ON THE ASYMPTOTICS OF WHITTAKER FUNCTIONS

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1. Introduction

Whittaker models are ubiquitous in the representation theory of quasi-split reductive groups over local fields. They comprise the bedrock for whole families of local zeta integrals of Rankin-Selberg type. In the analysis of the latter, it is imperative to know the asymptotics of Whittaker functions, in order to control the domain of convergence of the integrals. This question was studied extensively in the literature, especially in the context of $GL_n$ (e.g. [JS90b], [JS90a], [JPSS79], [CPS]). In the Archimedean case a fairly complete answer is given in [Wal92]. On the other hand, to the best of the authors’ knowledge, the connection between the asymptotics of the Whittaker functions and the exponents of the representation in the $p$-adic case is not made explicit in the literature. The purpose of this short note is to partially fill this gap. The precise statement is given in Theorem 1 below, and is motivated by the results above. (Cf. [JS90b, Proposition 2.2], [JS90a, §2], [JPSS79]). As a consequence we realize the inner product of generic square-integrable and more generally, tempered representations, on the Whittaker model. For simplicity we work with split groups.

After completing an early version of this note we learned that Yiannis Sakellaridis and Akshay Venkatesh have launched an ambitious program to study the decomposition of the $L^2$-space of spherical varieties. In particular, they obtained asymptotic results of the kind which appear in this paper, in a very general setup. Although strictly speaking their current setup does not include the Whittaker case, there is no doubt that this can be eventually incorporated to the general scheme. In particular, Conjectures 1 and 2 below are probably within reach. Nevertheless, we believe that the Whittaker case is both sufficiently important and elementary to merit its own exposition.

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2. Preliminaries

Notation. Throughout let \( F \) be a non-archimedean local field of characteristic zero with valuation \( v \) and normalized absolute value \( |\cdot| = q^{-v(\cdot)} \). If \( X \) is a variety over \( F \) we will often denote its \( F \)-points by \( X \) as well. For any algebraic group \( H \) over \( F \) we denote its center by \( Z(H) \), its derived group by \( H^{\text{der}} \) and the connected component of the identity by \( H^0 \). Let \( X^*(H) \) be the lattice of rational characters of \( H \) and set

\[
H^1 = \cap_{\chi \in X^*(H)} \ker |\chi|.
\]

If \( H \) is a subgroup of \( G \) we denote its centralizer by \( C_G(H) \).

If \( T \) is a torus then \( T^1 \) is the maximal compact subgroup of \( T \). Suppose that \( T \) is split. Then the lattice \( X_*(T) \) of co-characters of \( T \) can be identified with \( \text{Hom}(X^*(T), \mathbb{Z}) \) and the map \( H : T \to X_*(T) \) defined by \( H(\mu(a)) = v(a)\mu, \mu \in X_*(T), a \in F^* \) induces an isomorphism \( T/T^1 \to X_*(T) \). Any \( \lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \) defines a character \( |\lambda| : T \to \mathbb{R}_+ \) satisfying \( |\lambda|(\mu(a)) = |a|^{\langle \lambda, \mu \rangle} \) for any \( \mu \in X_*(T), a \in F^* \). Moreover, any continuous character \( T \to \mathbb{R}_+ \) arises this way for a unique \( \lambda \in X^*(T) \otimes \mathbb{R} \). In particular, to any continuous character \( \chi : T \to \mathbb{C}^* \) we can attach \( \text{Re} \chi \in X^*(T) \otimes \mathbb{R} \) such that \( |\chi(t)| = |\text{Re} \chi(t)|(t) \) for all \( t \in T \).

We say that a function \( f : H \to \mathbb{C} \) is smooth if it is invariant under right translation by an open subgroup of \( H \). The space of smooth functions on \( H \) will be denoted by \( C^\infty(H) \) and the subspace of compactly supported smooth functions by \( \mathcal{S}(H) \). Both spaces are smooth representations of \( H \) (by right translation).

Throughout \( G \) will be a connected reductive group which is split over \( F \). Fix a Borel subgroup \( B \) of \( G \) and a maximal torus \( T_0 \) contained in \( B \), both defined over \( F \). Let \( U_0 \) be the unipotent radical of \( B \), so that \( B = T_0U_0 \). We choose a maximal compact \( K \) of \( G \) such that \( G = BK \).

Roots and weights. We set \( a_0^* = X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}, a_0 = X_*(T_0) \otimes \mathbb{R} \) with the canonical pairing. Let \( \Delta_0 \subseteq X^*(T_0) \subseteq a_0^* \) denote the set of simple roots of \( T_0 \) on \( U_0 \) and \( \Delta_0^\vee \subseteq X_*(T_0) \subseteq a_0 \) the set of simple co-roots. The canonical bijection between \( \Delta_0 \) and \( \Delta_0^\vee \) will be denoted by \( \alpha \mapsto \alpha^\vee \). For each \( \alpha \in \Delta_0 \) let \( U_\alpha \) be the one-parameter unipotent group on which \( T_0 \) acts by \( \alpha \). We extend \( H : T_0 \to X_*(T_0) \) to \( G \) by requiring that \( H \) is left-\( U_0 \) and right-\( K \) invariant.

Henceforth, \( P = MU \) will be a parabolic subgroup containing \( B, U \) its unipotent radical and \( M \) its Levi part containing \( T \). The simple roots \( \Delta_0^P \) of \( T_0 \) on \( U_0 \cap M \) is a subset of \( \Delta_0 \). The torus \( T_0^M = T_0 \cap M^{\text{der}} \) is a maximal (split) torus in \( M^{\text{der}} \) and \( Z(M) = \cap_{\chi \in \Delta_0^P} \ker \chi \). Setting \( T_M = Z(M)^0 \) we have \( M = C_G(T_M) \). Moreover, \( M^1 \cap T_M \) is finite and \( T_M^1 \) is of finite index in \( M \). We write \( a_M^* = X^*(T_M) \otimes \mathbb{R} \) and \( (a_0^M)^* = X^*(T_0^M) \otimes \mathbb{R} \). Let \( a_M \) and \( a_0^M \) be the dual spaces. The set \( \Delta_0^P \) is a basis for \( (a_0^M)^* \) and the restriction maps \( X^*(T_0) \to X^*(T_M), X^*(T_0) \to X^*(T_0^M) \) induce an isomorphism

\[
a_0^* = a_M^* \oplus (a_0^M)^*.
\]
Similarly, the restriction map $X^*(M) \to X^*(T_M)$ induces an isomorphism $a_1^* = X^*(M) \otimes \mathbb{R}$.

Let $\Delta_P \subseteq a_1^*$ be the “simple roots” of $T_M$ on $U$, i.e. the restrictions of the roots $\Delta_0 \setminus \Delta_0^P$ to $T_M$. Similarly, let $\Delta_V^P \subseteq a_M$ be the “simple co-roots” pertaining to $P$.

We let $\Delta_0$ be the dual basis of $\Delta_0^P$ in $(a_0^M)^*$. Then $\Delta_P = \Delta_0 \cap a_1^*$ is a dual basis of $\Delta_0^P$ in $(a_0^M)^* \cap (a_0^M)^*$. Similarly, let $\Delta_P^V$ be the dual basis of $\Delta_P$ in $a_1^*$. There is a canonical bijection between $\Delta_P$ and $\Delta_P^V$ which will be denoted by $\varpi \mapsto \varpi^\vee$.

**Remark 1.** Suppose that $\alpha \in \Delta_P$ is the restriction of $\beta \in \Delta_0 \setminus \Delta_0^P$. Then in $a_0^*$ we have $\alpha = \beta + \sum_{\gamma \in \Delta_0^P} m_\gamma \gamma$ where $m_\gamma \geq 0$ for all $\gamma \in \Delta_0^P$. Indeed let $\lambda = \alpha - \beta \in (a_0^M)^*$. Then for any $\gamma \in \Delta_0^P$

$$0 = \langle \alpha, \gamma^\vee \rangle = \langle \beta, \gamma^\vee \rangle + \langle \lambda, \gamma^\vee \rangle \leq \langle \lambda, \gamma^\vee \rangle .$$

Therefore, $\lambda$ is in the closure of the positive Weyl chamber of $(a_0^M)^*$, and in particular, in the closed obtuse chamber.

If $P_i \supseteq B$, $i = 1, 2$ we will denote by $P_1 \bullet P_2$ the parabolic subgroup generated by $P_1$ and $P_2$. Its Levi subgroup $M_1 \bullet M_2$ is the group generated by $M_1$ and $M_2$. We have $Z(M_1 \bullet M_2) = Z(M_1) \cap Z(M_2)$, $\Delta_0^{P_1 \bullet P_2} = \Delta_0^{P_1} \cup \Delta_0^{P_2}$, $\Delta_P^{P_1 \bullet P_2} = \Delta_P^{P_1} \cap \Delta_P^{P_2}$, $a_0^{P_1 \bullet P_2} = a_0^{P_1} \oplus a_0^{P_2}$ and $a_{P_1 \bullet P_2} = a_{P_1} \cap a_{P_2}$. Similarly for the dual spaces.

**Some elementary Lemmas.** Let $V$ be a representation space of a locally compact abelian group $A$. For any character $\chi$ of $A$ and $n \in \mathbb{N}$ we write

$$V_{\chi,n} = \{u \in V : (a_1 - \chi(a_1)) \ldots (a_n - \chi(a_n))u = 0 \text{ for all } a_1, \ldots, a_n \in A\}.$$

We also set $V_\chi = \cup_{n=0}^\infty V_{\chi,n}$ for the generalized $\chi$-eigenspace and $V_{A, \text{fin}} = \oplus_{\chi \in A} V_\chi$ for the subspace of $A$-finite vectors. In particular, for $V = \text{functions on } A$, we write $\mathcal{F}(A) = V_{A, \text{fin}}$ for the space of $A$-finite functions on $A$.

Let $\mathbb{G}_m$ denote the multiplicative group and $\mathbb{G}_a$ the additive group. For any parabolic subgroup $P$ we set $\mathbb{M}_P = \prod_{\alpha \in \Delta_0} \begin{cases} \mathbb{G}_m & \alpha \in \Delta_0^P \\ \mathbb{G}_a & \text{otherwise} \end{cases}$. In particular, $\mathbb{M}_G = \mathbb{G}_m^\Delta$. For any $P' \subseteq P$, $\mathbb{M}_P$ is an open subvariety of $\mathbb{M}_{P'}$. We view $\mathcal{S}(\mathbb{M}_P)$ as the subspace of $\mathcal{S}(\mathbb{M}_{P'})$ of those functions which are supported in $\mathbb{M}_P$, that is, vanish on $\mathbb{M}_{P'} \setminus \mathbb{M}_P$. Let $\tau : T_0 \to \mathbb{M}_G$ be the homomorphism defined by $\tau(t) = \alpha(t)$.

For any $P$ and $\chi \in \widehat{T}_M$ we denote by $\mathfrak{F}_{P,\chi} = \mathfrak{F}_{P,\chi}^G$ (resp. $\mathfrak{F}_{P,\chi,n}$) the space of functions on $T_0$ which are linear combinations of functions of the form $\xi(t)\varphi(\tau(t))$ where $\xi \in \mathcal{F}(T_0)_\chi$ (resp. $\xi \in \mathcal{F}(T_0)_{\chi,n}$) and $\varphi \in \mathcal{S}(\mathbb{M}_P)$.

For any $\varpi \in \Delta_0$ we write $T_{\varpi} = T_{P_{\varpi}}$ where $P_{\varpi}$ is the maximal parabolic subgroup corresponding to $\varpi$.

We have the following elementary properties.

**Lemma 1.**

1. For any $\chi \in \widehat{T}_G$, $\mathfrak{F}_{G,\chi}$ is the space of smooth functions on $T_0$ which are compactly supported modulo $T_G$ and have generalized eigenvalue $\chi$ under $T_G$.

2. Suppose that $P' \subseteq P$ and $\chi |_{T_M} = \chi$. Then $\mathfrak{F}_{P,\chi} \subseteq \mathfrak{F}_{P',\chi'}$. 


(3) Let $\varpi \in \hat{\Delta}_P$, $t_0 \in T_\varpi$ and $D_{t_0}\phi(t) = \phi(t_{t_0}) - \chi(t_0)\phi(t)$. Then for any $n > 0$ if $\phi \in \mathcal{F}_{P,\chi,n}$ then $D_{t_0}\phi \in \mathcal{F}_{P,\chi,n-1} + \mathcal{F}_{P',\chi|T_{M_{M'}}}$ where $P'$ is the parabolic subgroup containing $P$ as a co-rank one subgroup such that $\hat{\Delta}_{P'} = \hat{\Delta}_P \setminus \{\varpi\}$.

(4) Suppose that $\phi_i \in \mathcal{F}_{P,\chi_i}$, $i = 1, 2$. Then $\phi_1\phi_2 \in \mathcal{F}_{P \bullet P_2,\chi_1\chi_2|T_{M_1 \bullet M_2}}$.

**Proof.** The first part follows from the fact that the kernel of $\tau$ is $Z(G)$ and its image is an open subgroup of $(F^*)^\Delta_\varphi$.

For the second part, suppose that $\phi \in \mathcal{F}_{P,\chi}$ is of the form $\xi(t)\varphi(\tau(t))$ where $\xi \in \mathcal{F}(T_0)_{\chi_1}$ with $\chi_1 \in \widehat{\mathcal{M}_{M'}}$ extending $\chi$ and $\varphi \in \mathcal{S}(\mathcal{M}_P)$. Then $\tilde{\chi} := \chi'\chi_1^{-1}$ is trivial on $T_M$ and therefore we can write $\tilde{\chi}(t)^{-1} = \tilde{\varphi}(\tau(t))$ where $\tilde{\varphi}$ is smooth function on $\mathcal{M}_P$ depending only on the coordinates in $\Delta_\varphi$. Thus, we can write $\phi(t) = \xi'(t)\varphi'(\tau(t))$ where $\xi' = \xi\tilde{\chi} \in \mathcal{F}(T_0)_{\chi'}$ and $\varphi' = \varphi\tilde{\varphi} \in \mathcal{S}(\mathcal{M}_P') \subseteq \mathcal{S}(\mathcal{M}_P)$.

For the third part, suppose that $\phi(t) = \xi(t)\varphi(\tau(t))$ where $\xi \in \mathcal{F}(T_0)_\chi$ and $\varphi \in \mathcal{S}(\mathcal{M}_P)$. Then

$$D_{t_0}\phi(t) = [\xi(t) - \chi(t_0)\xi(t_{t_0})] \varphi(\tau(t)) - \chi(t_0)\xi(t_{t_0})[\varphi(\tau(t)\tau(t_0)) - \varphi(\tau(t))].$$

Since $\tau(t_0)$ is 1 in all coordinates except $\alpha$, we have $\varphi(-\tau(t_0)) - \varphi(\cdot) \in \mathcal{S}(\mathcal{M}_P')$.

Finally, the last part is immediate. 

We point out that in order to deal with general quasi-split groups (i.e., non-split $T_0$) we need to modify the definition of $\mathcal{F}_{P,\chi}$.

A sequence $a_n$, $n \in \mathbb{Z}$ of complex numbers is called **eventually polynomial exponential** if there exists a finite subset $\Lambda$ of $\mathbb{C}/\frac{2\pi i}{\log q} \mathbb{Z}$ and polynomials $P_\lambda$, $\lambda \in \Lambda$ such that $a_n = 0$ for $n \ll 0$ and

$$a_n = \sum_{\lambda \in \Lambda} P_\lambda(n)q^{\lambda n} \quad (1)$$

for $n \gg 0$. Clearly, $\Lambda$ is determined by $\{a_n\}$ if we assume that $P_\lambda$ is non-zero for all $\lambda \in \Lambda$. We call $\Lambda$ the set of exponents of the sequence $a_n$. Alternatively, $a_n$ is eventually polynomially exponential of the above form if and only if $a_n = 0$ for $n \ll 0$ and there exists $m \geq 0$ such that the sequence $\prod_{\lambda \in \Lambda}(T - q^\lambda)^{a_n}(a_n)$ has only finitely many non-zero terms where $T$ is the right shift operator on sequences. In particular, if $a_n = 0$ for $n \ll 0$ and there exists $h > 0$ and $\lambda \in \mathbb{C}/\frac{2\pi i}{\log q} \mathbb{Z}$ such that $b_n = a_{n+h} - q^\lambda a_n$ is eventually polynomial exponential then the same is true for $a_n$. Moreover, the exponents of $a_n$ are contained in the union of the exponents of $b_n$ and the set $\frac{\lambda + \frac{2\pi i}{\log q} j}{h}$, $0 \leq j < h$.

For any $\varpi \in \hat{\Delta}_0$ the group $T_GT_\varpi^1 \setminus T_\varpi \simeq X_*(T_G \setminus T_\varpi)$ is isomorphic to $\mathbb{Z}$. We fix an element $t_\varpi \in T_\varpi$ which lies above a generator of $T_GT_\varpi \setminus T_\varpi$. We also fix a generating set $\omega_1^\gamma, \ldots, \omega_d^\gamma$ for $X_*(T_G \setminus T_0)$.

**Lemma 2.** Let $\phi \in \mathcal{F}_{P,\chi}$ and suppose that $\phi$ is $T_G$-invariant. (In particular, $\chi|_{T_G} \equiv 1.)$ Then
The integral
\[
 I(\phi, \lambda) = \int_{T_G \setminus T_0} \phi(t) q^{-\langle \lambda, H(t) \rangle} \, dt
\]
converges in the cone
\[
 \{ \lambda \in a_{0, C}^* : \langle \text{Re} \lambda + \text{Re} \chi, \varpi \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_P \}
\]
and defines a rational function in \( q^{\langle \lambda, \omega_i \rangle} \), \( i = 1, \ldots, d \) with poles at most along the hyperplanes
\[
 q^{\langle \lambda, H(t_\varpi) \rangle} = \chi(t_\varpi)
\]
with \( \varpi \in \hat{\Delta}_P \) satisfying \( \chi|_{T^1_\varpi} \equiv 1 \).

Proof. The convergence of \( I(\phi, \lambda) \) reduces to the convergence of
\[
 \sum_{\mu \in X_*(T_M) : \langle \alpha, \mu \rangle < N} (1 + \|\mu\|^2)^k q^{\langle \text{Re} \chi + \text{Re} \lambda, \mu \rangle}
\]
for any \( N \) and \( k \). This in turn boils down to the convergence of
\[
 \sum_{n < N} (1 + n^2)^k q^{n\langle \text{Re} \chi + \text{Re} \lambda, \varpi \rangle}
\]
for all \( \varpi \in \hat{\Delta}_P \). Similarly, the integrand in the definition of \( a_n \) is compactly supported in the domain of integration. Therefore, \( a_n \) is well-defined and it is easy to see that \( a_n = 0 \) for \( n \ll 0 \).

Suppose that \( \phi \in \mathcal{F}_{P, \chi, m} \). We will prove the two statements by induction on \( m \) and \( r \) where \( r \) is the co-rank of \( P \). If \( m = 0 \) then \( \phi = 0 \) and there is nothing to prove. If \( r = 0 \) then \( P = G \) and \( \phi \in \mathcal{S}(T_G \setminus T_0) \). In this case \( a_n = 0 \) for almost all \( n \) and \( I(\phi, \lambda) \) reduces to a polynomial in \( q^{\pm \langle \lambda, \mu_i \rangle} \).

For the induction step, suppose that \( P \neq G \) and \( m > 0 \). For any \( \varpi \in \hat{\Delta}_P \) and \( t_0 \in T^1_\varpi \) we have
\[
 I(D_{t_0} \phi, \lambda) = (q^{\langle \lambda, H(t_0) \rangle} - \chi(t_0))I(\phi, \lambda)
\]
\[
 a_n(D_{t_0} \phi) = a_n - v(\omega(t_0))(\phi) - \chi(t_0) a_n(\phi)
\]
The Lemma follows from Lemma 1 and the induction hypothesis applied to \( D_{t_0} \phi \) by taking \( t_0 \in T^1_\varpi \) such that \( \chi(t_0) \neq 1 \) if \( \chi|_{T^1_\varpi} \neq 1 \) and \( t_0 = t_\varpi \) otherwise. \( \square \)

We will also need the following elementary result.
Lemma 3. Suppose that the sequence \( a_n, n \in \mathbb{Z} \) is eventually polynomial exponential and \( a_n \geq 0 \) for all \( n \). Let \( \Lambda \) and \( P_\lambda \) be as in (1). Suppose further that \( \Re \lambda \leq 0 \) for all \( \lambda \in \Lambda \). Then

1. The highest coefficient of \( P_0 \) is non-negative.
2. \( \deg P_\lambda \leq d := \deg P_0 \) for all \( \lambda \in \Lambda^* := \Lambda \cap i\mathbb{R}/2\pi i\mathbb{Z} \).
3. \( a_n = O(n^d) \).
4. The series
   \[
   f(s) = \sum_{n \in \mathbb{Z}} a_n q^{-ns}
   \]
   converges for \( \Re s > 0 \) and is a rational function in \( q^s \).
5. At \( s = 0 \), \( f \) has a pole of order \( r = d + 1 \) and we have
   \[
   \sum_{n \leq m} a_n \sim \lim_{s \to 0} \frac{(s \log q)^r f(s)}{r!} m^r \quad \text{as } m \to \infty
   \]

Proof. The Lemma is easy if \( P_\lambda \equiv 0 \) for all \( \lambda \in \Lambda^* \). Assume that this is not the case and let \( l = \max_{\lambda \in \Lambda^*} \deg P_\lambda \geq 0 \). Let \( h_\lambda \) be the coefficient of \( x^l \) in \( P_\lambda \). We have

\[
(2) \quad b_n := \sum_{\lambda \in \Lambda^*} h_\lambda q^{\lambda n} = c_n + d_n
\]

where \( c_n = a_n/n^l \geq 0 \) and \( d_n \to 0 \) as \( n \to \infty \). For \( m \geq 0 \) consider

\[
x_m = \sum_{n=0}^{m} b_n, \quad y_m = \sum_{n=0}^{m} |b_n|, \quad z_m = \sum_{n=0}^{m} |b_n|^2.
\]

By summing the geometric series \( |x_m - h_0m| \) is bounded. By (2) \( |x_m - y_m| = o(m) \). Therefore, \( |y_m - h_0m| = o(m) \). On the other hand,

\[
|b_n|^2 = \sum_{\lambda, \lambda' \in \Lambda^*} h_\lambda h_{\lambda'} q^{\lambda - \lambda'} n
\]

so that \( z_m - Hm \) is bounded where \( H = \sum_{\lambda \in \Lambda^*} |h_\lambda|^2 > 0 \). Since \( b_n \) is bounded, \( z_m \) is majorized by a constant multiple of \( y_m \). We conclude that \( h_0 > 0 \) so that \( \deg P_0 = l \). The first three parts of the Lemma follow immediately. The last two parts follows readily from the identities

\[
\sum_{n=0}^{\infty} \binom{n}{k} x^n = x^k (1 - x)^{-(k+1)}, \quad |x| < 1, \quad \sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1},
\]

and the fact that for any polynomial \( P \) and \( \lambda \in \mathbb{C} \setminus \frac{2\pi i}{\log q} \mathbb{Z} \) with \( \Re \lambda \leq 0 \) we have

\[
|\sum_{n=0}^{m} P(n)q^{\lambda n}| = \begin{cases} O(m^{\deg P}) & \text{Re } \lambda = 0 \\ O(1) & \text{Re } \lambda < 0 \end{cases}
\]

as \( m \to \infty \). \( \square \)
Germs. For any parabolic subgroup $P$ and $\epsilon > 0$ let
\[ M_{<\epsilon} = \{ m \in M : |\alpha|(m) < \epsilon \} \]
We set an equivalence relation on $C^\infty(M)$ by saying that $f_1$ and $f_2$ have the same germ (at 0) if there exists $\epsilon$ such that $f_1(m) = f_2(m)$ on $M_{<\epsilon}$. This equivalence relation clearly respects the action by $M$ and therefore the space $G(M)$ of equivalence classes is also a representation space of $M$. Similarly, we can define the set $(T_M)_{<\epsilon}$ and germs of functions on $T_M$.

**Lemma 4.** The map $f \mapsto [f]$ sending $f$ to its germ (i.e., its equivalence class) induces an isomorphism of $T_M$-modules
\[ \Gamma : \mathcal{F}(T_M) \rightarrow \mathcal{G}(T_M)_{T_M} \text{-fin} \]
\[ \Gamma_{\chi,n} : \mathcal{F}(T_M)_{\chi,n} \rightarrow \mathcal{G}(T_M)_{\chi,n} \]
for any $\chi \in \widehat{T}_M$ and $n$. Moreover, let $V$ be an open subgroup of $T_M^1$, $\chi \in \widehat{T}_M/V$, $n \in \mathbb{N}$, and $B$ a finite set of generators of $T_M/V$. Then there exists $\delta > 0$ with the following property. Suppose that $f \in C^\infty(T_M)$ is $V$-invariant and $\prod_{i=1}^r (R(b_i) - \chi(b_i))f(t) = 0$ for all $t \in (T_M)^\epsilon$ and $b_1, \ldots, b_n \in B$. Then $f(t) = \Lambda^{-1}([f])(t)$ for all $t \in (T_M)^{<\delta}$.

**Proof.** It is easy to see that $\mathcal{F}(T_M)_{\chi,n}$ is spanned by $\chi(t)Q(H(t))$ where $Q$ is a polynomial on $a_M$ of degree $< n$. Any such function is determined by its germ. Therefore, $\Gamma$ is injective. The surjectivity and the last part of the Lemma amount to saying that every multi-sequence $a_n$, $n \in \mathbb{Z}^r$ which satisfies difference equations whenever $\sum_{i=1}^r n_i > N$ will be polynomial exponential on the set $\sum_{i=1}^r n_i > N + c$ for some $c$. We omit the details. \(\square\)

**Corollary 1.** The map $f \mapsto [f]$ induces a bi-$M$-equivariant isomorphism
\[ \iota_M : C^\infty(M)_{T_M} \text{-fin} \rightarrow \mathcal{G}(M)_{T_M} \text{-fin} \]

**Proof.** Let $f \in C^\infty(M)$ and for each $m \in M$ let $f_m(t) = f_m(tm)$, $t \in T_M$. Suppose that $f$ is $T_M$-finite and $\iota_M(f) = 0$. Then for any $m \in M$, $f_m \in \mathcal{F}(T_M)$ and $\Gamma(f_m) = 0$. By the Lemma we conclude that $f_m \equiv 0$ and therefore $f \equiv 0$. Thus $\iota_M$ is injective.

To show surjectivity, suppose that $f \in C^\infty(M)$ and $[f] \in \mathcal{G}(M)_{\chi,n}$ for some $\chi \in \widehat{T}_M$ and $n$. Then $[f_m] \in \mathcal{G}(T_M)_{\chi,n}$ for all $m \in M$. Define $\tilde{f}(m) = \Gamma^{-1}([f_m])(1)$. It is clear that $\tilde{f} \in C^\infty(M)$. Since $\Gamma$ is $T_M$-equivariant, we have $\tilde{f}(tm) = \Gamma^{-1}([f_m])(t)$ for all $t \in T_M$. Therefore, $\tilde{f}_m \in \mathcal{F}(T_M)_{\chi,n}$ for all $m \in M$. Thus $\tilde{f} \in C^\infty(M)_{\chi,n}$. Finally, by the last part of the Lemma there exists $\epsilon > 0$ such that
\[ \tilde{f}(m) = f_m(1) = f(m) \]
for all $m \in M_{<\epsilon}$. It follows that $\iota_M(\tilde{f}) = [f]$ as required. \(\square\)

3. The main result

Let $\pi$ be a smooth representation of $G$. For any parabolic subgroup $P \supset B$ let $J_P(\pi)$ denote the Jacquet module of $\pi$ with respect to $P$, viewed as a smooth representation of $M$. Let $E_P(\pi)$ denote the set of cuspidal exponents of $\pi$ along $P$ i.e. those $\chi \in \widehat{T}_M$ such
that $J_P(\pi)_\chi$, the $\chi$-generalized eigenspace of $J_P(\pi)$, contains a supercuspidal constituent. Set $\mathcal{E}(\pi) = \cup_P \mathcal{E}_P(\pi)$. If $\pi$ is of finite length then $\mathcal{E}(\pi)$ is finite.

Fix a non-degenerated character $\psi : U_0 \to \mathbb{C}^*$ of $U_0$, that is $\psi|_{U_\alpha} \neq 1$ for all $\alpha \in \Delta_0$. Let $\Omega(G)$ be the $G$-space of smooth function $f : G \to \mathbb{C}$ such that $W(n g) = \psi(n) W(g)$ for all $n \in U_0$, $g \in G$, with $G$ acting by right translation.

Suppose that $\pi$ is an irreducible generic representation of $G$. That is, $\pi$ can be realized as a subspace $\mathcal{W}(\pi)$ of $\Omega(G)$. The space $\mathcal{W}(\pi)$ is uniquely determined by the equivalence class of $\pi$ and is called the Whittaker model of $\pi$. (Cf. [Sha74], [GK75], [BZ76], [Rod73])

**Theorem 1.** Let $(\pi, \mathcal{W}(\pi))$ be a subrepresentation of $\Omega(G)$ of finite length. Then any $W \in \mathcal{W}(\pi)$ can be written as

$$W(utk) = \psi(u) \sum_{P \supseteq B} \delta^\frac{1}{2}_{P}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}(t, k) \quad t \in T_0, u \in U_0, k \in K$$

where $\phi_{P,\chi}(\cdot, k) \in \mathfrak{F}_{P,\chi}$ for all $k \in K$ and $\phi_{P,\chi}$ is invariant under an open subgroup of $K$.

**Proof.** We will prove the Theorem by induction on the semi-simple rank of $G$, the case where $G$ is a torus being trivial. Of course, we are primarily interested in irreducible representations. However, we need the finite length assumption to make the induction work.

First note that by considering finitely many translates of $W$ it is enough to prove the statement for $k = 1$. Consider the map

$$W \in \mathcal{W}(\pi) \mapsto [\delta^{-\frac{1}{2}}_P W]|_M.$$ 

We claim that it factors through the Jacquet module and gives rise to an intertwining map

$$\kappa_M : J_P(\pi) \to \mathcal{G}(M).$$

Indeed, let $u \in U$. Then for $m \in M_{<\epsilon}$ for $\epsilon$ sufficiently small we have

$$W(mu) = W(mum^{-1}m) = \psi(mum^{-1}) W(m) = W(m)$$

so that the germs of $W$ and $\pi(u) W$ coincide. The equivariance property is clear (because of the $\rho$-shift in the definition of the Jacquet functor).

Since $J_P(\pi)$ is of finite length, it is finite under $T_M$. Therefore, we get an $M$-equivariant map

$$\Xi_M := \iota^{-1}_M \circ \kappa_M : J_P(\pi) \to C^\infty(M)_{T_M, \text{fin}}$$

where $\iota_M$ is as in Corollary 1. Clearly the image lies in the space $\Omega(M)$ of Whittaker functions on $M$.

Next, observe that $W(t) = 0$ if $|\alpha(t)| \gg 1$ for some $\alpha \in \Delta_0$. Indeed, fix $u \in U_\alpha$ so that $\psi(u) \neq 1$. By the property of $t$, $u' = t^{-1}ut$ is very close to 1, and therefore $W$ is right invariant under $u'$. Hence,

$$W(t) = W(tu') = W(ut) = \psi(u) W(t)$$

so that $W(t) = 0$ as required.
We are now ready for the induction step. By passing to a direct summand we may assume that $\mathcal{W}(\pi) = \mathcal{W}(\pi)_\mu$ for some $\mu \in \mathcal{T}_G$. For any $\emptyset \neq I \subset \Delta_0$ let $P_I = M_I U_I$ be the proper parabolic subgroup of $G$ such that $\Delta_0 \setminus \Delta_0^{P_I} = I$ and let $T_I = Z(M_I)^0$. Let $j_I : \pi \to J_{P_I}(\pi)$ be the canonical projection. By the transitivity of the Jacquet functor and the induction hypothesis applied to $J_{P_I}(\pi)$ we can write

$$[\Xi_{M_I} j_I(\mathcal{W})](t) = \sum_{P \subseteq P_I} \delta^{\frac{1}{2}}_{\mathcal{P}(M_I)}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}^I(t)$$

where $\phi_{P,\chi}^I(\cdot) \in \mathfrak{F}_{P,\chi}^{M_I}$. In other words, there exists $\epsilon > 0$ such that

$$W(t) = \sum_{P \subseteq P_I} \delta^{\frac{1}{2}}_{\mathcal{P}}(t) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}^I(t)$$

provided that $|\alpha|(t) < \epsilon$ for all $\alpha \in \Delta_P$. Note that both sides of (3) vanish if $|\alpha(t)|$ is large for some $\alpha \in \Delta_0^{P_I}$. Therefore, it follows from Remark 1 that for an appropriate $\epsilon > 0$, (3) holds whenever $|\alpha(t)| < \epsilon$ for all $\alpha \in I$. Fix such $\epsilon$ which works for all $I \neq \emptyset$. Set

$$\phi_{P,\chi}(t) = \sum_{\emptyset \neq I \subseteq \Delta_0 \setminus \Delta_0^{P_I}} (-1)^{|I|-1} \phi_{P,\chi}^I(t) \prod_{\alpha \in I} 1_{<\epsilon}(|\alpha(t)|)$$

where $1_{<\epsilon}$ denotes the characteristic function of $(0,\epsilon)$. Then $\phi_{P,\chi} \in \mathfrak{F}_{P,\chi}$ and by inclusion-exclusion

$$\sum_{P \subseteq G} \sum_{\chi \in \mathcal{E}_P(\pi)} \delta^{\frac{1}{2}}_{\mathcal{P}}(t) \phi_{P,\chi}(t) = \begin{cases} W(t) & |\alpha(t)| < \epsilon \text{ for some } \alpha \in \Delta_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q$ be such that $\mathcal{E}_Q(\pi) \neq \emptyset$ and take any $\omega \in \mathcal{E}_Q(\pi)$. Then $\omega|_{\mathcal{T}_G} = \mu$ and therefore

$$\phi'(t) := \delta^{-\frac{1}{2}}_{\mathcal{P}}(t) W(t) \prod_{\alpha \in \Delta_0} 1_{\geq\epsilon}(|\alpha(t)|) \in \mathfrak{F}_{G,\mu} \subseteq \mathfrak{F}_G$$

by Lemma 1. The conclusion of the Theorem holds upon replacing $\phi_{Q,\omega}$ by $\phi_{Q,\omega} + \phi'$. □

**Question.** What is the kernel of the maps $\kappa_M$?

Suppose that $\pi$ is irreducible, generic and has a unitary central character. For $W_1, W_2 \in \mathcal{W}(\pi)$ define

$$I(W_1, W_2, \lambda) = \int_{T_G U_0 \setminus G} q^{-\langle \lambda, H(g) \rangle} W_1(g) \overline{W_2(g)} \, dg.$$

**Corollary 2.** The integral $I(W_1, W_2, \lambda)$ is absolutely convergent whenever

$$\langle \text{Re } \lambda + 2 \text{ Re } \chi, \varpi' \rangle > 0$$

for all $P$, $\varpi' \in \hat{\Delta}_P$ and $\chi \in \mathcal{E}_P(\pi)$. It extends to a rational function in $q^{\langle \lambda, \omega' \rangle}$, $i = 1, \ldots, d$ with poles at most along

$$q^{\langle \lambda, H(t) \rangle} = \chi_1 \chi_2(t)$$
where $\chi_i \in \mathcal{E}_{P_i}(\pi)$, $i = 1, 2$ and $\varpi \in \hat{\Delta}_{P_1 \bullet P_2}$ satisfy $\chi_1 \equiv \chi_2$ on $T^1_{\varpi}$. Here $\omega^1, \ldots, \omega^d$ and $t_{\varpi}$ are as in Lemma 2.

Proof. The integrand of $I(W_1, W_2, \lambda)$ is evidently left $T_G U_0$-invariant. We use the integration formula
\[
\int_{T_G U_0 \setminus G} f(g) \, dg = \int_K \int_{T_G \setminus T_0} f(tk) \delta_B(t) \, dt \, dk
\]
for any left $T_G U_0$-invariant continuous function on $G$. The Corollary immediately follows from Theorem 1 and Lemma 2. \hfill \Box

**Corollary 3.** Suppose that $\pi$ is square-integrable and generic. Then the form
\[
(W_1, W_2) = \int_{T_G U_0 \setminus G} W_1(g) \overline{W_2(g)} \, dg
\]
defines a non-zero $G$-invariant inner product on $\mathcal{W}(\pi)$.

Proof. We only need to check convergence. This follows from Corollary 2 and the characterization of square-integrable representations by the positivity of their exponents. \hfill \Box

We conjecture the following converse to Corollary 3. It is related to the question above.

**Conjecture 1.** Let $\pi$ be a generic representation, and suppose that
\[
\int_{T_G U_0 \setminus G} \|W(g)\|^2 \, dg < \infty
\]
for any $W \in \mathcal{W}(\pi)$. (It is enough to require this for a single $0 \neq W \in \mathcal{W}(\pi)$.) Then $\pi$ is square-integrable.

Finally, we extend Corollary 3 to the tempered case.

**Corollary 4.** Suppose that $\pi$ is tempered. Fix $\omega = \sum_{\alpha \in \Delta_0} n_\alpha \alpha \in X^*(T_0)$ with $n_\alpha \geq 0$. Then there exists $r = r(\pi, \omega) \in \mathbb{N}$ such that for any $W_1, W_2 \in \mathcal{W}(\pi)$
\[
\int_{g \in T_G U_0 \setminus G : \langle \omega, H(g) \rangle \leq n} W_1(g) \overline{W_2(g)} \, dg \sim [W_1, W_2]^r \text{ as } n \to \infty
\]
where $[\cdot, \cdot]$ is a non-zero invariant inner product on $\mathcal{W}(\pi)$.

Proof. As before, by Corollary 2 $I(W_1, W_2, s\omega)$ is absolutely convergent for $\text{Re}(s) > 0$ and extends to a rational function in $q^s$. Let $r = r(W)$ be the order of the pole of $I(W, W, s\omega)$ at $s = 0$. Fix $W \in \mathcal{W}(\pi)$ and consider
\[
a_n = a_n(W) := \int_{U_0 T_G \setminus G} \delta_{\langle \omega, H(g) \rangle, n} |W(g)|^2 \, dg.
\]
By Theorem 1 and Lemma 2 the sequence $a_n$ satisfies the conditions of Lemma 3. Note that
\[
I(W, W, s\omega) = \sum a_n q^{-ns}
\]
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for $\Re s > 0$. By Lemma 3 we have

$$\int_{g \in T_G U_0 \setminus G: \langle \omega, H(g) \rangle \leq n} |W(g)|^2 \, dg \sim \lim_{s \to 0} \frac{(s \log q)^r I(W, W, s\omega)}{r!} n^r$$

as $n \to \infty$. Since $a_n(W_1 + W_2) \leq 2(a_n(W_1) + a_n(W_2))$ for all $W_1, W_2 \in \mathcal{W}(\pi)$ and $n \in \mathbb{Z}$, it also follows from Lemma 3 that $r(W_1 + W_2) \leq \max(r(W_1), r(W_2))$.

Let $x \in G$ and let $W_x(g) = W(gx)$. Since $H(g) - H(gx)$ is bounded independently of $g$, there exists $C$ such that

$$\left| \int_{T_G U_0 \setminus G} 1_{\leq n}(\langle \omega, H(g) \rangle) |W_x(g)|^2 \, dg - \int_{T_G U_0 \setminus G} 1_{\leq n}(\langle \omega, H(g) \rangle) |W(g)|^2 \, dg \right|$$

$$\leq \sum_{|m-n| \leq C} a_m.$$

By Lemma 3 the right-hand side is $O(n^{r(W)-1})$ as $n \to \infty$. It follows once again from Lemma 3 that $r(W_x) = r(W)$ and

$$\lim_{s \to 0} s^r I(W_x, W_x, s\omega) = \lim_{s \to 0} s^r I(W, W, s\omega).$$

By irreducibility, $r = r(W)$ is independent of $W \neq 0$. The Corollary now follows by polarization. \hfill \Box

Example. Consider $G = PGL_2$. Let $\pi$ be the unramified tempered representation

$$\text{Ind}_B^G \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mapsto |a|^\lambda \right), \quad \lambda \in \mathbb{R}.$$

Then the unramified Whittaker function is given by

$$|a|^{-\frac{1}{2}} W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} \frac{1-q^{-2\lambda-1}}{1-q^{-\lambda}} |a|^\lambda + \frac{1-q^{2\lambda+1}}{1-q^{\lambda}} |a|^{-\lambda} & |a| \leq 1 \text{ and } \lambda \notin \frac{2\pi i}{\log q} \mathbb{Z}, \\
(1-q^{-1})(1+v(a)) & |a| \leq 1 \text{ and } \lambda \in \frac{2\pi i}{\log q} \mathbb{Z}, \\
0 & |a| > 1. \end{cases}$$

It follows that for any positive $\omega$, $r = 1$ if $\lambda \notin \frac{2\pi i}{\log q} \mathbb{Z}$ and $r = 3$ otherwise.

In general, for any split group $G$ and a regular unitary unramified character $\chi$ of $T_0$, $r(\text{Ind}_B^G \chi, \omega)$ is the semi-simple rank of $G$ (regardless of $\omega$). This follows readily from the Casselman-Shalika formula for the unramified Whittaker function [CS80].

One can contemplate the following Conjecture related to Conjecture 1.

Conjecture 2. Suppose that $\pi$ is the generic constituent of $\text{Ind}_P^G \tau$ where $\tau$ is a square-integrable generic representation of a Levi part of $P$. Suppose that the Plancherel measure on the component

$$\text{Ind}_P^G \tau q^{(\lambda, H(\cdot))}, \quad \lambda \in i\mathbb{A}_p^*/\frac{2\pi i}{\log q} X^*(M)$$
is given by $\mu(\tau, \lambda) \, d\lambda$ where $\mu(\tau, \lambda) = |c_\tau(\lambda)|^{-2}$. Then $r(\pi, \omega)$ is equal to the sum of the co-rank of $P$ plus the order of zero of $\mu(\tau, s\omega)$ at $s = 0$.

Remark 2. Let $\tilde{G}$ be the metaplectic cover of $Sp_{2n}$. One can define parabolic subgroups as the inverse images of parabolic subgroups of $Sp_{2n}$. The Jacquet functors are defined in an analogous way and satisfy the usual properties. In particular, they control the asymptotics of the matrix coefficients of the representation and the criterion for square-integrability is the same as for linear groups. Details will appear in a forthcoming paper of Szpruch. The notions of non-degenerate characters and generic representations are also defined and the uniqueness of Whittaker model is proved in [Szp07]. The results of this paper as well as the proofs immediately carry over to $\tilde{G}$.

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