \mathrm{Ind}\,(\mu \times \sigma_\chi, \bar{G}^*, \bar{G})$. Let $V_{F_1,a}, V_{F_2,a}$ be defined as in the complementary series case. We let

$$\begin{aligned}
&(F_1, F_2)\\
&\quad = \sum_{a\in k^*/(k^*)^2} \lim_{\nu\to\frac{1}{2}} \frac{1}{\Delta_{\psi,D,1/2}(D\tau)} \int V_{F_1,a}(x)\\
&\qquad\qquad \times \int \overline{V}_{F_2,a}(y+x)\chi_\psi(y)|y|^{\nu-1}\,dy\,dx\\
&\quad = \sum_{a\in k^*/(k^*)^2} \lim_{\nu\to\frac{1}{2}} \frac{1}{\Delta_{\psi,D,1/2}(D\tau)} \int V_{F_1,a}(x)\\
&\qquad\qquad \times \int \overline{V}_{F_2,a}(y+x)\chi_\psi(y)(|y|^{\nu-1} - |y|^{1/2})\,dy\,dx.
\end{aligned}$$

Here $\chi_{\psi^D}(y) = (-1,y)\gamma(y,\psi^D)\gamma(1,\psi^D)^{-1} = (-1,y)\gamma(y,\psi^D)$ and we have used Lemma 9.9 to obtain the last equality. It is easy to check that all integrals converge absolutely.

Remark 9.10. Notice that the functions F_1, F_2 are fixed in this limit and are in the special representation and not in the respective complementary series representation for $\nu < 1/2$. This means that the integral is not $\bar{G}$ invariant for every ν hence we need to check that both the limit exists and that the limit is $\bar{G}$ invariant.

The computation now follows [Go] p. 1.66. By using L'Hospital's rule we get that

$$\begin{aligned}
(9.14)\quad (F_1,F_2) &= c \sum_{a\in k^*/(k^*)^2}\\
&\quad \times \int V_{F_1,a}(x) \int \overline{V_{F_2,a}(y)\chi_{\psi^D}(y-x)}|y-x|^{-1/2}\,\mathrm{val}\,(y)\,dy\,dx
\end{aligned}$$

for some nonzero constant c. It remains to show that this inner form is invariant by $\bar{G}$. The invariance under $\bar{B}$ is easy to check and it is enough to show the invariance under w. The argument is the same as in [Go] p. 1.67 and is omitted. We remark again that (298) is replaced with Lemma 9.9. It remains to translate the inner product formula in (9.14) into an inner product in the Whittaker or Kirillov model of π. This is done using the limit defintion of the inner product above as in [Go] formula (300). That is:

$$\langle F_1, F_2\rangle = \sum_{b\in k^*/(k^*)^2} \lim_{\nu\to\frac{1}{2}} |b|^{-\frac{1}{2}}\Delta_{\psi,D,\nu}(b)\Delta_{\psi,D,\nu}(D\tau)^{-1}\int W_{F_1}^{\mu_b}(t(a))\overline{W_{F_2}^{\mu_b}(t(a))}\, d^*a$$

where $W_F^{\mu_b}(g) = \int V_{F,b}(x)\psi(-\, b^{-1}x)\, dx$ is a (ψ, μ_b) Whittaker functional. We notice that if $b \in D(k^*)^2$ then $W_F^{\mu_b}(g) = 0$ identically. Thus we get:

$$\langle F_1, F_2\rangle = \sum_{b\in k^*/(k^*)^2} c_{\mu_b}\int W_{F_1}^{\mu_b}(t(a))\overline{W_{F_2}^{\mu_b}(t(a))}\, d^*a \tag{9.15}$$

where $c_{\mu_D} = 0$ and $c_{\mu_b} = \lim_{\nu\to\frac{1}{2}} |b|^{-\frac{1}{2}}\Delta_{\psi,D,\nu}(b)/\Delta_{\psi,D,\nu}(D\tau)$ for bD not a square. Explicitly when $|2| = |D| = 1$ and ψ is unramified, we have $c_{\mu_D} = 0$, $c_{\mu_{D\tau}} = 1$, $c_{\mu_\varpi} = c_{\mu_{\tau\varpi}} = 2(1+q)^{-1}$.

For the unique quotient representation θ_χ of $I(\chi)$, the representation $Ind(\mu \times \theta_\chi, \bar{G}^*, \bar{G})$ is the theta representation considered in [G-PS, 1]. It has only one nontrivial (ψ, μ_b) Whittaker model, that is when $\mu_b = \mu_D$. In this case we can just set the coefficients in (9.1) as follows: $\lambda_{l^\mu} = 0$ if $\mu \neq \mu_D$, and $\lambda_{l^{\mu_D}} = 1$. The fact that this gives a $\bar{G}$ invariant Hermitian form can be verified directly using the model for theta representation ((1.2.1) in [G-PS, 1]).

9.7. Inner product formula on $\bar{S}$. We now translate the results above to $\bar{S}$. We use these results in a subsequent work. Let $\pi = Ind(\mu \times \sigma, \bar{G}^*, \bar{G})$ as in subsection 9.2. As a representation of G, $V_\pi \cong \oplus_{\mu'\in\Omega(\omega_\pi)} V_{\mu'}$ (Lemma 7.1). If $(\mu', \mu) = \delta_{\mu'}$, then $V_{\mu'}$ is the space of functions $F(g)$: $\bar{G} \mapsto V_\sigma$, where $F(g) = 0$ if $\det(g) \neq \delta_{\mu'} \mod (k^*)^2$.

The subspace V_μ is isomorphic to V_σ as a representation of $\bar{S}$. Explicitly, the isomorphism is given by $F \mapsto F(e)$. Thus any $\bar{G}$ invariant Hermitian form on V_π restricts to a $\bar{S}$ invariant Hermitian form on V_σ. Let $l^{\mu'}$ be the linear forms as in Theorem 9.1. Let $L^{\delta_{\mu'}}$ be a linear form on V_σ defined by:

$$L^{\delta_{\mu'}}(v) = l^{\mu'}(\pi(\delta_{\mu'}v)),\, v \in V_\mu. \tag{9.16}$$

Then $L^{\delta_{\mu'}}$ is a $\psi^{\delta_{\mu'}}$-Whittaker functional on V_μ, i.e.,

$$L^{\delta_{\mu'}}(\sigma(n(x))v) = \psi(\delta_{\mu'}x)L^{\delta_{\mu'}}(v).$$

Such a linear functional is unique up to a constant.

THEOREM 9.11. *Let $\langle , \rangle$ be a $\bar{S}$ invariant Hermitian form on V_σ. There exist scalars λ_{δ_i} where δ_i is a set of representatives of the square classes in k^*, such that*

$$(9.17) \qquad \langle v_1, v_2\rangle = \sum_{\delta_i} \lambda_{\delta_i} \int_{k^*} L^{\delta_i}(\sigma(s(a))v_1)\overline{L^{\delta_i}(\sigma(s(a))v_2)}\, d^*a.$$

Proof. Restrict the inner product in Theorem 9.1 to $V_\mu \cong V_\sigma$. Note that $l^{\mu'}(\pi(t(a)v) = 0$ for $v \in V_\mu$ and $a \neq (\mu, \mu')$ mod $(k^*)^2$. Thus for $v_1, v_2 \in V_\mu$,

$$\langle v_1, v_2\rangle = \sum_{\mu' \in \Omega(\omega_\pi)} |2|/2\lambda_{l\mu'} \int_{k^*} l^{\mu'}(\pi(t(\delta_{\mu'}\alpha^2))v_1)\overline{l^{\mu'}(\pi(t(\delta_{\mu'}\alpha^2))v_2)}\, d^*\alpha.$$

By the definition of $L^{\delta_{\mu'}}$, $< v_1, v_2 >$ has the form

$$\sum_{\delta_i} \lambda_{\delta_i} \int_{k^*} L^{\delta_i}(\pi(t(\alpha^2))v_1)\overline{L^{\delta_i}(\pi(t(\alpha^2))v_2)}\, d^*a.$$

Since the characters μ' are unitary, and $\pi(t(\alpha^2))v = \mu(\alpha)\sigma(s(\alpha))v$ when $v \in V_\mu$, we get the form in the theorem. □

We now derive the explicit formula for principal series, complementary series and special representations of $\bar{S}$. In the last two cases, we assume $|2| = |D| = 1$ and ψ is unramified.

Case 1. σ is a unitary principal series representation of $\bar{S}$. Let F as in subsection 9.3. Then $F \mapsto \phi_F(h) = F(h, e)$ defines the isomorphism between V_μ and V_σ. From (9.3), we have an inner product formula

$$(9.18) \qquad \langle \phi_1, \phi_2\rangle = \int_k \phi_1(wn(x))\overline{\phi_2(wn(x))}\, dx.$$

Let $L^{b^{-1}}(\phi_F) = l^{\mu_b}(\pi(t(b^{-1})F)$ for $F \in V_\mu$. Then

$$\begin{aligned} L^b(\phi_F) = l(b^{-1}, b, \pi(t(b))F) &= \int^+ F(wn(x), t(b^{-1})t(b))\psi(-bx)\, dx \\ &= \int^+ \phi_F(wn(x))\psi(-bx)\, dx. \end{aligned}$$

The proof of Theorem 9.11 and equation (9.5) give

$$(9.19) \qquad \langle \phi_1, \phi_2\rangle = \sum_{b\in k^*/(k^*)^2} |2b|/2 \int_{k^*} L^b(\sigma(s(a))\phi_1)\overline{L^b(\sigma(s(a))\phi_2)}\, d^*a$$

with

$$L^b(\phi) = \int^{+} \phi(wn(x))\psi(-bx)\,dx. \tag{9.20}$$

Case 2. σ is a complementary series representation $Ind(\chi_{\psi^D}|\,|^{\nu}, \bar{B}_S, \bar{S})$. Using the notation as in Case 1, from (9.9), we get

$$\langle \phi_1, \phi_2 \rangle = \int_k A(\phi_1)(wn(x))\overline{\phi_2(wn(x))}\,dx \tag{9.21}$$

with A being the intertwining operator:

$$A(\phi)(h) = \int \phi(wn(y)h)\,dy.$$

Define L^b as above. From the proof of Theorem 9.11 and equation (9.12), we get:

$$\langle \phi_1, \phi_2 \rangle = \Delta_{\psi,D,\nu}(D) \sum_{b \in k^*/(k^*)^2} \frac{c_b}{2} \int L^b(\sigma(s(a))\phi_1)\overline{L^b(\sigma(s(a))\phi_2)}\,d^*a. \tag{9.22}$$

where $c_b = c_{\mu_b}$ is as defined in equation (9.13).

Case 3. σ is a special representation $V(\chi)$ as in subsection 9.6. Define L^b as above. From the proof of Theorem 9.11 and equation (9.15), we get that there is a $\bar{S}$ invariant Hermitian form on $V(\chi)$:

$$\langle \phi_1, \phi_2 \rangle = \sum_{b \in k^*/(k^*)^2} \frac{c_b}{2} \int L^b(\sigma(s(a))\phi_1)\overline{L^b(\sigma(s(a))\phi_2)}\,d^*a, \tag{9.23}$$

where $c_b = c_{\mu_b}$ is as in subsection 9.6.

10. Bessel distributions over $\bar{G}$. In this section we define Bessel distributions for irreducible unitary representations of $\bar{G}$. We shall show using the inner product formula in the Whittaker model (9.1) that these Bessel distributions are given by the Bessel functions defined in (8.3). The argument is similar to the one given in [B] for Bessel distributions on G and to the argument given in the proof of Theorem 6.3. Although our definition of Bessel distributions here depends on the inner product on our unitary representation, it is possible, as in [B], to generalize the definition and the proof to every admissible irreducible representation of $\bar{G}$.

Let (π, V) be a smooth, irreducible, genuine, unitary representation of $\bar{G}$ and let $\langle v, w \rangle$ be a nonzero $\bar{G}$ invariant inner product on V. Such inner product is unique up to a positive multiple. For every smooth functional l: $V \to \mathbf{C}$ there exists a unique vector $v_l \in V$ such that

$$l(w) = \langle w, v_l \rangle \quad \text{for every } w \in V. \tag{10.1}$$

Let $\Omega(\pi)$ be the set of characters μ of $\bar{Z}$ that support a nontrivial (μ, ψ) Whittaker functional on V (see (8.1)). For each $\mu \in \Omega(\pi)$ we let l^μ, be a nonzero (ψ, μ) Whittaker functional on V. Such a functional is unique up to scalar. By Theorem 9.1 we can and will normalize all the l^μs such that

$$\langle v, w \rangle = \sum_{\mu \in \Omega(\pi)} \int W_v^\mu(t(a)) \overline{W_w^\mu(t(a))}\, d^*a \tag{10.2}$$

for all $v, w \in V$. Here $W_v^\mu(\bar{g}) = l^\mu(\pi(\bar{g})v)$.

Now fix $\mu \in \Omega(\pi)$. Let $f \in C_c^\infty(\bar{G})$ be a genuine function. We define the Bessel distribution $J_{\pi,\psi}^\mu$ to be

$$J_{\pi,\psi}^\mu(f) = \overline{l^\mu(v_{\pi(\check{f})l^\mu})}.$$

Here $\check{f}(\bar{g}) = f(\bar{g}^{-1})$ and $\pi(\check{f})l^\mu$ is the smooth linear functional on V defined by

$$\pi(\check{f})l^\mu(v) = \int_{\bar{G}} f(\bar{g}^{-1}) l^\mu(\pi(\bar{g}^{-1})v)\, d\bar{g} = \int_{\bar{G}} f(\bar{g}) l^\mu(\pi(\bar{g})v)\, d\bar{g}. \tag{10.3}$$

As we have related the Hermitian form with the linear forms l^μ by (10.2), the distribution is independent of our choice of the Hermitian form or l^μ—it is only dependent on the choice of measure on $\bar{G}$ and k^*.

LEMMA 10.1. *Fix $\mu \in \Omega(\pi)$ as above and Let $v \in V$ be such that $W_v^{\mu'}(t(a)) = 0$ for all $\mu' \neq \mu$ and all $a \in k^*$. Let $f \in C_c^\infty(\bar{G})$. Then*

$$\int_{k^*} J_\pi^{\mu,\psi}(\rho_r(t(a))f) W_v^\mu(t(a))\, d^*a = \int_{\bar{G}} f(\bar{g}) W_v^\mu(\bar{g})\, d\bar{g},$$

where $(\rho_r(\bar{g}')f)(\bar{g}) = f(\bar{g}\bar{g}')$.

Proof. It is easy to check that $\pi[(\rho_r(\bar{g}')f)^\vee] = \pi(\bar{g}')\pi(\check{f})$. Hence we have

$$\int_{k^*} J_\pi^{\mu,\psi}(\rho_r(t(a))f) W_v^\mu(t(a))\, d^*a = \int W_v^\mu(t(a)) \overline{W_{v_{\pi(\check{f})l^\mu}}^\mu(t(a))}\, d^*a.$$

By the choice of v and by (10.2) this last quantity equals $\langle v, v_{\pi(\check{f})l^\mu} \rangle$ and by (10.1) and (10.3) this equals

$$\int_{\bar{G}} f(\bar{g}) W_v^\mu(\bar{g})\, d\bar{g}. \qquad \square$$

THEOREM 10.2. *Let $J^{\mu}_{\pi,\psi}$ be the distribution defined above and $j^{\mu}_{\pi,\psi} = j^{\mu,\mu}_{\pi}$ be the Bessel function defined in (8.3). Then*

$$J^{\mu}_{\pi,\psi}(f) = \int_{\bar{G}} j^{\mu}_{\pi,\psi}(\bar{g}) f(\bar{g})\, d\bar{g}$$

for all $f \in C^{\infty}_{c}(\bar{G})$. Here $j^{\mu}_{\pi,\psi}$ is defined on the open Bruhat cell and can be taken to be zero outside the cell.

Proof. Fix $f \in C^{\infty}_{c}(\bar{G})$. There exists m such that $\rho_r(t(a))f = f$ if $a \in 1 + P^m$. By [G-PS, 1], there exists a vector $v \in V$ such that $W^{\mu}_{v}(t(a)) = q^m \chi(a)$ for all $a \in k^*$ where χ is the characteristic function of $1+P^m$ and such that $W^{\mu'}_{v}(t(a)) = 0$ for all $\mu' \neq \mu$ and $a \in k^*$. Now

$$\begin{aligned}
&\int_{\bar{G}} j^{\mu}_{\pi,\psi}(\bar{g}) f(\bar{g})\, d\bar{g} \\
&\quad = \int_{k^*} \left(\int_{\bar{G}} j^{\mu}_{\pi,\psi}(\bar{g})(\rho_r(t(a))f)(\bar{g})\, d\bar{g} \right) W^{\mu}_{v}(t(a))\, d^*a \\
&\quad = \int_{k^*} \left(\int_{\bar{G}} j^{\mu}_{\pi,\psi}(\bar{g}) f(\bar{g}t(a))\, d\bar{g} \right) W^{\mu}_{v}(t(a))\, d^*a \\
&\quad = \int_{k^*} \left(\int_{\bar{G}} j^{\mu}_{\pi,\psi}(\bar{g}t(a)^{-1}) f(\bar{g})\, d\bar{g} \right) W^{\mu}_{v}(t(a))\, d^*a \\
&\quad = \int_{\bar{G}} f(\bar{g}) \left(\int_{k^*} j^{\mu}_{\pi,\psi}(\bar{g}t(a)^{-1}) W^{\mu}_{v}(t(a))\, d^*a \right) d\bar{g} \\
&\quad = \int_{\bar{G}} f(\bar{g}) W^{\mu}_{v}(\bar{g})\, d\bar{g}.
\end{aligned}$$

Note that the $\bar{G}$ integration is in fact taking place on the open and dense set $\bar{B}w\bar{B}$, hence we can use Proposition 8.3 and our special choice of v to obtain the last equality. By Lemma 10.1 this last integral equals

$$\int_{k^*} J^{\mu}_{\pi,\psi}(\rho_r(t(a))f) W^{\mu}_{v}(t(a))\, d^*a$$

which is $J^{\mu}_{\pi,\psi}(f)$. □

11. Bessel distributions on $\overline{SL_2(k)}$. In this section we show that the Bessel distributions on $\bar{S} = \overline{SL_2(k)}$ are given by Bessel functions which are restrictions of Bessel functions from $\bar{G} = \overline{GL_2(k)}$. We shall keep the notations of Section 10.

Let (σ, W) be an irreducible, admissible, unitary, genuine representation of $\bar{S}$. We assume σ has a nontrivial ψ-Whittaker functional. We note that any σ has a nontrivial ψ^D-Whittaker functional for some $D \in k^*$.

Let $\pi = \operatorname{Ind}(\mu \times \sigma, \bar{G}^*, \bar{G})$ the representation of $\bar{G}$ associated to σ as in subsection 9.2. From subsection 9.7, we see that W inherits a $\bar{S}$ invariant inner product which we again denote by $\langle , \rangle$. Let the $l^{\mu'}$ be as normalized in Section 10. Let $L(v) = l^{\mu}(v)$ for $v \in V_{\mu}$, then L is a nonzero ψ Whittaker functional on W (see subsection 9.7). Given a function $\phi \in C_c^{\infty}(\bar{S})$ we define the Bessel distribution $J_{\sigma,\psi}$ to be

$$J_{\sigma,\psi}(\phi) = \overline{L(w_{\sigma(\check{\phi})L})}. \tag{11.1}$$

Here $\check{\phi}(\bar{s}) = f(\bar{s}^{-1})$ and $\overline{L(\cdot)}$ is the complex conjugate of $L(\cdot)$. The vector $w_{\sigma(\check{\phi})L} \in W$ is defined as in (10.1). (Recall that if l is a smooth functional on W then $w_l \in W$ is defined by the equality $l(w) =< w, w_l >$ for all $w \in W$.)

THEOREM 11.1. *If π is the representation of $\bar{G}$ induced from σ of $\bar{S}$ as above, then*

$$J_{\sigma,\psi}(\phi) = \int_{\bar{S}} j^{\mu}_{\pi,\psi}(\bar{s})\phi(\bar{s})\, d\bar{s}$$

where $j^{\mu}_{\pi,\psi}$ is the (ψ, μ) Bessel function of π as defined in (8.3).

Proof. Fix $\phi \in C_c^{\infty}(\bar{S})$. Let $\bar{Z}^2$ be the center of $\bar{G}$ and let $U \subset \bar{Z}^2$ be a small open subgroup around the identity element e such that $-e \notin U$ and such that μ is trivial on U. Let $X = \bar{S}U$. Then X is an open set in $\bar{G}$ and every $x \in X$ is uniquely written in the form $x = \bar{s}u$ for $\bar{s} \in \bar{S}$ and $u \in U$. We define $\Phi \in C_c^{\infty}(\bar{G})$ by $\Phi(\bar{g}) = \operatorname{vol}(U)^{-1}\phi(\bar{s})$ if $\bar{g} \in X$ and $\bar{g} = \bar{s}u$; and $\Phi(\bar{g}) = 0$ if $\bar{g} \notin X$. Then it is clear that $w_{\sigma(\check{\phi})L} = v_{\pi(\check{\Phi})L^{\mu}}$ as linear forms on V. Thus we have

$$J_{\sigma,\psi}(\phi) = J^{\mu}_{\pi,\psi}(\Phi).$$

Now by Theorem 10.2 we have that

$$J^{\mu}_{\pi,\psi}(\Phi) = \int_{\bar{G}} \Phi(\bar{g}) j^{\mu}_{\pi,\psi}(\bar{g})\, d\bar{g}.$$

Writing $d\bar{g} = d\bar{s}\, d\bar{z}$ on X and noticing that $j^{\mu}_{\pi,\psi}(\bar{s}u) = j^{\mu}_{\pi,\psi}(\bar{s})\mu(u) = j^{\mu}_{\pi,\psi}(\bar{s})$ for every $\bar{s} \in \bar{S}$ and $u \in U$ we have

$$\begin{aligned} &\int_{\bar{G}} \Phi(\bar{g}) j^{\mu}_{\pi,\psi}(\bar{g})\, d\bar{g} \\ &= \int_{\bar{S}} \int_U \Phi(\bar{s}u) j^{\mu}_{\pi,\psi}(\bar{s}u)\, d\bar{s}\, du \\ &= \int_{\bar{S}} \phi(\bar{s}) j^{\mu}_{\pi,\psi}(\bar{s})\, d\bar{s} \int_U \operatorname{vol}(U)^{-1}\, du \\ &= \int_{\bar{S}} \phi(\bar{s}) j^{\mu}_{\pi,\psi}(\bar{s})\, d\bar{s}. \end{aligned} \tag{11.2}$$

□

It is clear that the restriction of $j^{\mu}_{\pi,\psi}$ to $\bar{S}$ is independent of μ when $\pi = \mathrm{Ind}\,(\mu \otimes \sigma, \bar{G}^{*}, \bar{G})$. This follows from the theorem above or from the definition of $j^{\mu}_{\pi,\psi}$. Hence we define

$$j_{\sigma,\psi}(\bar{s}) = j^{\mu}_{\pi,\psi}(\bar{s}).$$

Equivalently, restricting the equation (8.3) to the case $v \in V_{\mu}$, we see $j_{\sigma,\psi}$ is defined by the equality

$$\int^{+} W(\bar{s}n(x))\psi^{-1}(x)\,dx = j_{\sigma,\psi}(\bar{s})W(e) \tag{11.3}$$

where W is in the ψ Whittaker model of σ and $\bar{s} \in \bar{B}_S w \bar{B}_S$. The Bessel function $j_{\sigma,\psi}$ is determined by the choice of measure on k.

From Corollary 7.6, we get the asymptotic behavior of $j_{\sigma,\psi}(\bar{s})$.

Corollary 11.2. *Let ω be the central character of σ. There exists a constant $C = C_{\sigma}$ such that for $|x^{-1}| > C$, then*

$$j_{\sigma,\psi}(ws(x)) = |x|^{-\frac{1}{2}} \times \left(\omega(-1)(-1,-x)\gamma(x,\psi)\psi\left(\frac{2}{x}\right)+\gamma(-x,\psi)\psi\left(-\frac{2}{x}\right)\right). \tag{11.4}$$

Proof. Let $a = x^{-2}$. Then $t(a)w = \left(\begin{pmatrix} x^{-1} & \\ & x^{-1}\end{pmatrix}, (-1,x)\right) ws(x)$. From equation (7.9), we see when $|x^{-1}| > C$,

$$\begin{aligned} j_{\sigma,\psi}(ws(x)) &= J(W,t(a)w)\mu(x^{-1})^{-1}(-1,x)/W(e) \\ &= |x|^{-\frac{1}{2}}\left(\omega(-1)(-1,-x)\gamma(x^{-1},\psi)\psi\left(\frac{2}{x}\right)+\gamma(-x^{-1},\psi)\psi\left(-\frac{2}{x}\right)\right). \end{aligned}$$

The corollary then follows from the properties of Weil's constant. □

We end the section by stating the analogue of Corollary 6.4 and Lemma 6.5. The proofs are similar and will be omitted.

Corollary 11.3. *We fix a Haar measure dg on $\bar{S}$ such that $dg = |a|^2 d^{*}a\,dx\,dy$ on the set of elements of the form $n(x)ws(a)n(y)$ where $a \in k^{*}$ and $x, y \in k$. Then*

$$J_{\sigma,\psi}(\phi) = \int_k j_{\sigma,\psi}(ws(a))O_{\psi}(\phi, ws(a))|a|^2\,d^{*}a$$

where $O_{\psi}(\phi, g) = \int_k \int_k \phi(n(x)gn(y))\psi(x+y)\,dx\,dy$ is the orbital integral of ϕ as defined in [J].

LEMMA 11.4. *Assume that* (σ, W) *is unitary. Let* $\{v_i\colon\ i = 1, 2, \ldots\}$ *be an orthonormal basis of K types with respect to the Hermitian form chosen above. Then*

$$J_\sigma(\phi) = \sum_i L(\sigma(\phi)v_i)\overline{L(v_i)}. \tag{11.5}$$

This lemma with Corollary 6.6 relates the local distributions in Jacquet's relative trace formula with the Bessel distributions and relative Bessel distributions discussed here.

12. Bessel functions for principal and complementary series. In this section we shall compute the Bessel functions for principal complementary series and for special representations of $\bar{S} = \overline{SL_2(k)}$. Our computations and methods are very similar to the ones in Section 5. It is clear that the Bessel function $j_{\sigma,\psi}(g)$ is determined by its value at $g = ws(a)$, $a \in k^*$.

12.1. Principal and complementary series. Let χ be a character of k^* such that $|\chi(x)| = |x|^\rho$ with ρ satisfying $0 \le \rho < \frac{1}{2}$. Let $\sigma_\chi = \mathrm{Ind}\,(\bar{B}_{\bar{S}}, \bar{S}, \chi)$. Then V_{σ_χ}, the space of σ_χ is the space of all smooth functions $\phi\colon\ \bar{S} \to \mathbf{C}$ satisfying

$$\phi([n(x)s(a)\bar{s}, \epsilon]) = \epsilon|a|\chi_\psi(a)\chi(a)\phi(\bar{s}, 1), \quad \text{for all } x, a \in k \text{ and } \bar{s} \in \bar{S}. \tag{12.1}$$

Fix $\phi \in V_{\sigma_\chi}$. We define $\phi_m \in V_{\sigma_\chi}$ by

$$\phi_m(g) = \int_{|x|\le q^m} \phi(gn(x))\psi^{-1}(x)\,dx.$$

Let L be a Whittaker functional on V_{σ_χ}. Then for $g \in \bar{B}wN$ we have (see (11.3)):

$$\lim_{m\to\infty} L(\sigma_\chi(g)\phi_m) = L(\phi)j_{\sigma_\chi,\psi}(g) \tag{12.2}$$

where $j_{\sigma_\chi,\psi}(g)$ is the Bessel function of σ_χ.

Let $\phi \in V_{\sigma_\chi}$ be defined by

$$\phi(g, \epsilon) = \begin{cases} \epsilon|a|\chi_\psi(a)\chi(a)\psi(x) & \text{if } g = n(y)s(a)wn(x), \text{ and } |x| \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ_m is given by

$$\phi_m(g, \epsilon) = \begin{cases} \epsilon|a|\chi_\psi(a)\chi(a)\psi(x) & \text{if } g = n(y)s(a)wn(x), \text{ and } |x| \le q^m; \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$L(\phi) = \int^{+} \phi(wn(y))\psi^{-1}(y)\,dy. \tag{12.3}$$

Then by [W1] this integral converges and gives a Whittaker functional on V_{σ_χ}. It is easy to see that $L(\phi) = 1$. Hence if $\bar{g} \in \bar{B}_{\bar{S}} w \bar{B}_{\bar{S}}$ then by (12.2), $j_{\sigma_\chi,\psi}(\bar{g}) = \lim_{m\to\infty} W_{\phi_m}(g)$ where

$$W_\phi(g) = \int^{+} \phi(wn(y)g)\psi^{-1}(y)\,dy = L(\sigma_\chi(g)\phi).$$

To compute $W_{\phi_m}(ws(a))$ we shall use the following equation in $\bar{S}$

$$wn(y)ws(a) = n\left(-\frac{1}{y}\right) s(a^{-1}y^{-1}) wn\left(-\frac{1}{ya^2}\right)(y,a)(y,y).$$

It follows that

$$W_{\phi_m}(ws(a)) = \int^{+}_{|y^{-1}a^{-2}|\le q^m} |ya|^{-1}\chi_\psi\left(\frac{1}{ya}\right)\chi\left(\frac{1}{ya}\right)\psi\left(-\frac{1}{ya^2} - y\right)(y,a)(y,y)\,dy.$$

Changing variables $u = (ya)^{-1}$ we get

$$W_{\phi_m}(ws(a)) = \int^{-}_{|\frac{u}{a}|\le q^m} |a|^{-1}\chi_\psi(u)\chi(u)\psi\left(-\frac{u}{a} - \frac{1}{ua}\right)(u,a)(u,u)\,d^*u.$$

Writing $\chi_\psi(u) = (u,u)\gamma(u,\psi)\gamma(1,\psi)^{-1}$ and from the fact that

$$(u,a)\gamma(u,\psi)\gamma(1,\psi)^{-1} = \gamma(ua,\psi)\gamma(a,\psi)^{-1}$$

we get that $\chi_\psi(u)(u,a)(u,u) = \gamma(ua,\psi)\gamma(a,\psi)^{-1}$. Hence we have

$$j_{\sigma_\chi,\psi}(ws(a)) = |a|^{-1}\gamma(a,\psi)^{-1}\int^{+}_{-} \chi(u)\gamma(ua,\psi)\psi\left(-\frac{u}{a} - \frac{1}{ua}\right)d^*u. \tag{12.4}$$

12.2. Special representations. Let $\xi \in k^*$ be such that $\xi \notin (k^*)^2$. Let χ_ξ be the quadratic character of k^* associated with the quadratic extension $k[\sqrt{\xi}]$. Let χ be a character of k^* defined by $\chi(x) = |x|^{\frac{1}{2}}\chi_\xi(x)$. We let V_{σ_χ} be the space of smooth functions ϕ satisfying (12.1) and satisfying

$$\int^{+} \phi(wn(y))\psi(-xy)\,dy = 0$$

for all $x \in \xi k^2$. Then by [W1] V_{σ_χ} is invariant under the action of $\bar{S}$ by left translations and gives an irreducible representation of $\bar{S}$ which we denote by σ_χ.

By [W1] σ_χ has a Whittaker functional for every character ψ^x with $x \notin \xi(k^*)^2$. (Here $\psi^x(y) = \psi(yx)$.) In particular, σ_χ has a ψ Whittaker functional which is given again by (12.3). We can define again the function $\phi \in V_{\sigma_\chi}$ as above and the same computations yield the formula

$$j_{\sigma_\chi,\psi}(ws(a)) = |a|^{-1}\gamma(a,\psi)^{-1}\int_{-}^{+}\chi(u)\gamma(ua,\psi)\psi\left(-\frac{u}{a}-\frac{1}{ua}\right)d^*u. \tag{12.5}$$

13. Bessel identities. In this section we establish a local nonarchimedean correspondence between irreducible unitary representations of $G = PGL_2(k)$ and $\bar{S} = \overline{SL_2(k)}$. This correspondence is in fact the Waldspurger correspondence which was established in [W1] using theta correspondence techniques.

Fix an additive character ψ of k, let $D \in k^*$. Let σ be a representation of $\bar{S}$. If σ has nontrivial ψ^D-Whittaker, we will define its Bessel function j_{σ,ψ^D} with respect to ψ^D by the equation (11.3), where in the equation ψ is replaced by ψ^D (however the measure remains self dual for ψ).

Define a *transfer factor*

$$\Delta_{D,\psi}(x) = \gamma(x,\psi^D)\psi\left(\frac{2D}{x}\right)|x|^{\frac{1}{2}}. \tag{13.1}$$

Let π be an irreducible admissible unitary representation of G.

Definition 13.1. We say that an irreducible admissible unitary representation σ of $\bar{S}$ corresponds to π if the following equality (Bessel identity) holds

$$i_{\pi,\psi}\left(n\left(\frac{x}{4D}\right)w_0\right) = \frac{\Delta_{D,\psi}(x)\epsilon(\pi,\frac{1}{2},\psi)|2D|}{L(\pi,\frac{1}{2})}j_{\sigma,\psi^D}(ws(x)) \tag{13.2}$$

for all $x \in k^*$.

Remark 13.2. Given σ, if there is a π such that the above is satisfied, then since the values of $i_{\pi,\psi}(n(x)w_0)$ ($x \in k^*$) determine the relative Bessel distribution of π (Corollary 6.4), the representation π is uniquely determined. Similarly given π there is at most one σ satisfying the given equation. If a correspondence as above exists then it is one-to-one.

The factor $|2D|$ in the equation relates to our choice of Haar measure. We note that the value of $\epsilon(\pi,\frac{1}{2}) = \epsilon(\pi,\frac{1}{2},\psi)$ is independent of ψ from [J-L].

The following theorem is the main theorem of this paper. Let π_χ be as in Section 5, and σ_{χ,ψ^D} be as in Section 12 with ψ replaced by ψ^D, (i.e., χ_ψ replaced by χ_{ψ^D} in (12.1)). Let $r^-_{\psi^D}$ be the supercuspidal Weil representation defined in [G-PS, 1] (with ψ replaced by ψ^D).

THEOREM 13.3. *For each irreducible admissible unitary representation π of G there exists a corresponding representation σ of $\bar{S}$ satisfying the Bessel identity (13.2). The following diagram describes the correspondence:*

G	$\bar{S}$	
π_χ	σ_{χ,ψ^D}	$\lvert\chi\rvert = \lvert\,\rvert^\rho,\ 0 \le \rho < \frac{1}{2}$
π_χ	σ_{χ,ψ^D}	$\lvert\chi\rvert = \lvert\,\rvert^{\frac{1}{2}},\ \chi \neq \lvert\,\rvert^{\frac{1}{2}}$
$\pi_{\lvert\,\rvert^{\frac{1}{2}}}$	$r^-_{\psi^D}$	
supercuspidal	*supercuspidal*	

Remark 13.4. We took the approach of using the Bessel identity to define the correspondence. It is easy to check that $\sigma = \Theta(\pi, \psi^D)$, where $\Theta(*, \psi^D)$ is the theta correspondence used in [W1]. Thus the theorem is equivalent to the following: if $\sigma = \Theta(\pi, \psi^D)$, then (13.2) holds.

Proof. When $\pi = \pi_\chi$ is a principal series, a complementary series or a special representation for $\chi \neq \lvert\,\rvert^{\frac{1}{2}}$ then the Bessel identities (13.2) for $\pi = \pi_\chi$ and $\sigma = \sigma_{\chi,\psi^D}$ follow from Theorem 5.1 and (12.4), (12.5), the fact that $\epsilon(\pi_\chi, \frac{1}{2}, \psi) = \chi(-1)$ and $\gamma(\frac{a}{D}, \psi) = \lvert D\rvert^{\frac{1}{2}} \gamma(a, \psi^D)$. For $\chi = \lvert\,\rvert^{\frac{1}{2}}$ the Bessel identities follow from our formula for $i_{\pi_{\lvert\,\rvert^{\frac{1}{2}}},\psi}$ in Theorem 5.3 and for the formula for the Bessel representation of a supercuspidal Weil representation given in ([G-PS, 1] Proposition 4.4.2).

Now assume that π is supercuspidal.

LEMMA 13.5. *When π is supercuspidal, there exists a number field K and a place ν_0, and a cuspidal automorphic representation Π of $PGL_2(\mathbf{A}_K)$, such that $K_{\nu_0} = k$ and $\Pi_{\nu_0} = \pi$, and $L(\Pi, \frac{1}{2}) \neq 0$.*

Proof. By Lemma 6.5 in [Ar-Cl] we can find a number field K and a place ν_0 of K such that $K_{\nu_0} = k$ and such that there exists an automorphic cuspidal representation $\Pi = \otimes \Pi_\nu$ of $PGL_2(\mathbf{A}_K)$ such that $\Pi_{\nu_0} = \pi$. Moreover, we can assume that the number of finite places ν where Π_ν is not a principal series representation is larger or equal to two. As in the proof of Lemme 41 in [W3] we can choose a global quadratic character χ_ξ such that $\chi_{\xi,\nu_0} = 1$ and such that $\epsilon(\Pi \otimes \chi_\xi, \frac{1}{2}) = 1$. Replacing Π with $\Pi \otimes \chi_\xi$ we get that there exists an automorphic cuspidal representation Π such that $\Pi_{\nu_0} = \pi$ and such that $\epsilon(\Pi, \frac{1}{2}) = 1$. By Theorem 4 in [W3] there exists $\xi \in K$ such that ξ is a square in K_{ν_0} such that $L(\Pi \otimes \chi_\xi, \frac{1}{2}) \neq 0$. Hence, by replacing again Π with $\Pi \otimes \chi_\xi$ we get an automorphic cuspidal representation Π such that $\Pi_{\nu_0} = \pi$ and such that $L(\Pi, \frac{1}{2}) \neq 0$. □

By [J] there exists a cuspidal automorphic representation $\Pi' = \otimes \Pi'_\nu$ of $\bar{S}(\mathbf{A}_K)$ such that $I_\Pi(\phi) = J_{\Pi'}(\phi')$ for all matching ϕ and ϕ'. Recall that the matching condition for ϕ and ϕ' is described by equations of orbital integrals. The functions $\phi = \otimes\phi_\nu$ and $\phi' = \otimes\phi'_\nu$ match if for each ν, [J]

$$O_\psi\left(\phi_\nu, n\left(\frac{a}{4D}\right)w_0\right) = O_{\psi^D}(\phi'_\nu, ws(a))|a|\Delta_{D,\psi}(a)^{-1}(1-q^{-1}). \tag{13.3}$$

(In the definition of O_{ψ^D}, the measure is self-dual for ψ.) Since $I_\Pi = \otimes I_{\Pi_\nu,\psi}$ and $J_{\Pi'} = \otimes J_{\Pi'_\nu,\psi^D}$ we can isolate the ν_0 component by fixing ϕ_ν and ϕ'_ν for all $\nu \neq \nu_0$. Hence we get that

$$\lambda I_{\Pi_{\nu_0},\psi}(\phi_{\nu_0}) = J_{\Pi'_{\nu_0},\psi^D}(\phi'_{\nu_0}) \tag{13.4}$$

where λ is some nonzero scalar and the equality holds for every matching ϕ_{ν_0} and $\phi_{\nu'_0}$. Put $\sigma = \Pi'_\nu$. It follows from Corollary 6.4 and Corollary 11.3 that $i_{\pi,\psi}$ and j_{σ,ψ^D} satisfy for some constant λ'

$$\lambda' i_{\pi,\psi}\left(n\left(\frac{x}{4D}\right)w_0\right) = \Delta_{D,\psi}(x) j_{\sigma,\psi^D}(ws(x)). \tag{13.5}$$

However, when $|x|$ is small, from the asymptotics of $i_{\pi,\psi}$ and j_{π',ψ^D} (Lemma 4.1 and Corollary 11.2), we have (note that the formula for asymptotic of j_{σ,ψ^D} is obtained by replacing ψ with ψ^D in Corollary 11.2):

$$\begin{aligned}&\frac{j_{\sigma,\psi^D}(ws(x))}{i_{\pi,\psi}(n(\frac{x}{4D})w_0)}\\ &\quad= \frac{L(\pi,\frac{1}{2})|x|^{-\frac{1}{2}}(\omega(-1)(-1,-x)\times\gamma(x,\psi^D)\psi(\frac{2D}{x}) + \gamma(-x,\psi^D)\psi(-\frac{2D}{x}))}{\epsilon(\pi,\frac{1}{2},\psi) + \psi(\frac{4D}{x})}\end{aligned} \tag{13.6}$$

where $|x|$ is small and ω is the central character of π'. Clearly we have

$$\lambda'\epsilon\left(\pi,\frac{1}{2},\psi\right) = \Delta_{D,\psi}(x)L\left(\pi,\frac{1}{2}\right)|x|^{-\frac{1}{2}}\gamma(-x,\psi^D)\psi\left(-\frac{2D}{x}\right)$$

and

$$\lambda'\psi\left(\frac{4D}{x}\right) = \Delta_{D,\psi}(x)L\left(\pi,\frac{1}{2}\right)|x|^{-\frac{1}{2}}\omega(-1)(-1,-x)\gamma(x,\psi^D)\psi\left(\frac{2D}{x}\right).$$

The first equation gives

$$\lambda'\epsilon\left(\pi,\frac{1}{2},\psi\right) = L\left(\pi,\frac{1}{2}\right)\gamma(-x,\psi^D)\gamma(x,\psi^D) = L\left(\pi,\frac{1}{2}\right)|2D|^{-1}$$

which implies the Bessel identity. As a bonus, both equations together give:

$$\epsilon\left(\pi, \frac{1}{2}, \psi\right) \omega(-1)\gamma(x, \psi^D)(-1, -x) = \gamma(-x, \psi^D)$$

which implies $\epsilon(\pi, \frac{1}{2}, \psi) = \frac{\omega(-1)}{\chi_{\psi^D}(-1)}$. This equality is the assertion 10 in section IV of [W2]. □

From the Bessel identity in the previous Theorem, we can derive the identity between Bessel distributions. Define $J_{\sigma,\psi^D}(\phi')$ as in (11.1), with ψ replaced by ψ^D, but with the additive measure still being self-dual for ψ.

THEOREM 13.6. *When ϕ and ϕ' satisfy the equation (13.3), for π and π' correspond as in Theorem 13.3, we have*

$$I_{\pi,\psi}(\phi) = J_{\sigma,\psi^D}(\phi')\epsilon\left(\pi, \frac{1}{2}\right)|2D|^{-1}(1-q^{-1})L\left(\pi, \frac{1}{2}\right)^{-1}. \tag{13.7}$$

Proof. From Corollary 6.4, (13.3) and the previous Theorem, we get

$$\begin{aligned} I_{\pi,\psi}(\phi) &= \int_k i_{\pi,\psi}(n(x)w_0)O_\psi(\phi, n(x)w_0)\,dx \\ &= |4D|^{-1}\int_k i_{\pi,\psi}\left(n\left(\frac{x}{4D}\right)w_0\right)O_\psi\left(\phi, n\left(\frac{x}{4D}\right)w_0\right)\,dx \\ &= |2|^{-1}\epsilon\left(\pi, \frac{1}{2}\right)(1-q^{-1})L\left(\pi, \frac{1}{2}\right)^{-1}\int_k j_{\sigma,\psi^D}(ws(x))O_{\psi^D}(\phi', ws(x))|x|\,dx. \end{aligned}$$

The integral in the last equation equals $J_{\sigma,\psi^D}(\phi')|D|^{-1}$ from Corollary 11.3, the extra fact $|D|^{-1}$ is due to the measure not being self dual for ψ^D. Thus we get the theorem. □

For convenience, we have set the multiplicative measure to be $d^*x = \frac{dx}{|x|}$. In [B-M] and [J], the multiplicative measure is $d^*x = (1-q^{-1})^{-1}\frac{dx}{|x|}$. If we define $I_{\pi,\psi}$ and J_{σ,ψ^D} with this change of measure, then the matching between ϕ and ϕ' in [J] is given by

$$O_\psi\left(\phi_\nu, n\left(\frac{a}{4D}\right)w_0\right) = O_{\psi^D}(\phi'_\nu, ws(a))|a|\Delta_{D,\psi}(a)^{-1}$$

and when ϕ and ϕ' match:

$$I_{\pi,\psi}(\phi) = J_{\sigma,\psi^D}(\phi')\epsilon\left(\pi, \frac{1}{2}\right)|2D|^{-1}L\left(\pi, \frac{1}{2}\right)^{-1}. \tag{13.8}$$

14. Appendix. We justify the change of order of integrals in the computation of the proof of Theorem 5.1. We also compute a certain limit that is needed for the computation of the relative Bessel function for the special representation π_χ where $\chi = ||^{\frac{1}{2}}$.

Our starting point is the integral

$$(14.1)\quad I_{\pi_\chi,\psi}\left(n(x)w_0, \frac{1}{2}\right) = \int^{+}\int^{+,-} \chi\left(-\frac{a}{z^2}\right)|a|^{\frac{1}{2}}|z|^{-1}\psi\left(\frac{a}{z} - z + ax\right) dz\, d^*a.$$

If $|\chi(x)| = |x|^r$ with $|r| < \frac{1}{2}$ or $\chi^2(x) = |x|^{\pm 1}$ then this integral converges. We justify the first change of order of integration:

LEMMA 14.1. *Assume $|\chi(x)| = |x|^r$ with $r \neq 0$ and $-\frac{1}{2} < r < \frac{1}{2}$ as above. Then*

$$(14.2)\qquad \int^{+}\int^{+,-} \chi\left(-\frac{a}{z^2}\right)|a|^{\frac{1}{2}}|z|^{-1}\psi\left(\frac{a}{z} - z + ax\right) dz\, d^*a$$
$$= \int^{+}\int^{+} \chi\left(-\frac{a}{z^2}\right)|a|^{\frac{1}{2}}|z|^{-1}\psi\left(\frac{a}{z} - z + ax\right) d^*a\, dz.$$

Proof. We first show that the right-hand side converges. Write $f(a,z) = \chi(-\frac{a}{z^2})|a|^{\frac{1}{2}}|z|^{-1}\psi(\frac{a}{z} - z + ax)$. Then $|f(a,z)| = |a|^{r+\frac{1}{2}}|z|^{-1-2r}$. Fixing z we can see that the integral in a converges absolutely at $a = 0$ and stabilizes at infinity if $\frac{1}{z} \neq x$. Let $g(z) = \int^{+} f(a,z)\, d^*a$. Changing variables $a \to a(\frac{1}{z} + x)^{-1}$ we get that

$$(14.3)\qquad g(z) = \gamma_\chi \chi\left(\frac{1}{z} + x\right)^{-1} \chi(z)^{-2}\left|\frac{1}{z} + x\right|^{-\frac{1}{2}} |z|^{-1}\psi(-z)$$

where

$$\gamma_\chi = \int^{+} \chi(-a)|a|^{\frac{1}{2}}\psi(a)\, d^*a.$$

Now when $|z|$ is small $|g(z)| = |\gamma_\chi||z|^{r-2r+\frac{1}{2}-1} = |\gamma_\chi||z|^{-r-\frac{1}{2}}$ hence the integral of $g(z)$ converges at $z = 0$. When $|z|$ is large then

$$g(z) = \gamma_\chi \chi(x)|x|^{-\frac{1}{2}}\chi(z)^{-2}|z|^{-1}\psi(-z)$$

hence the integral stabilizes. When z is close to $-x^{-1}$ then

$$|g(z)| = \lambda|z + x^{-1}|^{-r-\frac{1}{2}}$$

for some constant λ, hence the integral converges.

The left-hand side of (14.2) is

$$\lim_{m\to\infty}\int_{|a|\le q^m}\lim_{n\to\infty}\int_{q^{-n}\le|z|\le q^n} f(a,z)\,dz\,d^*a.$$

We first show that

$$\int_{|a|\le q^m}\lim_{n\to\infty}\int_{q^{-n}\le|z|\le q^n} f(a,z)\,dz\,d^*a = \lim_{n\to\infty}\int_{|a|\le q^m}\int_{q^{-n}\le|z|\le q^n} f(a,z)\,dz\,d^*a. \tag{14.4}$$

We would like to use the Dominated Convergence Theorem. Let

$$h_n(a)=\int_{q^{-n}\le|z|\le q^n}|f(a,z)|\,dz.$$

We first notice that if $|a|$ is bounded from above then the dz integral stabilizes for $|z|>C$ for some large C independent of a. Hence we can write

$$h_n(a)=\int_{q^{-n}\le|z|\le C}|f(a,z)|\,dz.$$

When z is small

$$f(a,z)=\chi\left(-\frac{a}{z^2}\right)|a|^{\frac12}|z|^{-1}\psi\left(\frac{a}{z}+ax\right).$$

Hence we have that

$$\begin{aligned}|h_n(a)| \;\le\; &|\chi(-a)||a|^{\frac12}\left(\left|\int_{q^{-n}\le|z|\le D}\chi(z^{-2})|z|^{-1}\psi\left(\frac{a}{z}\right)\,dz\right|\right.\\ &\left.+\left|\int_{D<|z|\le C}\chi(z^{-2})|z|^{-1}\psi\left(\frac{a}{z}-z\right)\,dz\right|\right)\\ \le\; &|\chi(-a)||a|^{\frac12}\left(\left|\int_{q^{-n}\le|z|\le D}\chi(z^{-2})|z|^{-1}\psi\left(\frac{a}{z}\right)\,dz\right|+K\right)\end{aligned}$$

for some positive constant K independent of a. Now

$$\left|\int_{q^{-n}\le|z|\le D}\chi(z^{-2})|z|^{-1}\psi\left(\frac{a}{z}\right)\,dz\right| = \left|\int_{D^{-1}\le|z|\le q^n}\chi(z^2)\psi(az)\,d^*z\right|.$$

If $|\chi(z)| = |z|^r$ with $r < 0$ then we have

$$\left| \int_{D^{-1} \leq |z| \leq q^n} \chi(z^2)\psi(az)\, d^*z \right| \leq \int_{|z| \geq D^{-1}} |\chi(z)|^2\, d^*z \leq K_1$$

where K_1 is independent of a. If $|\chi(z)| = |z|^r$ with $r > 0$ then

$$\begin{aligned} &\left| \int_{D^{-1} \leq |z| \leq q^n} \chi(z^2)\psi(az)\, d^*z \right| \\ &\qquad = |\chi(a)|^{-2} \left| \int_{|a|D^{-1} \leq |z| \leq |a|q^n} \chi(z^2)\psi(z)\, d^*z \right| \leq K_2|\chi(a)|^{-2} \end{aligned}$$

for some positive constant K_2. The existence of K_2 follows from the fact that the last integral stabilizes for large z hence we can bound it by $\int_{|z|<E} |\chi(z)|^2\, d^*z = \int_{|z|<E} |z|^{2r}\, d^*z = K_2$. Combining all cases we can see that $|h_n(a)| \leq K'|a|^{-r+\frac{1}{2}} + K''|a|^{r+\frac{1}{2}}$ for some nonnegative constants K' and K''. Using our assumption on r we can now apply the Dominated Convergence Theorem.

Now we can apply the Fubini Theorem to (14.4) to conclude that the left-hand side of (14.2) equals

$$lim_{m\to\infty} \int^{+,-} \int_{|a| \leq q^m} f(a,z)\, d^*a\, dz.$$

Set $g_m(z) = \int_{|a|\leq q^m} f(a,z)\, d^*a$ and $g(z) = \int^+ f(a,z)\, d^*a$ as above. It is clear that $g_m(z) \to g(z)$. To conlude the proof we need to show that $lim_{m\to\infty} \int^{+,-} g_m(z)\, dz = \int^+ g(z)\, dz$. We have

$$\begin{aligned} g_m(z) &= \int_{|a|\leq q^m} \chi\left(-\frac{a}{z^2}\right) |a|^{\frac{1}{2}} |z|^{-1} \psi\left(\frac{a}{z} - z + ax\right) d^*a \\ &= \chi(z^{-2})|z|^{-1}\chi^{-1}\left(\frac{1}{z} + x\right) \left|\frac{1}{z} + x\right|^{-\frac{1}{2}} \psi(z) \int_{|a|\leq|\frac{1}{z}+x|q^m} \chi(-a)|a|^{\frac{1}{2}}\psi(a)\, d^*a, \end{aligned}$$

where we have made the change of variables $a \to a(\frac{1}{z} + x)$. If $|z| > |x|^{-1}$ then this integral stabilizes for large m independently of z. In particular we have that for such m and z, $g_m(z) = g(z)$ as defined above. Hence to prove that

$$lim_{m\to\infty} \int^{+,-} g_m(z)\, dz = \int^+ g(z)\, dz$$

it is enough to prove that

$$lim_{m\to\infty} \int^{-}_{|z|\le|x|^{-1}} g_m(z)\,dz = \int_{|z|\le|x|^{-1}} g(z)\,dz.$$

However,

$$|\,g_m(z)| \le K_2|z|^{-2r-1}\left|\frac{1}{z}+x\right|^{-r-\frac{1}{2}} = K_2|x|^{-r-\frac{1}{2}}|z|^{-r-\frac{1}{2}}|z+x^{-1}|^{-r-\frac{1}{2}}$$

for some positive constant K_2, hence we can use the Dominated Convergence Theorem to conclude the proof of the Lemma. □

We now make a change of variables $a \to az^2$ in the right-hand side of (14.2) and proceed with changing the order of integration again.

LEMMA 14.2. *Assume that* $|\chi(x)| = |x|^r$ *with* $0 < r < \frac{1}{2}$. *Then*

$$\int^{+}\int^{+} \chi(-a)|a|^{\frac{1}{2}}\psi(az - z + az^2x)\,d^*a\,dz = \int^{+}\int^{+} \chi(-a)|a|^{\frac{1}{2}}\psi(az - z + az^2x)\,dz\,d^*a. \tag{14.5}$$

Proof. The convergence of the right-hand side follows from the computation in the proof of Theorem 5.1. The proof here is similar to the proof of Lemma 14.1. Let $q(a,z) = \chi(-a)|a|^{\frac{1}{2}}\psi(az - z + az^2x)$. The left-hand side of (14.5) can be written as

$$\lim_{m\to\infty}\int_{|z|\le q^m}\left(\lim_{n\to\infty}\int_{|a|\le q^n} q(a,z)\,d^*a\right)\,dz.$$

We first prove that

$$\int_{|z|\le q^m}\left(\lim_{n\to\infty}\int_{|a|\le q^n} q(a,z)\,d^*a\right)\,dz = \lim_{n\to\infty}\int_{|z|\le q^m}\left(\int_{|a|\le q^n} q(a,z)\,d^*a\right)\,dz. \tag{14.6}$$

Let $p_n(z) = \int_{|a|\le q^n} q(a,z)\,d^*a$. Then

$$\begin{aligned}|\,p_n(z)| &= \left|\int_{|a|\le q^n}\chi(-a)|a|^{\frac{1}{2}}\psi(az+az^2x)\,d^*a\right|\\ &= |\chi^{-1}(z+z^2x)|z+z^2x|^{-\frac{1}{2}}\left|\int_{|a|\le|z+z^2x|q^n}\chi(-a)|a|^{\frac{1}{2}}\psi(a)\,d^*a\right|\end{aligned}$$

$$\leq K_3|z|^{-r-\frac{1}{2}}|z+x^{-1}|^{-r-\frac{1}{2}}$$

for some positive constant K_3. Hence we again use the Dominated Convergence Theorem to establish our equality. Using this equality and the Fubini Theorem we have that the left-hand side of (14.5) equals

$$\lim_{m\to\infty}\int^{+}\left(\int_{|z|\leq q^m} q(a,z)\,dz\right)\,d^*a.$$

Let $r(a) = \int^{+} q(a,z)\,dz$ and $r_m(a) = \int_{|z|\leq q^m} q(a,z)\,dz$. We need to prove that $\int^{+} r(a)\,d^*a = \lim_{m\to\infty}\int^{+} r_m(a)\,d^*a$. Now

$$\begin{aligned} r_m(a) &= \chi(-a)|a|^{\frac{1}{2}}\int_{|z|\leq q^m}\psi(az-z+az^2x)\,dz \\ &= \chi(-a)|a|^{\frac{1}{2}}\psi\left(-\frac{(a-1)^2}{4ax}\right)\int \Phi_m(z)\psi\left(ax\left(z+\frac{a-1}{2ax}\right)^2\right)\,dz \end{aligned}$$

where Φ_m is the characteristic function of P^{-m}. Set $y = \frac{a-1}{2ax}$ and $\Phi_{m,y}(z) = \Phi_m(z-y)$. Then using (2.2) we have

$$r_m(a) = \chi(-a)|a|^{\frac{1}{2}}\psi\left(-\frac{(a-1)^2}{4ax}\right)|ax|^{-\frac{1}{2}}\gamma(ax,\psi)\int\hat{\Phi}_{m,y}(z)\psi(-a^{-1}x^{-1}z^2)\,dz.$$

Assume for the moment that ψ is unramified. Similar arguments will appply for the case that ψ is ramified. If ψ is unramified then $\hat{\Phi}_{m,y}(z) = q^m\psi(-yz)\Phi_{-m}(z)$. When $|a| > 1$ then $|y| = |2x|^{-1}$. Hence if we take m sufficiently large then $\hat{\Phi}_{m,y}(z) = q^m\Phi_{-m}(z)$ and we get that $r_m(a) = r(a)$. It follows that it is enough to prove that

$$\int_{|a|\leq 1} r(a)\,d^*a = \lim_{m\to\infty}\int_{|a|\leq 1} r_m(a)\,d^*a.$$

However, $|r_m(a)| \leq C|a|^r$ for some constant C and we can use again the Dominated Convergence Theorem to conclude the proof. □

Combining (14.2) with (14.5) and the computation in the proof of Theorem 5.1 we have proved that

$$(14.7)\quad \int^{+}\int^{+,-}\chi\left(-\frac{a}{z^2}\right)|a|^{\frac{1}{2}}|z|^{-1}\psi\left(\frac{a}{z}-z+ax\right)\,dz\,d^*a$$
$$= \psi\left(\frac{1}{2x}\right)|x|^{-\frac{1}{2}}\int^{+,-}\chi(-a)\gamma(ax,\psi)\psi\left(-\frac{a}{4x}-\frac{1}{4ax}\right)\,d^*a$$

for all χ such that $|\chi(x)| = |x|^r$ with $0 < r < \frac{1}{2}$. We now extend this to the unitary principal series and the unitary special representations by taking limits on both sides.

LEMMA 14.3. *Let $\chi(x) = \chi_0(x)|x|^r$ with $|\chi_0(x)| = 1$ for all $x \in k^*$. The equality (14.7) holds for χ such that $r = 0$ and for χ such that $r = 1/2$, $\chi_0^2 = 1$ and $\chi_0 \neq 1$.*

Proof. Fix χ_0 as above and let $0 < r < 1/2$. We will take the limits from both sides of (14.7) as $r \to 0$ and as $r \to 1/2$ to get the desired equailty. We start with the left-hand side. It follows from the proof of Proposition 4.3 that for a fixed x the d^*a integral of the left-hand side of (14.7) takes place in a compact set of k independent of r. Now it follows that the dz integral takes place in a compact set of k independent of a and r. Hence we can write

$$\begin{aligned}
&\int^{+} \int^{+,-} \chi\left(-\frac{a}{z^2}\right) |a|^{\frac{1}{2}} |z|^{-1} \psi\left(\frac{a}{z} - z + ax\right) dz\, d^*a \\
&\quad = \int_{|a|\le C_1} \int^{-}_{|z|\le C_2} \chi\left(-\frac{a}{z^2}\right) |a|^{\frac{1}{2}} |z|^{-1} \psi\left(\frac{a}{z} - z + ax\right) dz\, d^*a \\
&\quad = \int_{|a|\le C_1} \chi_0(-a)|a|^{r+1/2}\psi(ax) \left(\int_{D_2<|z|\le C_2} \chi_0^{-2}(z)|z|^{-2r}\psi\left(\frac{a}{z} - z\right) d^*z\right) d^*a \\
&\quad\quad + \int_{|a|\le C_1} \chi_0(-a)|a|^{r+1/2}\psi(ax) \left(\int^{-}_{|z|\le D_2} \chi_0^{-2}(z)|z|^{-2r}\psi\left(\frac{a}{z}\right) d^*z\right) d^*a.
\end{aligned}$$

We can take the limit when $r \to 0$ or $r \to 1/2$ of the first summand and use the Dominated Convergence Theorem. We now analyze the second summand. If $\chi_0^2 \neq 1$ then the inner integral in absolute value is less than or equal to a constant times $|a|^{-2r}$. Hence if $r \to 0$ we can apply the Dominated Convergence Theorem to the second summand to get our first equality. So assume $\chi_0^2 = 1$. We have

$$\begin{aligned}
&\int_{|a|\le C_1} \chi_0(-a)|a|^{r+1/2}\psi(ax) \left(\int^{-}_{|z|\le D_2} \chi_0^{-2}(z)|z|^{-2r}\psi\left(\frac{a}{z}\right) d^*z\right) d^*a \\
&\quad = \int_{|a|\le C_1} \chi_0(-a)|a|^{r+1/2}\psi(ax) \left(\int^{+}_{|z|\ge D_2^{-1}} |z|^{2r}\psi(az)\, dz\right) d^*a.
\end{aligned}$$

Looking at the inner integral which can be computed explicitly for $r \ge 0$ we get that when r is positive and $|a|$ is small it is of order $O(|a|^{-2r}\operatorname{Log}(|a|^{-1}))$, hence we can use the Dominated Convergence Theorem for the outer integral. This finishes the $r = 0$ case. For r close to $1/2$ we notice that

$$\int^{+}_{|z|\ge D_2^{-1}} |z|^{2r}\psi\left(\frac{a}{z}\right) dz = \lambda_1(r)|a|^{-2r} + \lambda_2(r)$$

where $\lambda_1(r)$ and $\lambda_2(r)$ are continuous in a neighborhood of $r = 1/2$. By our assumptions we have that $\chi_0 \neq 1$ hence the d^*a integration takes place on a set of the form $|a| = q^k$ for some integer k which is independent of r. Hence we can use the Dominated Convergence Theorem for the outer integral to conclude that the limit of the left-hand side integral of (14.7) when χ_0 is fixed as above and $r \to 0$ or $r \to 1/2$ exists and is exactly (14.7) with $r = 0$ or $r = 1/2$.

For a fixed x, the right-hand side of (14.7) takes place in a compact set of k^* independent of r hence we can also take the limit here and get the required equality. □

We now turn our attention to the special representation π_χ where $\chi(x) = |x|^{\frac{1}{2}}$. Since we get the same representation, hence the same relative Bessel function when $\chi = ||^{\frac{1}{2}}$ and $\chi = ||^{-\frac{1}{2}}$ we shall use the second choice of χ. We now compute (14.2) for the case when $\chi_r(x) = |x|^r$, $-\frac{1}{2} < r < 0$. We also assume that ψ is unramified.

Let

$$I(r,x) = I_{\pi_{\chi_r},\psi}\left(n(x)w_0, \frac{1}{2}\right) = \int^+ \int^+ |a|^{r+\frac{1}{2}} |z|^{-2r-1} \psi\left(\frac{a}{z} - z + ax\right) dz\, d^*a.$$

By Lemma 14.1 we have that for $-1/2 < r < 0$

$$I(r,x) = \gamma_{\chi_r} \int^+ \left|\frac{1}{z} + x\right|^{-r-\frac{1}{2}} |z|^{-2r-1} \psi(-z)\, dz$$

where

$$\gamma_{\chi_r} = \int^+ \chi_r(-a)|a|^{\frac{1}{2}} \psi(a)\, d^*a = \frac{1 - q^{r-\frac{1}{2}}}{1 - q^{-r-\frac{1}{2}}}.$$

LEMMA 14.4.

$$\lim_{r \to -1/2} I(r,x) = \psi(x^{-1}) + 1.$$

Proof. Let

$$Z(r) = \int^+ \left|\frac{1}{z} + x\right|^{-r-\frac{1}{2}} |z|^{-2r-1} \psi(-z)\, dz.$$

Then $Z(r) = A(r) + B(r) + C(r)$ where

$$A(r) = |x|^{-r-\frac{1}{2}} \int^+_{|z|>|x|^{-1}} |z|^{-2r-1} \psi(-z)\, dz,$$

$$B(r) = \int_{|z|<|x|^{-1}} |z|^{-r-\frac{1}{2}} \psi(-z)\, dz,$$

$$C(r) = \int_{|z|=|x|^{-1}} \left|\frac{1}{z}+x\right|^{-r-\frac{1}{2}} |z|^{-2r-1}\psi(-z)\,dz$$
$$= |x|^{2r+1}\int_{|z|=|x|^{-1}} \left|\frac{1}{z}+x\right|^{-r-\frac{1}{2}} \psi(-z)\,dz.$$

We have

$$A(r) = \begin{cases} |x|^{-r-\frac{1}{2}}\left[\frac{(1-q^{-1})(1-|x|^{2r})}{1-q^{2r}} - q^{-2r-1}\right] & \text{if } |x| \geq q^{-1}; \\ 0 & \text{if } |x| \leq q^{-1}. \end{cases}$$

$$B(r) = \begin{cases} \frac{(1-q^{-1})q^{r-\frac{1}{2}}|x|^{r-\frac{1}{2}}}{1-q^{r-\frac{1}{2}}} & \text{if } |x| \geq q^{-1}; \\ \frac{q^{-r-\frac{1}{2}}-1}{q^{r-\frac{1}{2}}-1} & \text{if } |x| \leq q^{-2}. \end{cases}$$

The domain of the integral giving $C(r)$ is $P^m - P^{m-1}$ where $|x| = q^{-m}$. We change variables $u = z + x^{-1}$. The domain for u is

$$(x^{-1}+P^m) - (x^{-1}+P^{m-1}) = P^m - (x^{-1}+P^{m-1}).$$

On the domain of integration we have $|z^{-1}+x| = |z+x^{-1}||xz^{-1}| = |u||x|^2$. Hence, $C(r) = D(r) - E(r)$ where

$$D(r) = \psi(x^{-1})\int_{|u|\leq|x|^{-1}} |u|^{-r-\frac{1}{2}}\psi(-u)\,du$$
$$E(r) = |x|^{r+\frac{1}{2}}\int_{|u|<|x|^{-1}} \psi(-u)\,du.$$

We have

$$D(r) = \begin{cases} \frac{(1-q^{-1})|x|^{r-\frac{1}{2}}}{1-q^{r-\frac{1}{2}}} & \text{if } |x| \geq 1; \\ \psi(x^{-1})\frac{q^{-r-\frac{1}{2}}-1}{q^{r-\frac{1}{2}}-1} & \text{if } |x| \leq q^{-1}. \end{cases}$$

$$E(r) = \begin{cases} q^{-1}|x|^{r-\frac{1}{2}} & \text{if } |x| \geq q^{-1}; \\ 0 & \text{if } |x| \leq q^{-2}. \end{cases}$$

It is now easy to show that $\lim_{r\to-\frac{1}{2}} I(r,x) = \lim_{r\to-\frac{1}{2}} \gamma_r Z(r)$ equals

$$\lim_{r\to-\frac{1}{2}} \frac{1-q^{r-\frac{1}{2}}}{1-q^{-r-\frac{1}{2}}}(A(r)+B(r)+D(r)+E(r)) = \begin{cases} \psi(x^{-1})+1 & \text{if } |x| < 1; \\ 2 & \text{if } |x| \geq 1. \end{cases}$$

Since $\psi(x^{-1}) = 1$ when $|x| \geq 1$ we can write the last limit as $\psi(x^{-1}) + 1$ in both cases. □

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