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A Whittaker–Plancherel inversion formula for $SL(2,\mathbb{R})$ [☆]

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Abstract

We prove a Whittaker–Plancherel inversion formula which gives a Whittaker coefficient of a function on $SL(2,\mathbb{R})$ in terms of certain Bessel coefficients of this function. The Bessel coefficients come from Bessel functions attached to irreducible unitary tempered representations. The Kuznecov transform and Kuznecov inversion formula play a central role in the proof of this Whittaker–Plancherel inversion formula.

Keywords: Whittaker–Plancherel; Bessel functions; Bessel distributions; Kuznecov transform; Orbital integrals

1. Introduction

Let $G = SL(2,\mathbb{R})$ and $N = \{n(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right):\ x \in \mathbb{R}\}$.

Let $\lambda \in \mathbb{R}^*$ and let ψ_λ be a character on N defined by

$$\psi_\lambda\big(n(x)\big) = e^{2\pi i\lambda x}. \tag{1.1}$$

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A Whittaker space for G is a space of functions $W : G \to \mathbb{C}$ invariant under right translations by elements of G, and satisfying $W(ng) = \psi_{-\lambda}(n)W(g)$ for every $g \in G$ and $n \in N$.

Whittaker spaces are attached to infinite dimensional irreducible admissible representations of G and play a major role in the theory of automorphic forms. (See, for example, [11].) In this paper we consider the space of all smooth Whittaker functions which are compactly supported mod N. We will prove a Whittaker–Plancherel inversion formula for such functions. The inversion formula will give the value of a Whittaker function W at the identity element in terms of an integral and a sum over the Bessel coefficients of this Whittaker function. Here Bessel coefficients are defined to be integrals of W against a Bessel function of an irreducible unitary representation which we define below and in Section 8.

Each Whittaker function W which is compactly supported mod N can be obtained in the following way. Let $f \in C_c^\infty(G)$ and let dx be the standard Lebesgue measure on $\mathbb{R}$. Let $W_f^\lambda(g) = \int_{\mathbb{R}} f(n(x)g)\psi_\lambda(n(x))\,dx$. When $g = e$ we get that

$$W(f) = W_f^\lambda(e) = \int_{-\infty}^{\infty} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{2\pi i\lambda x}\,dx$$

is a Fourier–Whittaker coefficient of f. Our formula and main theorem is the following:

Theorem 1.1. *Let $f \in C_c^\infty(G)$. Then*

$$\begin{aligned}(2\pi)^2 &\int_{-\infty}^{\infty} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{2\pi i\lambda x}|\lambda|^{1/2}\,dx \\ &= \frac{1}{2}\int_0^\infty J_{\pi_{ir}^+,\psi_\lambda}(f)\tanh(\pi r/2)r\,dr + \frac{1}{2}\int_0^\infty J_{\pi_{ir}^-,\psi_\lambda}(f)\operatorname{cotanh}(\pi r/2)r\,dr \\ &\quad + \sum_{n=1}^{\infty} nJ_{\pi_n^\lambda,\psi_\lambda}(f). \end{aligned} \tag{1.2}$$

Here π_{ir}^+ is the unitary principal series representation of G with trivial central character and π_{ir}^- is the unitary principal series with nontrivial central character; π_n^λ is the unitary discrete series representation of G with ψ_λ Whittaker model as in Definition 7.1, that is, π_n^λ is the unitary discrete series representation with highest weight $-n-1$ or lowest weight $n+1$ depending on λ. (For a detailed description of these representations see Section 7.) J_{π,ψ_λ} is the normalized Bessel distribution of π as defined in [4]. (See also Section 8.) We will give a formula for J_{π,ψ_λ} below.

This formula has a number theoretic analog. When f is a function on G (noncompactly supported) which comes from a Maass cusp form there is a similar definition of $W(f) = W_f^\lambda(e)$ which gives a value related to the (λth) Fourier coefficient of f. It is a well-known problem in number theory to estimate these Fourier coefficients. Sum formulas for these Fourier coefficients which are quite similar to our formulas were obtained in [5–7,14]. Our formula is derived from the Kuznecov transform and inversion formula which were proved in [14] for the purpose of estimating these coefficients. Our formula puts the Kuznecov transform into a representation theoretic context. It should generalize to quasi-split real reductive groups.

A general decomposition formula for the space of $L^2(N \setminus G)$ Whittaker functions of a real reductive group G was obtained in [18]. Our formula, in contrast, is a pointwise formula and does not follow from this decomposition formula. A more direct analog to our formula is Harish-Chandra–Plancherel formula for $SL(2,\mathbb{R})$ which is also a pointwise formula (see, for example, [13, Theorem 11.6]). While it is sometimes possible to use a pointwise formula to prove a decomposition formula (see, for example, [13, p. 387, remark after (11.9)] or [18, 14.12]), it is not easy to get a pointwise formula from a decomposition formula. Hence our formula does not follow from the Whittaker–Plancherel formula in [18].

This is the first example of a full Whittaker–Plancherel inversion formula. A spherical Whittaker–Plancherel inversion formula was obtained by Wallach in the general setting of a real reductive group. Our proof is completely different from the proof in [18] since we do not use Harish-Chandra's Plancherel formula. In particular, our proof brings to the front objects and tools which we think are interesting by themselves: Bessel functions and Bessel distributions of representations, orbital integrals and the Kuznecov transform.

1.1. Bessel distributions

To give a self contained statement of our main theorem we will now give a formula for J_{π,ψ_λ}. Let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let A be a subgroup of G defined by

$$A = \left\{ s(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{R}^* \right\}.$$

Let $B = NA$. Let (π, H) be an irreducible unitary representation of G on a Hilbert space H and let J_{π,ψ_λ} be the normalized Bessel distribution as defined in [4]. By [4] there exists a real analytic function $j_{\pi,\psi_\lambda} : BwB \to \mathbb{C}$ which is locally integrable on G such that

$$J_{\pi,\psi_\lambda}(f) = \int_G f(g) j_{\pi,\psi_\lambda}(g)\, d_\lambda g, \quad f \in C_c^\infty(G). \tag{1.3}$$

Here $d_\lambda g$ is a Haar measure on G which depends on λ and is given by (2.1). Moreover, j_π satisfies:

$$j_{\pi,\psi_\lambda}(n_1 g n_2) = \psi_\lambda(n_1)\psi_\lambda(n_2) j_{\pi,\psi_\lambda}(g), \quad g \in BwB,\ n_1, n_2 \in N. \tag{1.4}$$

It follows that j_{π,ψ_λ} is determined by its values on elements of the form

$$s(z)w = \begin{pmatrix} 0 & -z \\ z^{-1} & 0 \end{pmatrix}.$$

It was proved in [4] that the values of j_{π,ψ_λ} are the same as those computed in [7]. We have

$$j_{\pi_{ir}^+,\psi_\lambda}\big(s(z)w\big) = -\frac{\pi|\lambda|^{1/2}}{\sin(\pi ir/2)}|z|\big(J_{ir}\big(4\pi|\lambda z|\big) - J_{-ir}\big(4\pi|\lambda z|\big)\big),$$

$$j_{\pi_{2d-1},\psi_\lambda}\big(s(z)w\big) = (-1)^d 2\pi|\lambda|^{1/2}|z|J_{2d-1}\big(4\pi|\lambda z|\big). \tag{1.5}$$

Here J_ν is the classical J-Bessel function. The values of j_{π,ψ_λ} for the case of nontrivial central character are

$$j_{\pi_{ir}^-,\psi_\lambda}\big(s(z)w\big) = -\frac{\pi i|\lambda|^{1/2}}{\cos(\pi ir)}\operatorname{sgn}(z)|z|\big(J_{ir}\big(4\pi|\lambda z|\big) + J_{-ir}\big(4\pi|\lambda z|\big)\big),$$

$$j_{\pi_{2d},\psi_\lambda}\big(s(z)\big) = (-1)^{d+1} 2\pi i|\lambda|^{1/2}\operatorname{sgn}(z)|z|J_{2d}\big(4\pi|\lambda z|\big). \tag{1.6}$$

Remark 1.2. Using (1.5), (1.6) and (1.4) we have given a complete definition of j_{π,ψ_λ} hence of J_{π,ψ_λ} for the representations π appearing in formula (1.2). Hence our statement of Theorem 1.1 is now complete.

In order to make our notations and proofs simpler and our formula shorter we will look at the special case where $\lambda = 1$ and $f(g) = f(-g)$ for all $g \in G$. The general case stated in (1.2) will follow easily from this case and from the case $f(g) = -f(g)$. In essence, we are restricting to the case of $PSL(2,\mathbb{R})$. We let $\psi = \psi_1$ and $\pi_n = \pi_n^1$.

1.2. The formula in the case of $PSL(2,\mathbb{R})$

Theorem 1.3. *Let $f \in C_c^\infty(G)$. Assume that $f(g) = f(-g)$ for all $g \in G$. Then*

$$(2\pi)^2 \int_{-\infty}^{\infty} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{2\pi ix}\,dx$$

$$= \frac{1}{2}\int_0^\infty J_{\pi_{ir}^+,\psi}(f)\tanh(\pi r/2)r\,dr + \sum_{d=1}^{\infty}(2d-1)J_{\pi_{2d-1},\psi}(f). \tag{1.7}$$

2. Orbital integrals and the Kuznecov transform

In this section we study the right-hand side of (1.7). In particular, the Bessel distributions appearing in the formula will lead us to the Kuznecov transform of a function defined on the positive real axis. A key ingredient in our proof of formula (1.7) will be the Kuznecov inversion formula which we will recall here.

2.1. Orbital integrals

Let $BwB = NAwN$ be the open Bruhat cell. We fix a normalized Haar measure $dg = d_1g$ on G where

$$d_\lambda g = |\lambda|\,dx\,dy\,|z|^{-2}\frac{dz}{|z|}, \qquad dg = d_1g = dx\,dy\,|z|^{-2}\frac{dz}{|z|} \tag{2.1}$$

are defined on the set of elements of the form $n(x)s(z)wn(y)$. Here dx, dy, dz are standard Lebesgue measures on $\mathbb{R}$. For a function $f \in C_c^\infty(G)$ we define

$$O_f(z) = \int_{\mathbb{R}} \int_{\mathbb{R}} f\big(n(x)s(z)wn(y)\big)e^{2\pi i(x+y)}\,dx\,dy. \tag{2.2}$$

It is easy to check (see [12]) that the above integral converges absolutely for all $z \in \mathbb{R}$ and gives a smooth function of z. It is also easy to see that $O_f(z)$ vanishes around $z = 0$. On the other hand, the asymptotic behavior of $O_f(z)$ at ∞ is nontrivial and will play an essential role in the proof. This asymptotic behavior was described by Jacquet in [10]. We will study these asymptotic expansions in Section 3.

Let $j : G \to \mathbb{C}$ be a locally integrable function satisfying (1.4). Then

$$\begin{aligned}
\int_G f(g)j(g)\,dg &= \int f\big(n(x)s(z)wn(y)\big)j\big(n(x)s(z)wn(y)\big)|z|^{-2}\,dx\,dy\,\frac{dz}{|z|} \\
&= \int_{-\infty}^{\infty} j\big(s(z)w\big)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f\big(n(x)s(z)wn(y)\big)e^{2\pi i(x+y)}\,dx\,dy\right)|z|^{-2}\,\frac{dz}{|z|} \\
&= \int_{-\infty}^{\infty} j\big(s(z)w\big)O_f(z)|z|^{-2}\,\frac{dz}{|z|}.
\end{aligned}$$

Assume further that $f(g) = f(-g)$. Then $O_f(z) = O_f(-z)$ for all $z \in \mathbb{R}^*$. Assume also that $j(s(z)w) = j(s(-z)w)$ for all $z \in \mathbb{R}^*$. Then it follows from the above equation that

$$\int_G f(g)j(g)\,dg = 2\int_0^{\infty} j\big(s(z)w\big)O_f(z)z^{-2}\,\frac{dz}{z}. \tag{2.3}$$

Applying (2.3) for the case when $j = j_{\pi,\psi}$ is a Bessel function of an irreducible unitary tempered representation of G with trivial central character as above we get

$$\begin{aligned}
J_{\pi_{ir}^+,\psi}(f) &= 2\int_0^{\infty} -\frac{\pi}{\sin(\pi ir/2)}\big(J_{ir}(4\pi z) - J_{-ir}(4\pi z)\big)O_f(z)z^{-1}\,\frac{dz}{z}, \\
J_{\pi_{2d-1},\psi}(f) &= \int_0^{\infty} (-1)^d 4\pi J_{2d-1}(4\pi z)O_f(z)z^{-1}\,\frac{dz}{z}.
\end{aligned}$$

We set

$$G_f(z) = z^{-1}O_f(z/4\pi). \tag{2.4}$$

Then we have

$$J_{\pi_{ir}^{+},\psi}(f) = -\frac{8\pi^2}{\sin(\pi i r/2)} \int_0^\infty G_f(z)\big(J_{ir}(z) - J_{-ir}(z)\big)\,\frac{dz}{z},$$

$$J_{\pi_{2d-1},\psi}(f) = (-1)^d 16\pi^2 \int_0^\infty G_f(z) J_{2d-1}(z)\,\frac{dz}{z}. \tag{2.5}$$

These formulas immediately bring into mind the Kuznecov transform.

2.2. *The Kuznecov transform*

Let $G(x)$ be a function defined on $x > 0$ which is absolutely integrable on the positive real line with respect to the measure dx/x. For $r > 0$ define the Kuznecov transform

$$\tilde{G}(r) = \int_0^\infty G(x)\big(J_{ir}(x) - J_{-ir}(x)\big)\,\frac{dx}{x}$$

and

$$c_n(G) = \int_0^\infty G(y) J_{2n-1}(x)\,\frac{dx}{x}.$$

Since $J_{ir}(x) - J_{-ir}(x)$ and $J_{2n-1}(x)$ are of order $O(x^{-1/2})$ at ∞ and $O(1)$ at $x = 0$ it follows that these integrals converge absolutely. The following theorem follows immediately from the proof of [14, Theorem in the appendix on p. 337]. See in particular [14, p. 340, (A.22)–(A.25)].

Theorem 2.1. [14] *Assume that $G(x)$ is continuous on $x > 0$ and that $G(x)$ is absolutely integrable on the positive real line with respect to the measure dx/x. Assume that for some $B > 2$, $\tilde{G}(r) = O(r^{-B} e^{\pi r/2})$ as $r \to \infty$. Then*

$$G(x) = -\int_0^\infty \tilde{G}(r)\big(J_{ir}(x) - J_{-ir}(x)\big)\frac{r\,dr}{2(\sinh(\pi r))} + \sum_{n>0} 2(2n-1) c_n(G) J_{2n-1}(x). \tag{2.6}$$

Corollary 2.2. *Assume that $G(x)$ is continuous on $x > 0$, $G(x)$ vanishes around $x = 0$ and that for some $A > 0$, $G(x) = O(x^{-A})$ as $x \to \infty$. Assume also that for some $B > 2$, $\tilde{G}(r) = O(r^{-B} e^{\pi r/2})$ as $r \to \infty$. Then the Kuznecov inversion formula* (2.6) *holds for G.*

One of our main technical results of this paper is that $G_f(x)$ satisfies the conditions of the above corollary for every $f \in C_c^\infty(G)$. To prove that we will need to study the asymptotic expansions of the orbital integrals $O_f(z)$. These were studied by Jacquet in [10]. In particular, it was proved in [10] that the left-hand side of (1.7) appears in the asymptotic expansion of $O_f(z)$.

3. Asymptotic expansion of orbital integrals

In this section we will obtain an asymptotic expansion of the orbital integral $O_f(z)$ and its derivatives when $z \to \infty$. The results in this section follow from the results in [10, Section 7]. Since our context is slightly different and for the sake of completeness we will provide the proof here. Let $f \in C_c^\infty(G)$. We can extend f to a function on $GL(2,\mathbb{R})$ and consequently to a function on $M_2(\mathbb{R})$ which is smooth and compactly supported and which agrees with f on G. (This can be done, for example, by choosing a smooth and compactly supported function on the center of $GL(2,\mathbb{R})$ which is supported on a small neighborhood of the identity and by multiplying this function with our original function on these coordinates.) We shall call this extended function f. Writing $O_f(z)$ in matrix coordinates we get

$$O_f(z) = \int f\begin{pmatrix} xz^{-1} & -z + xyz^{-1} \\ z^{-1} & yz^{-1} \end{pmatrix} e^{2\pi i(x+y)}\,dx\,dy.$$

We change variables $xz^{-1} \mapsto x$ and $yz^{-1} \mapsto y$ to get

$$O_f(z) = |z|^2 \int f\begin{pmatrix} x & z(xy-1) \\ z^{-1} & y \end{pmatrix} e^{2\pi iz(x+y)}\,dx\,dy.$$

Then we use Fourier inversion formula on the $(1,2)$ variable to get

$$O_f(z) = |z|^2 \int \hat{f}_{1,2}\begin{pmatrix} x & u \\ z^{-1} & y \end{pmatrix} e^{2\pi iz(u(xy-1)+x+y)}\,dx\,dy. \tag{3.1}$$

Here $\hat{f}_{1,2}$ is the Fourier transform of f in the $(1,2)$ coordinate and is defined by

$$\hat{f}_{1,2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{-\infty}^{\infty} f\begin{pmatrix} a & t \\ c & d \end{pmatrix} e^{-2\pi itb}\,dt.$$

It follows that $\hat{f}_{1,2}$ is smooth and that it is compactly supported in the a, c and d variables and that it is rapidly decreasing in the b variable. Hence the integral in (3.1) takes place on a set where x and y are bounded. That is, there exist $C > 0$ such that the integral takes place on the set $|x| \leqslant C$ and $|y| \leqslant C$. We claim that the main term in the asymptotic expansion of this integral will come from the region where u is bounded and that the region where u is unbounded will contribute a rapidly decreasing function in z. To make this precise we make a change of variables in the integral: $m = u(xy-1)$, $t = x$ and $s = y$. This transformation is nonsingular when $xy \neq 1$, hence by using a partition of the identity and by dividing the integral to two integrals, one of them taking place on a small neighborhood of the set $\{(x,y,u)\colon\ xy = 1\}$ we obtain that

$$\begin{aligned} O_f(z) &= |z|^2 \int f_1\begin{pmatrix} x & u \\ z^{-1} & y \end{pmatrix} e^{2\pi iz(u(xy-1)+x+y)}\,dx\,dy \\ &\quad + |z|^2 \int f_2\begin{pmatrix} t & m/(ts-1) \\ z^{-1} & s \end{pmatrix} e^{2\pi iz(m+t+s)}\,dt\,ds. \end{aligned}$$

Here f_1 is a smooth function which is supported on a small neighborhood around the set above (defined by the equation $xy = 1$) and such that x and y are bounded. The function f_2 is compactly supported in the t and s variables away from the set $\{(t,s)\colon\ ts = 1\}$ and is rapidly decreasing in the m variable. Using an integration by parts arguments, it follows that the second integral gives a rapidly decreasing function in z and does not contribute to the main term of asymptotic expansion. Hence, we can assume that $\hat{f}_{1,2}$ is supported on a neighborhood of the set $\{(x,y,u)\colon\ xy = 1,\ |x| < C,\ |y| < C\}$. Notice that this means that x and y are bounded away from 0. Using another change of variables: $m = uxy - x$, $t = u$, $s = y$ and arguing as above we can further assume that $\hat{f}_{1,2}$ is supported on a small neighborhood of the set $uy - 1 = 0$. Since y is away from 0, it means that u is bounded.

To summarize, we have proved that the main term of asymptotic expansion of $O_f(z)$ comes from the integral of $\hat{f}_{1,2}$ in (3.1) where we integrate on a set where x, y, u are bounded. That is, we can assume that $\hat{f}_{1,2}$ is compactly supported in all coordinates. In particular, we can use a Taylor expansion in the $(2,1)$ coordinate to evaluate the integral above: The first term of this expansion is

$$|z|^2 \int \hat{f}_{1,2}\begin{pmatrix} x & u \\ 0 & y \end{pmatrix} e^{2\pi i z(u(xy-1)+x+y)}\, dx\, dy\, du.$$

This is an oscillatory integral with the phase function

$$\phi(u,x,y) = u(xy-1) + x + y.$$

There are two critical points $(-1,1,1)$ and $(1,-1,-1)$ and the Hessian is nonsingular at both points hence ϕ is a Morse function (see [17]). The values of $\hat{f}_{1,2}$ at the critical points are

$$\hat{f}_{1,2}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \int_{-\infty}^{\infty} f\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^{2\pi i t}\, dt = W(f),$$

$$\hat{f}_{1,2}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \int_{-\infty}^{\infty} f\begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} e^{-2\pi i t}\, dt = \int_{-\infty}^{\infty} f\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} e^{-2\pi i t}\, dt = W(f).$$

It follows from the theory of stationary phase [17, Theorem 2.9] that

$$O_f(z) = z^{1/2}\bigl(\cos(4\pi z) - \sin(4\pi z)\bigr)W(f) + O\bigl(z^{-1/2}\bigr). \tag{3.2}$$

More precisely, using the asymptotic expansion in [17, Theorem 2.9] and taking into account the other terms of the Taylor expansion in the $(2,1)$ variable we can find the asymptotic expansion of $O_f(z)$. Similar calculations for the derivatives of $O_f(z)$ will yield:

Theorem 3.1. [10, Section 7] *Let $f \in C_c^\infty(G)$ and assume that $f(g) = f(-g)$ for all $g \in G$. Then there exist constants $\alpha_i, \beta_i \in \mathbb{C}$ such that for every nonnegative integer n and for large z we have*

$$O_f(z) = z^{1/2}\big(\cos(4\pi z) - \sin(4\pi z)\big)W(f) + \left(\sum_{i=1}^{n}\big(\alpha_i \cos(4\pi z) + \beta_i \sin(4\pi z)\big)z^{-n+1/2}\right) + O\big(z^{-n-1/2}\big). \tag{3.3}$$

Moreover, the derivatives of $O_f(z)$ satisfy an asymptotic expansion which comes from taking derivatives in the above asymptotic expansion.

If we set $G_f(z) = z^{-1}O_f(z/4\pi)$ as above then by (3.2) we have:

Corollary 3.2. [10] *Let f be as in the theorem above. Then*

$$W(f) = \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2} G_f(z)}{\cos(z) - \sin(z)} \tag{3.4}$$

where the limit is taken on z which is away from $\pi/4 + k\pi$, that is, on a set of z such that $\cos(z) - \sin(z)$ *is bounded away from* 0.

4. Proof of Theorem 1.3

We are now ready to prove Theorem 1.3. We will leave two questions of convergence to the next section. The first point is that our function $G_f(z)$ that was defined in (2.4) satisfies the conditions of Corollary 2.2. That is, we can apply the Kuznecov inversion formula for $G_f(z)$. The second is a change of order of a limit and integration that we will point out in the course of the proof.

Let f be in $C_c^\infty(G)$. By (3.4) we have that

$$W(f) = \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2} G_f(z)}{\cos(z) - \sin(z)}.$$

By (2.6) we have that

$$G_f(z) = -\int_0^\infty \tilde{G}_f(r)\big(J_{ir}(z) - J_{-ir}(z)\big)\frac{r}{2\sinh(\pi r)}\,dr + \sum_{n>0} 2(2n-1)c_n(G_f)J_{2n-1}(z). \tag{4.1}$$

Hence, we have

$$W(f) = \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)}\left(-\int_0^\infty \tilde{G}_f(r)\big(J_{ir}(z) - J_{-ir}(z)\big)\frac{r\,dr}{2\sinh(\pi r)}\right) + \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \sum_{n>0} 2(2n-1)c_n(G_f)J_{2n-1}(z).$$

We shall now switch the order of integration and limit. This will be justified using the dominated convergence theorem in the next section. Hence we get

$$W(f) = -\sqrt{\pi}\int_0^\infty \tilde{G}_f(r)\left(\lim_{z\to\infty}\frac{z^{1/2}\big(J_{ir}(z)-J_{-ir}(z)\big)}{\cos(z)-\sin(z)}\right)\frac{r\,dr}{\sinh(\pi r)}$$
$$+ 2\sqrt{\pi}\sum_{n>0} 2(2n-1)c_n(G_f)\left(\lim_{z\to\infty}\frac{z^{1/2}J_{2n-1}(z)}{\cos(z)-\sin(z)}\right). \tag{4.2}$$

It follows from the asymptotic expansions of the classical Bessel functions [15, (5.11.6)] that

$$\lim_{z\to\infty}\frac{z^{1/2}(J_{ir}(z)-J_{-ir}(z))}{\cos(z)-\sin(z)} = -\frac{2i\sinh(\pi r/2)}{\sqrt{\pi}},$$
$$\lim_{z\to\infty}\frac{z^{1/2}J_{2n-1}(z)}{\cos(z)-\sin(z)} = \frac{(-1)^n}{\sqrt{\pi}}.$$

Hence we have that

$$W(f) = i\int_0^\infty \frac{1}{\cosh(\pi r/2)}\tilde{G}_f(r)r\,dr + 4\sum_{n=1}^\infty (2n-1)(-1)^n c_n(G_f).$$

By (2.5) we have that $\tilde{G}_f(r) = -i\frac{\sinh(\pi r/2)}{8\pi^2}J_{\pi^+_{ir},\psi}(f)$ and that $c_n(G_f) = \frac{(-1)^n}{16\pi^2}J_{\pi_{2n-1},\psi}(f)$. Thus,

$$(2\pi)^2 W(f) = \frac{1}{2}\int_0^\infty J_{\pi^+_{ir},\psi}(f)\tanh(\pi r/2)r\,dr + \sum_{n=1}^\infty (2n-1)J_{\pi_{2n-1},\psi}(f).$$

5. Estimates and convergence

To conclude the proof of Theorem 1.3 we need to show that $G_f(z)$ satisfies the requirements of Corollary 2.2 so that Kuznecov inversion holds and that the change of order of the limit and integration which we employed in the proof is justified. We will need some estimates on the Kuznecov transform of $G_f(z)$. We start first with some classical estimates and integrals of Bessel functions.

5.1. Integrals of Bessel functions

We will need the following integrals [9, p. 328, (10), (11)]. Assume that $-\mathrm{Re}(\nu) < \mathrm{Re}(s) < 1/2$. Then

$$C(s,\nu) = \int_0^\infty J_\nu(x)\cos(x)x^{s-1}\,dx = \frac{2^{\nu-1}\Gamma(1/2-s)\Gamma(\nu/2+s/2)}{\Gamma(-\nu/2+1/2-s/2)\Gamma(1+\nu-s)},$$

$$S(s,\nu)=\int_0^\infty J_\nu(x)\sin(x)x^s\,dx=\frac{2^{\nu-1}\Gamma(1/2-s)\Gamma(\nu/2+1/2+s/2)}{\Gamma(-\nu/2+1-s/2)\Gamma(1+\nu-s)}. \tag{5.1}$$

Using the classical formulas

$$\Gamma(z)=\frac{\pi}{\sin(\pi z)\Gamma(1-z)},\qquad \Gamma(z)=\frac{\sqrt{\pi}\;\Gamma(2z)}{2^{2z-1}\Gamma(z+1/2)}$$

we get that

$$\begin{aligned}
C(s,\nu)&=\Gamma(1/2-s)\frac{2^{\nu-1}\sin(\pi(\nu/2+1/2+s/2))\Gamma(\nu/2+1/2+s/2)\Gamma(\nu/2+s/2)}{\pi\Gamma(1+\nu-s)}\\
&=2^s\Gamma(1/2-s)\frac{\sin(\pi(\nu/2+1/2+s/2))\Gamma(\nu+s)}{\sqrt{\pi}\Gamma(1+\nu-s)},\\
S(s,\nu)&=\Gamma(1/2-s)\frac{2^{\nu-1}\sin(\pi(\nu/2+s/2))\Gamma(\nu/2+s/2)\Gamma(\nu/2+1/2+s/2)}{\pi\Gamma(1+\nu-s)}\\
&=2^s\Gamma(1/2-s)\frac{\sin(\pi(\nu/2+s/2))\Gamma(\nu+s)}{\sqrt{\pi}\Gamma(1+\nu-s)}.
\end{aligned}$$

By [15, p. 7, ex. 7] we have that

$$\frac{\Gamma(\nu+a)}{\Gamma(\nu+b)}=\nu^{b-a}\big(1+O\big(|\nu|^{-1}\big)\big)$$

where a and b are constants and $|\arg(\nu)|<\pi-\delta$ for some positive constant δ. Hence we have proved the following lemma.

Lemma 5.1. *Let s be a constant and assume that $-\mathrm{Re}(\nu)<\mathrm{Re}(s)<1/2$. Assume also that $\mathrm{Re}(\nu)\geqslant 0$, hence, $|\arg(\nu)|\leqslant\pi/2$. Then*

$$\begin{aligned}
C(s,\nu)&=\frac{1}{\sqrt{\pi}}2^s\Gamma(1/2-s)\sin\big(\pi(\nu/2+1/2+s/2)\big)\nu^{2s-1}\big(1+O\big(|\nu|^{-1}\big)\big),\\
S(s,\nu)&=\frac{1}{\sqrt{\pi}}2^s\Gamma(1/2-s)\sin\big(\pi(\nu/2+s/2)\big)\nu^{2s-1}\big(1+O\big(|\nu|^{-1}\big)\big).
\end{aligned}\tag{5.2}$$

In particular, we get the following corollary.

Corollary 5.2. *Let $s\in\mathbb{R}$ be a constant.*

(a) *Let $x>0$ and assume that $x>-s$. Then for large x we have $C(s,x)=O(x^{2s-1})$, $S(s,x)=O(x^{2s-1})$.*
(b) *Let $\alpha\in\mathbb{R}$ be a constant and assume that $-\alpha<s<0$. Let $r\in\mathbb{R}$. Then for large $|r|$ we have $C(s,\alpha+ir)=O(e^{\pi|r|/2}|r|^{2s-1})$, $S(s,\alpha+ir)=O(e^{\pi|r|/2}|r|^{2s-1})$.*

We will be interested in the case $s=-1/2$ and $x=2n+1$ where $n\in\mathbb{N}$.

Lemma 5.3. $C(-1/2, 2n+1) = S(-1/2, 2n+1)$.

Proof. This follows immediately from the above computations and from the fact that $\sin(\pi((2n+1)/2 - 1/4)) = \sin(\pi((2n+1)/2 + 1/4))$. □

The following lemma is easy.

Lemma 5.4. *Let $\alpha \in \mathbb{R}$ be a constant and let $t \in \mathbb{R}$. Let $\nu = \alpha + ir$. Then when $|r| \to \infty$, $|\nu^{-2} - \bar{\nu}^{-2}| = O(|r|^{-3})$.*

Proposition 5.5. *Let $n \in \mathbb{N}$ be fixed. Then when $|t|$ is large,*

$$\begin{aligned}&C(-1/2, 2n+1+ir) - S(-1/2, 2n+1-ir) + S(-1/2, 2n+1+ir)\\&\quad - C(-1/2, 2n+1-ir) = O\big(|r|^{-3}e^{-\pi|r|/2}\big),\\&C(-1/2, 2n+ir) + S(-1/2, 2n-ir) - S(-1/2, 2n+ir)\\&\quad - C(-1/2, 2n-ir) = O\big(|r|^{-3}e^{-\pi|r|/2}\big).\end{aligned}$$

Proof. We claim that the summands $C(-1/2, 2n+1+ir) - S(-1/2, 2n+1-ir)$, $S(-1/2, 2n+1+ir) - C(-1/2, 2n+1-ir)$, $C(-1/2, 2n+ir) + S(-1/2, 2n-ir)$ and $-S(-1/2, 2n+1+ir) - C(-1/2, 2n+1-ir)$ have the required order. For example, by (5.2) we have that the first summand satisfies

$$\begin{aligned}&C(-1/2, 2n+1+ir) - S(-1/2, 2n+1-ir)\\&\quad = \frac{1}{\sqrt{2\pi}}\sin\big(\pi\big((2n+1)/2 + 1/2 - 1/4 + ir/2\big)\big)(2n+1+ir)^{-2}O\big(1+|r|^{-1}\big)\\&\qquad - \frac{1}{\sqrt{2\pi}}\sin\big(\pi(2n+1)/2 - 1/4 - ir/2\big)(2n+1-ir)^{-2}O\big(1+|r|^{-1}\big)\\&\quad = \bigg(\frac{1}{\sqrt{2\pi}}\sin\big(\pi(n+3/4+ir/2)\big)(2n+1+ir)^{-2}\\&\qquad - \frac{1}{\sqrt{2\pi}}\sin\big(\pi(n+1/4-ir/2)\big)(2n+1-ir)^{-2}\bigg)O\big(1+|r|^{-1}\big).\end{aligned}$$

Assume $r > 0$ is large. Then the leading term of $\sin(\pi(n+3/4+ir/2))$ is $-\frac{1}{2i}e^{-i\pi(n+3/4)}e^{\pi r/2}$ and the leading term of $\sin(\pi(n+1/4-ir/2)$ is $\frac{1}{2i}e^{i\pi(n+1/4)}e^{\pi r/2}$. Since $e^{-i\pi(n+3/4)} = -e^{i\pi(n+1/4)}$ we get that the leading term of $C(-1/2, 2n+1+ir) - S(-1/2, 2n+1-ir)$ is $\frac{1}{2i\sqrt{2\pi}}e^{i\pi(n+1/4)}e^{\pi r/2}(\nu^{-2} - \bar{\nu}^{-2})$. Hence by Lemma 5.4 we have that the leading term is of order $|r|^{-3}e^{-\pi|r|/2}$ as required. The computations for $r < 0$ and for the other summands are similar. □

We will also need some classical bounds on J-Bessel functions: Let $r, x \in \mathbb{R}$ be positive and large and let n be a large integer. Then

$$\left|J_{\pm ir}(x)\right| \ll e^{\pi r/2} x^{-1/2}, \tag{5.3}$$

$$\left|J_n(x)\right| \ll n^{1/6} x^{-1/2}, \tag{5.4}$$

$$\left|J_n(x)\right| \ll 1. \tag{5.5}$$

The first inequality follows from the asymptotic expansion [8, 7.13, (17)]. The second and third inequalities follow from the asymptotic expansions for five different regions of n and x. For $n > 2x$ we use [8, 7.13, (14)] or [19, 8.43, (1)]. For $x > 2n$ we use [8, 7.13, (11)] or [19, 8.43, (2)]. For $n/2 \leqslant x \leqslant 2n$ and $|n - x| \leqslant x^{1/3}$ we use [8, 7.13, (15)] and for $n/2 \leqslant x \leqslant 2n$ and $|n - x| > x^{1/3}$ we use [8, 7.4, (30), (31)]. The following lemma is also well known.

Lemma 5.6. *Let $\nu = \alpha + ir \in \mathbb{C}$ be such that $\alpha \geqslant 0, r \in \mathbb{R}$. Let $x > 0$. Then*

$$\left|J_\nu(x)\right| \leqslant \left|\frac{\sqrt{\pi} x^\alpha}{\Gamma(\nu + 1/2)}\right|, \tag{5.6}$$

$$\left|J_\nu(x)\right| \ll_\alpha e^{\pi |r|/2}. \tag{5.7}$$

Proof. The first inequality follows from taking absolute values in the integral representation

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi \cos\bigl(x \cos(\theta)\bigr) \sin^{2\nu}(\theta)\, d\theta$$

which is valid when $\alpha \geqslant 0$. The second inequality follows from the first inequality when $x \leqslant 1$. For $x > 1$ and $\alpha = 0$ we use (5.3) and for $\alpha > 0$ we use the integral representation [19, 6.21, (7)]. □

As a corollary from (5.3) and (5.7) we get the well-known inequality:

$$\left|J_{ir}(x)\right| \ll 1 \quad \text{for } |r| \leqslant 1. \tag{5.8}$$

5.2. Bounds on Kuznecov transforms

Let $F(x)$ be a function defined on $(0, \infty)$. Recall that we defined

$$\tilde{F}(r) = \int_0^\infty F(x)\bigl(J_{ir}(x) - J_{-ir}(x)\bigr) \frac{dx}{x}$$

and

$$c_n(F) = \int_0^\infty F(y) J_{2n-1}(y) \frac{dy}{y}.$$

If $F(x)$ is absolutely integrable on the positive real line with respect to the measure dx/x then it follows from (5.8) that

$$\tilde{F}(r) = O(1), \quad r \in [0, 1]. \tag{5.9}$$

The following lemma is implicit in [14, Appendix]. It is not sharp but it suffices for our purpose.

Lemma 5.7. *Let $F(x)$ be a smooth function on $(0, \infty)$ such that F vanishes around $x = 0$. Assume that there exists $\epsilon > 0$ such that as $x \to \infty$.*

$$\left|F(x)\right| + \left|F'(x)\right| + \left|F''(x)\right| + \left|F'''(x)\right| = O\left(x^{-3-\epsilon}\right).$$

Then $\tilde{F}(r) = O(|r|^{-3} e^{\pi|r|/2})$ and $c_n(F) = O(n^{-3})$.

Proof. The proof follows from integration by parts. We will use the relation

$$\frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right) = -x^{-\nu} J_{\nu+1}(x).$$

Assume $\operatorname{Re}(\nu) \geqslant 0$ and $\nu \neq 0$. Then we have

$$\begin{aligned}
\int_0^\infty F(x) J_\nu(x) \frac{dx}{x} &= \int_0^\infty x^{-1+\nu}\left(F(x) x^{-\nu} J_\nu(x)\right) dx \\
&= -\frac{1}{\nu} \int_0^\infty x^\nu F'(x) x^{-\nu} J_\nu(x)\, dx + \frac{1}{\nu} \int_0^\infty x^{\nu+1} F(x) x^{-\nu-1} J_{\nu+1}(x)\, dx \\
&= \frac{1}{\nu(\nu+1)} \int_0^\infty x^{\nu+1} F''(x) x^{-\nu} J_\nu(x)\, dx \\
&\quad - \frac{1}{\nu(\nu+1)} \int_0^\infty x^{\nu+2} F'(x) x^{-\nu-1} J_{\nu+1}(x)\, dx \\
&\quad - \frac{1}{\nu(\nu+2)} \int_0^\infty x^{\nu+2} F'(x) x^{-\nu-1} J_{\nu+1}(x)\, dx \\
&\quad + \frac{1}{\nu(\nu+2)} \int_0^\infty x^{\nu+3} F(x) x^{-\nu-2} J_{\nu+2}(x)\, dx.
\end{aligned} \tag{5.10}$$

Applying integration by parts once more we will get eight summands.

For our first statement we need to take $\nu = ir$. Then the first summand of these eight is estimated using (5.7):

$$\left| -\frac{1}{ir(ir+1)(ir+2)} \int_0^\infty F'''(x) J_{ir}(x) x^2\, dx \right| \ll |r|^{-3} e^{\pi|r|/2} \int_0^\infty \left|F'''(x)\right| x^2\, dx.$$

The other summands are estimated similarly. For our second statement we take $\nu = 2n+1$. Using (5.5) and a similar analysis of the eight summands will yield the bound n^{-3}. $\square$

Proposition 5.8. *Let $G(x)$ be a smooth function on $(0,\infty)$ such that G vanishes around $x = 0$. Assume that there exists a constant A and constants α_i, β_i, $i = 1, 2, \ldots,$ such that*

$$G(x) = Ax^{-1/2}\big(\cos(x) - \sin(x)\big) + \left(\sum_{i=1}^{n} \big(\alpha_i \cos(x) + \beta_i \sin(x)\big) x^{-i-1/2}\right) + O\big(x^{-n-3/2}\big) \tag{5.11}$$

for every $n \in \mathbb{N}$ and that the derivatives of G satisfy an asymptotic expansion which comes from taking derivatives in this asymptotic expansion. In particular

$$G'(x) = -Ax^{-1/2}\big(\cos(x) + \sin(x)\big) + \left(\sum_{i=1}^{n} \big(\alpha_i' \cos(x) + \beta_i' \sin(x)\big) x^{-i-1/2}\right) + O\big(x^{-n-3/2}\big), \tag{5.12}$$

where α_i', β_i' are constants that are determined by A, α_j, β_j, $j \in \mathbb{N}$. Then $\tilde{G}(r) = O(r^{-3} e^{\pi r/2})$ when $r \to \infty$ and $c_n(G) = O(n^{-3})$.

Proof. We will first prove that $c_n(G) = O(n^{-3})$. Let $\gamma(x)$ be a smooth function on $[0,\infty]$ that vanishes around $x = 0$ and such that $\gamma(x) = 1$ around ∞. We will also assume that $0 \leqslant \gamma(x) \leqslant 1$ for all x. Let $\delta(x) = 1 - \gamma(x)$. Then there exists a positive constant M such that $\delta(x) = 0$ for all $x > M$. Let

$$G_1(x) = Ax^{-1/2}\big(\cos(x) - \sin(x)\big) + x^{-3/2}\big(\alpha_1 \cos(x) + \beta_1 \sin(x)\big) + x^{-5/2}\big(\alpha_2 \cos(x) + \beta_2 \sin(x)\big).$$

We write $G(x) = \gamma(x) G_1(x) + G_2(x)$. Then $G_2(x)$ satisfies the conditions of Lemma 5.7, hence we have that $c_n(G_2) = O(n^{-3})$. We will now prove that $c_n(\gamma(x) G_1) = O(n^{-3})$. We have that $\gamma(x) G_1(x) = (1 - \delta(x)) G_1(x) = G_1(x) - \delta(x) G_1(x)$. By (5.1) we have that

$$c_n(G_1) = A\big(C(-1/2, 2n-1) - S(-1/2, 2n-1)\big) + \alpha_1 C(-3/2, 2n-1) + \beta_1 S(-3/2, 2n-1) + \alpha_2 C(-5/2, 2n-1) + \beta_2 S(-5/2, 2n-1).$$

By Lemma 5.3 and Corollary 5.2(a), we have that each summand is of order n^{-3}. We now consider the function $\delta(x) G_1(x)$. Every summand of this function is a constant multiple of

$R(x) = \delta(x)x^p \cos(x)$ or $Q(x) = x^p \sin(x)$. Using the fact that $\delta(x)$ is supported on $[0, M]$ and that $|\delta(x)| \leqslant 1$ and using (5.6) we have that for large n

$$\begin{aligned}\left|c_n(R)\right| &= \left|\int_0^\infty \delta(x)x^{p-1}\sin(x)J_{2n-1}(x)\,dx\right| \leqslant \frac{\sqrt{\pi}}{\Gamma(2n-1/2)}\int_0^M x^{p-1}x^{2n-1}\,dx \\ &= \frac{\sqrt{\pi}M^{2n+p-1}}{(2n+p-1)\Gamma(2n-1/2)}.\end{aligned}$$

Using Stirling's formula it is easy to see that $c_n(R)$ is rapidly decreasing as $n \to \infty$. Since the same is true for $|c_n(Q)|$ we get that $c_n(G) = O(n^{-3})$.

We will now prove that $\tilde{G}(r) = O(r^{-3}e^{\pi r/2})$ when $r \to \infty$. Set $F(x) = G(x)/x$. The asymptotic expansion of $F(x)$ and its derivatives follow directly from the asymptotic expansion of $G(x)$. Using the relation $J_\nu(x) = J_{\nu+2}(x) + 2J'_{\nu+1}(x)$ we can write

$$\begin{aligned}\tilde{G}(r) &= \int_0^\infty \frac{G(x)}{x}\big(J_{ir}(x) - J_{-ir}(x)\big)\,dx \\ &= \int_0^\infty F(x)\big(J_{2+ir}(x) - J_{2-ir}(x)\big)\,dx + 2\int_0^\infty F(x)\big(J'_{1+ir}(x) - J'_{1-ir}(x)\big)\,dx.\end{aligned}$$

Using integration by parts for the second summand and using the fact that $F(x)$ vanishes around $x = 0$ and is of order $x^{-3/2}$ at ∞ and that $J_\nu(x)$ has order $x^{-1/2}$ at ∞ we get that

$$\tilde{G}(r) = \int_0^\infty F(x)\big(J_{2+ir}(x) - J_{2-ir}(x)\big)\,dx - 2\int_0^\infty F'(x)\big(J_{1+ir}(x) - J_{1-ir}(x)\big)\,dx.$$

Applying the same type of arguments for the first summand and for the second summand we get

$$\begin{aligned}\tilde{G}(r) = {}&\int_0^\infty F(x)\big(J_{4+ir}(x) - J_{4-ir}(x)\big)\,dx - 4\int_0^\infty F'(x)\big(J_{3+ir}(x) - J_{3-ir}(x)\big)\,dx \\ &+ 4\int_0^\infty F''(x)\big(J_{3+ir}(x) - J_{3-ir}(x)\big)\,dx - 8\int_0^\infty F'''(x)\big(J_{3+ir}(x) - J_{3-ir}(x)\big)\,dx.\end{aligned}$$

The purpose of this manipulation was to increase the order of vanishing of the Bessel functions at $x = 0$. This will allow us to extend the asymptotic expansions of $F(x)$, $F'(x)$, $F''(x)$ and $F'''(x)$ all the way to $x = 0$ and to get a classical convergent integral as we did in the $c_n(G)$ case.

We shall now analyze each summand separately. Since the analysis is similar for all summands we will only look at $\int_0^\infty F'(x)(J_{3+ir}(x) - J_{3-ir}(x))\,dx$.

Let $\gamma(x)$ be as above and $\delta(x) = 1 - \gamma(x)$. Since $F(x) = G(x)/x$ and by (5.11), (5.12), we have that when $x \to \infty$,

$$F'(x) = Ax^{-3/2}\big(\cos(x)+\sin(x)\big) + \left(\sum_{i=1}^{n}\big(c_i\cos(x)+d_i\sin(x)\big)x^{-i-3/2}\right) \\ + O\big(x^{-n-5/2}\big).$$

Let

$$F_1(x) = Ax^{-3/2}\big(\cos(x)+\sin(x)\big) + x^{-5/2}\big(c_1\cos(x)+d_1\sin(x)\big) \\ + x^{-7/2}\big(c_2\cos(x)+d_2\sin(x)\big).$$

Notice that $\gamma(x)F_1(x)$ is a smooth function that vanishes around $x=0$ and that has similar asymptotic to F at ∞. Write $F'(x) = \gamma(x)F_1(x) + F_2(x)$. Then F_2 is smooth and vanishes around $x=0$ and is such that $F_2(x)$ and all its derivatives are of order $x^{-9/2}$ when $x\to\infty$. From a similar argument as in the proof of Lemma 5.7 it follows that $\int_0^\infty F_2(x)(J_{3+ir}(x)-J_{3-ir}(x))\,dx = O(r^{-3}e^{\pi r/2})$. To conclude our proof we will show that $\int_0^\infty \gamma(x)F_1(x)(J_{3+ir}(x)-J_{3-ir}(x))\,dx = O(r^{-3}e^{\pi r/2})$. To do that we write $\gamma(x)F_1(x) = F_1(x)-\delta(x)F_1(x)$. By (5.1) we have that

$$\int_0^\infty F_1(x)\big(J_{3+ir}(x)-J_{3-ir}(x)\big)\,dx \\ = A\big(C(-1/2,3+ir)-S(-1/2,3-ir)+S(-1/2,3+ir)-C(-1/2,3-ir)\big) \\ + c_1\big(C(-3/2,3+ir)-C(-3/2,3-ir)\big)+d_1\big(S(-3/2,3+ir)-S(-3/2,3-ir)\big) \\ + c_2\big(C(-5/2,3+ir)-C(-5/2,3-ir)\big)+d_2\big(C(-5/2,3+ir)-S(-5/2,3-ir)\big).$$

By Proposition 5.5 and Corollary 5.2(b), each summand is of order $r^{-3}e^{\pi r/2}$.

We now need to consider the integral of $\delta(x)F_1$. By the definition of $\delta(x)$ above we have

$$\left|\int_0^\infty \delta(x)F_1(x)\big(J_{3+ir}(x)-J_{3-ir}(x)\big)\,dx\right| \leqslant \int_0^M \big|F_1(x)\big(J_{3+ir}(x)-J_{3-ir}(x)\big)\big|\,dx.$$

Now $|F_1(x)| \ll_M x^{-7/2}$ on the interval $(0,M]$ and by (5.6), $|J_{3\pm ir}(x)| \ll |\frac{1}{\Gamma(3+1/2\pm ir)}|x^3$. It follows that the above integral is of order $|\frac{1}{\Gamma(3+1/2\pm ir)}|$ which by Sterling's formula is of order $|r|^{-3}e^{\pi|r|/2}$.

Hence we have shown that $\int_0^\infty F'(x)(J_{3+ir}(x)-J_{3-ir}(x))\,dx = O(|r|^{-3}e^{\pi|r|/2})$. Similar arguments for the integrals $\int_0^\infty F(x)(J_{4+ir}(x)-J_{4-ir}(x))\,dx$, the integral $\int_0^\infty F''(x)(J_{3+ir}(x)-J_{3-ir}(x))\,dx$ and $\int_0^\infty F'''(x)(J_{3+ir}(x)-J_{3-ir}(x))\,dx$ will yield the same bound and complete the proof. □

We can now state the Kuznecov inversion formula for functions $G_f(x)$ coming from orbital integrals that was promised after Corollary 2.2.

Corollary 5.9. *Let $f \in C_c^\infty(G)$ and for $x > 0$ let $G_f(x) = x^{-1} O_f(x/4\pi)$ where $O_f(x)$ is the orbital integral defined in* (2.2). *Then $\tilde{G}_f(r) = O(r^{-3} e^{\pi r/2})$ and in particular, G_f satisfies the Kuznecov inversion formula* (2.6).

Proof. It follows from [10] or [4] or by direct calculation that $O_f(x)$ and $G_f(x)$ vanish around $x = 0$. Hence, it follows from (3.3) and Proposition 5.8 that $G_f(x)$ satisfies the conditions of Corollary 2.2. □

5.3. Change of order of limit and integration

In the proof of Theorem 1.3 (see (4.2)) we had to change the order of

$$W(f) = \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \left(-\int_0^\infty \tilde{G}_f(r) \big(J_{ir}(z) - J_{-ir}(z) \big) \frac{r\,dr}{2\sinh(\pi r)} \right)$$
$$+ \lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \sum_{n>0} 2(2n-1) c_n(G_f) J_{2n-1}(z).$$

Since we take the limit $z \to \infty$ away from $\pi/4 + k\pi$, we have that $|\frac{1}{\cos(z)-\sin(z)}|$ is bounded uniformly by a constant. By (5.3) we have that for $|r| \geqslant 1$ and large z, $J_{ir}(z) - J_{-ir}(z) = O(|z|^{-1/2} e^{\pi|r|/2})$. Thus, for $|r| \geqslant 1$ and z away from $\pi/4 + k\pi$, we have that by Proposition 5.8,

$$\frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \tilde{G}_f(r) \big(J_{ir}(z) - J_{-ir}(z) \big) \frac{r\,dr}{2\sinh(\pi r)} = O\big(|r|^{-2}\big).$$

Hence we can use the dominated convergence theorem to move the limit into the integral from 1 to ∞.

For the integral from 0 to 1 we use that $J_{ir}(z) - J_{-ir}(z) = O(z^{-1/2})$ for $|r| \leqslant 1$ and large z. Also, $\frac{r}{2\sinh(\pi r)}$ is bounded for $|r| \leqslant 1$. Hence by (5.9) we have that for $|r| \leqslant 1$,

$$\frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \tilde{G}_f(r) \big(J_{ir}(z) - J_{-ir}(z) \big) \frac{r\,dr}{2\sinh(\pi r)} = O(1),$$

and we can use the dominated convergence theorem again.

5.4. Change of order of limit and summation

In (4.2) we changed the order of limit and summation in the following equation:

$$\lim_{z\to\infty} \frac{2\sqrt{\pi} z^{1/2}}{\cos(z) - \sin(z)} \sum_{n=1}^\infty 2(2n-1) c_n(G_f) J_{2n-1}(z)$$
$$= 2\sqrt{\pi} \sum_{n=1}^\infty 2(2n-1) c_n(G_f) \left(\lim_{z\to\infty} \frac{z^{1/2} J_{2n-1}(z)}{\cos(z) - \sin(z)} \right).$$

As above, we take the limit $z \to \infty$ away from $z = \pi/4 + k\pi$. By Proposition 5.8 we have that $c_n(G_f) = O(n^{-3})$ and by (5.3) we have $J_{2n-1}(z) = O(n^{1/6} z^{-1/2})$. Hence

$$2(2n-1)c_n(G_f)\frac{2\sqrt{\pi}z^{1/2}J_{2n-1}(z)}{\cos(z)-\sin(z)} = O\big(n^{-11/6}\big),$$

and we can use the dominated convergence theorem to justify the above change of order.

6. The general case

We will now prove the general case of Theorem 1.1. For that we need to consider odd functions and any character ψ_λ.

6.1. The case of odd functions

We consider now the case of functions $f \in C_c^\infty(G)$ satisfying $f(-g) = -f(g)$ for all $g \in G$. The Whittaker–Plancherel inversion formula for these functions is:

Theorem 6.1. *Let $f \in C_c^\infty(G)$. Assume that $f(-g) = -f(g)$ for all $g \in G$. Let $\psi(x) = e^{2\pi i x}$. Then*

$$(2\pi)^2 \int_{-\infty}^{\infty} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{2\pi i x}\, dx = \frac{1}{2}\int_0^\infty J_{\pi_{ir}^-,\psi}(f)\operatorname{cotanh}(\pi r/2) r\, dr + \sum_{d=1}^\infty 2d J_{\pi_{2d},\psi}(f). \tag{6.1}$$

The proof of this theorem will follow the exact same steps as the proof of Theorem 1.3. We shall omit the calculations and indicate the results: For a function f as above we can show that

$$O_f(z) = i z^{1/2}\big(\cos(4\pi z) + \sin(4\pi z)\big) W(f) + O\big(z^{-1/2}\big). \tag{6.2}$$

Moreover, we can show as in the proof of (3.3), that

$$\begin{aligned} O_f(z) = {} & i z^{1/2}\big(\cos(4\pi z) - \sin(4\pi z)\big) W(f) + \left(\sum_{i=1}^n \big(\gamma_i \cos(4\pi z) + \delta_i \sin(4\pi z)\big) z^{-n+1/2}\right) \\ & + O\big(z^{-n-1/2}\big). \end{aligned} \tag{6.3}$$

From here the proof proceeds as in Section 4. We will need a Kuznecov transform and inversion formula that is adjusted for this case and we state it now.

6.2. Another Kuznecov transform

Let $G(x)$ be a function defined on $x > 0$ which is absolutely integrable on the positive real line with respect to the measure dx/x. For $t > 0$ define

$$\check{G}(r) = \int_0^\infty G(x)\big(J_{ir}(x) + J_{-ir}(x)\big)\frac{dx}{x}$$

and

$$b_n(G)=\int_0^\infty G(y)J_{2n}(y)\,\frac{dy}{y}.$$

Theorem 6.2. [14, Appendix] *Assume that $G(x)$ is continuous on $x>0$, $G(x)$ vanishes around $x=0$ and that for some $A>0$, $G(x)=O(x^{-A})$ as $x\to\infty$. Assume also that for some $B>2$, $\check{G}(r)=O(r^{-B}e^{\pi r/2})$ as $r\to\infty$. Then*

$$G(x)=\int_0^\infty \check{G}(r)\big(J_{ir}(x)+J_{-ir}(x)\big)\frac{r\,dr}{2(\sinh(\pi r))}+\sum_{n>0}4nb_n(G)J_{2n}(x).$$

The rest of the proof of Theorem 1.3 follows the same steps as above and is omitted.

6.3. Proof of Theorem 1.1

We again assume that $\psi(x)=e^{2\pi ix}$. Let $f\in C_c^\infty(G)$. We can write $f=f_1+f_2$ where $f_1(g)=(f(g)+f(-g))/2$ is even and $f_2(g)=(f(g)-f(-g))/2$ is odd. It follows that the odd distributions vanish on f_1 and that the even distributions vanish on f_2. That is, $J_{\pi_{ir}^-,\psi}(f_1)=J_{\pi_{2n},\psi}(f_1)=0$ and $J_{\pi_{ir}^+,\psi}(f_2)=J_{\pi_{2n-1},\psi}(f_2)=0$. Hence we can use (1.7) and (6.1) to get

$$\begin{aligned}(2\pi)^2&\int_{-\infty}^{\infty} f\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}e^{2\pi ix}\\ &=\frac{1}{2}\int_0^\infty J_{\pi_{ir}^+,\psi}(f)\tanh(\pi r/2)r\,dr+\frac{1}{2}\int_0^\infty J_{\pi_{ir}^-,\psi}(f)\operatorname{cotanh}(\pi r/2)r\,dr\\ &\quad+\sum_{n=1}^{\infty} nJ_{\pi_n,\psi}(f).\end{aligned}\tag{6.4}$$

This formula is exactly formula (1.2) for $\lambda=1$. For the more general formula we fix $\lambda>0$. We let

$$s\big(\lambda^{-1/2}\big)=\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}.$$

For a function $f\in C_c^\infty(G)$ we define $f^\lambda(g)=f(s(\lambda^{-1/2})gs(\lambda^{1/2}))$. It is easy to see that $W^\lambda(f)=\lambda^{-1/2}W(f^\lambda)$ and that $J_{\pi,\psi_\lambda}(f)=\lambda^{-1/2}J_{\pi,\psi}(f^\lambda)$. Hence, formula (1.2) follows immediately from (6.4). For $\lambda<0$, we first consider the case $\lambda=-1$. Following the same steps as for the case $\lambda=1$ we can prove formula (1.2) for this case and obtain the general $\lambda<0$ in the same way.

7. The representations

The Bessel functions in (1.5) are attached to tempered unitary representations of $G = SL(2,\mathbb{R})$ which we denoted by $\pi_{ir}^{+}, \pi_{ir}^{-}, \pi_{n}^{\lambda}$. We shall now describe these representations.

7.1. Principal series

Let B be the Borel subgroup of G of upper triangular matrices. That is, $B = \{n(x)s(z)\colon x \in \mathbb{R},\ z \in \mathbb{R}^*\}$. We define characters $\chi_{ir}^{+}, \chi_{ir}^{-}$ on B by $\chi_{ir}^{+}(n(x)s(z)) = |z|^{ir}$, $\chi_{ir}^{-}(n(x)s(z)) = \operatorname{sgn}(z)|z|^{ir}$. We let $\pi_{ir}^{+} = \operatorname{Ind}_B^G \chi_{ir}^{+}$ and $\pi_{ir}^{+} = \operatorname{Ind}_B^G \chi_{ir}^{-}$ where the space of $\operatorname{Ind}_B^G \chi$ is the set of smooth functions $f : G \to \mathbb{C}$ satisfying $f(n(x)s(z)g) = |z|^{-1}\chi(s(z))f(g)$ for every $g \in G$, $x \in \mathbb{R}$, $z \in \mathbb{R}^*$. The action of G is by right translations.

7.2. Discrete series

For $n \in \mathbb{N}$ we define $\chi_{2n+1}(n(x)s(z)) = |z|^{2n+1}$ and $\chi_{2n}(n(x)s(z)) = \operatorname{sgn}(z)|z|^{2n+1}$. We let $\Pi_{2n+1} = \operatorname{Ind}_B^G \chi_{2n+1}$ and $\Pi_{2n} = \operatorname{Ind}_B^G \chi_{2n}$. Then the representation Π_d, $d \in \mathbb{N}$ has two irreducible subspaces. One subspace which we denote by π_d^1 has a Whittaker model with respect to ψ_λ, $\lambda > 0$ and the other subspace, π_d^{-1} has a Whittaker model with respect to ψ_λ, $\lambda < 0$. (One will have a maximal vector of weight $-d-1$ and the other will have a minimal weight $d+1$.) We define

Definition 7.1.

$$\pi_d^{\lambda} = \begin{cases} \pi_d^1 & \text{if } \lambda > 0; \\ \pi_d^{-1} & \text{if } \lambda < 0. \end{cases}$$

8. Bessel distributions

The Bessel distributions given in (1.3) were defined in an ad hoc way. In this section we explain how to define Bessel distributions on $SL(2,\mathbb{R})$ and how to normalize them. It turns out that Bessel functions on $SL(2,\mathbb{R})$ are restrictions of Bessel functions on $GL(2,\mathbb{R})$ hence our formulas in the introduction come from similar formulas on $GL(2,\mathbb{R})$ which were obtained in [4]. For additional results on Bessel distributions see [1–4].

8.1. Non-normalized Bessel distributions on $SL(2,\mathbb{R})$

Let (π, H) be an irreducible unitary representation of $G = SL(2,\mathbb{R})$ on a Hilbert space H and let $\langle\,,\rangle$ be a G invariant inner product on H. Let H_∞ be the space of smooth vectors in H with the usual topology. Let ψ_λ be a character of N as defined in (1.1). A ψ_λ Whittaker functional on H_∞ is a nontrivial continuous functional $L : H_\infty \to \mathbb{C}$ satisfying

$$L\big(\pi(n)v\big) = \psi_\lambda(n)L(v), \quad n \in N,\ v \in H^\infty.$$

If such a functional exists then by [16] it is unique and we say that π has a ψ_λ Whittaker model.

For every $f \in C_c^\infty(G)$ and $v \in H$ we define $\pi(f)v = \int_G f(g)\pi(g)v\,dg$ where dg is a Haar measure on G.

Assume that π has a ψ_λ Whittaker model and let L be a ψ_λ Whittaker functional on H_∞. Let $\{v_i\colon i \in \mathbb{N}\}$ be an orthonormal basis of smooth vectors of H. We define the (non-normalized) Bessel distribution J_{π,ψ_λ} by

$$J_{\pi,\psi_\lambda}(f) = \sum_{i=1}^{\infty} L\bigl(\pi(f)v_i\bigr)\overline{L(v_i)}. \tag{8.1}$$

It follows from [4, Appendix 4] that the distribution is independent of the choice of orthonormal basis. It does depend on the choice of Whittaker functional L, on the choice of G invariant inner product $\langle\,,\rangle$ and on the choice of Haar measure dg. Hence, J_{π,ψ_λ} is defined up to a positive constant. We shall not worry about the normalization of the Haar measure dg since we can fix one measure for all representations π. This choice will change formula (1.2) by a positive scalar. For example, our particular choice in (2.1) is responsible for the factor $(2\pi)^2$ which appears in the left-hand side of (1.2). As before we set $B = NA$ to be a Borel subgroup in G. The main result about Bessel distributions for $GL(2,\mathbb{R})$ and consequently for $G = SL(2,\mathbb{R})$ is a regularity theorem proved in [4]. (For a weaker result which applies for general quasi-split groups see [2].)

Theorem 8.1. [4] *Let π be an irreducible unitary representation of G. Assume that π has a ψ_λ Whittaker model. Then there exists a real analytic function $j_{\pi,\psi} : BwB \to \mathbb{C}$ which is locally integrable on G such that*

$$J_{\pi,\psi_\lambda} = \int_G j_{\pi,\psi}(g) f(g)\,dg. \tag{8.2}$$

We call j_{π,ψ_λ} the Bessel function of π. It is clear that $j_{\pi,\psi}$ is dependent on the choice of L and $\langle\,,\rangle$ but independent on the choice of dg. In order to normalize $j_{\pi,\psi}$ we will connect between L and $\langle\,,\rangle$. It is easier to do that in $GL(2,\mathbb{R})$. Hence we will normalize the Bessel functions on $GL(2,\mathbb{R})$ and restrict them to $SL(2,\mathbb{R})$ to get normalized Bessel functions on $SL(2,\mathbb{R})$.

8.2. Normalized Bessel distributions and functions on $GL(2,\mathbb{R})$

Let

$$t(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Let (Π, M) be an infinite dimensional irreducible unitary representation of $GL(2,\mathbb{R})$. Then there exist a nontrivial ψ_λ Whittaker functional L on M_∞. For every $v \in M_\infty$ and $g \in G$ we define $W_v(g) = L(\pi(g)v)$. It is well known that for every $v_1, v_2 \in V_\infty$, the integral

$$\langle v_1, v_2\rangle = \int_{\mathbb{R}^*} W_{v_1}\bigl(t(a)\bigr)\overline{W_{v_2}\bigl(t(a)\bigr)}\,\frac{da}{|a|} \tag{8.3}$$

is absolutely convergent and defines a G invariant inner product on M_∞. Hence, $\langle v_1, v_2\rangle$ is a restriction of a G invariant inner product on M. The normalized Bessel distribution J_{Π,ψ_λ} is defined by (8.1) where $\{v_i\}$ is an orthonormal basis of smooth vectors in M^∞ with respect to the inner product (8.3). It is easy to see that the distribution J_{Π,ψ_λ} is independent of the choice of Whittaker functional L. By [4] there exist a function j_{Π,ψ_λ} on the open Bruhat cell in $GL(2,\mathbb{R})$ such that (8.2) holds. It is easy to see that j_{Π,ψ_λ} is independent of the Haar measure dg hence it is uniquely defined. From the description of irreducible unitary representations of $SL(2,\mathbb{R})$ and from arguments similar to the ones in [4, Section 18], it follows that for every irreducible unitary representation π of $SL(2,\mathbb{R})$ with a ψ_λ Whittaker model there exists an irreducible unitary representation Π of $GL(2,\mathbb{R})$ such that

$$j_{\pi,\psi_\lambda}(g) = j_{\Pi,\psi_\lambda}(g), \quad g \in BwB.$$

Here j_{Π,ψ_λ} is a nonnormalized Bessel function on $GL(2,\mathbb{R})$. We now normalize $j_{\pi,\psi_\lambda}(g)$ to be the restriction of a normalized Bessel function j_{Π,ψ_λ} on $GL(2,\mathbb{R})$. It follows that the formulas for $j_{\pi,\psi_\lambda}(g)$ will be obtained from formulas for normalized j_{Π,ψ_λ}. In particular, our formulas in (1.5) were obtained by restricting the formulas in [4] to $SL(2,\mathbb{R})$.

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