Problem 1. Evaluate the following limit

1. \[ \lim_{x \to 0} \frac{\sin 7x}{\sin 5x}. \]
   \[ = \left( \lim_{x \to 0} \frac{7}{5} \cdot \frac{5x \sin 7x}{7x \sin 5x} = \frac{7}{5} \right) \cdot \left( \lim_{x \to 0} \frac{\sin 7x}{7x} \cdot \lim_{x \to 0} \frac{5x}{5x \sin 5x} = \frac{7}{5} \right) \]

2. \[ \lim_{x \to 0} \frac{2x - 1}{x}. \]
   \[ = (2^x)'|_{x=0} = \ln 2 \]

3. \[ \lim_{x \to 0} \frac{\cos 7x - 1}{\tan x}. \]
   \[ = \lim_{x \to 0} \frac{-2 \sin^2 \left( \frac{7x}{2} \right)}{\tan x} = \frac{-2}{\tan x} \cdot \left( \frac{7}{2} \right)^2 \cdot \frac{1}{x} \cdot \left( \frac{\sin \left( \frac{7x}{2} \right)}{\frac{7x}{2}} \right)^2 \cdot \frac{x}{\tan x} = 0 \]

4. \[ \lim_{x \to 0} \frac{\cos x}{x}. \]
   \[ = \pm \infty \]

Problem 2. Evaluate the derivative \( f'(x) \) of the following functions

1. \( e^{2x}(x-2)^2. \)
   solution. \[ f'(x) = 2e^{2x}(x-2)^2 + 2e^{2x}(x-2)(x-1). \]

2. \( \sin(\tan^{-1}(\ln x)). \)
   solution. \[ f'(x) = \cos(\tan^{-1}(\ln x)) \cdot \frac{1}{1 + (\ln x)^2} \cdot \frac{1}{x}. \]

3. \( \frac{(2x + 3)^{2/3}(x - 5)^{2/5}}{(x + 1)^{7/4}(x - 1)^{1/2}}. \)
   solution. Using logarithm differentiation. We first notice that
   \[ \ln f(x) = \frac{2}{3} \ln(2x + 3) + \frac{2}{5} \ln(x - 5) - \frac{7}{4} \ln(x + 1) - \frac{1}{2} \ln(x - 1) \]
   Differentiate both side we obtain that
   \[ f'(x) = f(x) \cdot \left( \frac{4}{3(2x + 3)} + \frac{2}{5(x - 5)} - \frac{7}{4(x + 1)} - \frac{1}{2(x - 1)} \right) \]
   \[ = \frac{(2x + 3)^{2/3}(x - 5)^{2/5}}{(x + 1)^{7/4}(x - 1)^{1/2}} \cdot \left( \frac{4}{3(2x + 3)} + \frac{2}{5(x - 5)} - \frac{7}{4(x + 1)} - \frac{1}{2(x - 1)} \right) \]
Problem 3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$e^{2y} + 2x = y$.

**solution.** Differentiating both sides of the above equation, we obtain $2y'e^{2y} + 2 = y'$. Hence

$$y' = \frac{2}{1 - 2e^{2y}} \quad (1)$$

Next differentiating (1), we have

$$y'' = \frac{8y'e^{2y}}{(1 - 2e^{2y})^2} = \frac{16e^{2y}}{(1 - 2e^{2y})^3}$$

Problem 4. Consider the functions

$f(x) = 2x^6 - 15x^4 + 24x^2$ on $[-2, 2]$.

1. Find the critical points of $f$ on the given interval.
2. Determine if they are local maximum, local minimum or neither.
3. Determine the absolute extreme values of $f$ on the given interval when they exist.

**solution.** $f'(x) = 12x^5 - 60x^3 + 48x = 12x(x^4 - 5x^2 + 4) = 12x(x^2 - 4)(x^2 - 1) = 12x(x + 1)(x - 1)(x - 2)(x + 2)$. So the critical points are $x = 0, \pm 1, \pm 2$.

By 1st derivative test, $x = 0, \pm 2$ are local minimum and $x = \pm 1$ are local maximum.

Notice that $f(\pm 2) = -16$, $f(\pm 1) = 11$ and $f(0) = 0$. So absolute minimum are $\pm 2$ and absolute maximum are $\pm 1$. □

Problem 5. A normal line at a point $P$ on a curve defined by the equation

$$\sqrt{x} + \sqrt{y} = 2$$

passes through $P = (1, 1)$ and is perpendicular to the line tangent to the curve at $P$. Determine an equation of the normal line at the given point.

1. Verify that the given $P$ lies on the curve.
2. Determine an equation of the normal line of the curve at the given point.

**solution.** First, by implicit differentiation, we obtain that

$$\frac{1}{3}x^{-2/3} + \frac{3}{4}y^{-1/4}y' = 0.$$  

That is,

$$\frac{1}{3} + \frac{3}{4}y'(1) = 0,$$

hence $y'(1) = -4/9$. Then the slope for the normal line is $-1/y'(1) = 9/4$ and an equation for the normal line is

$$y - 1 = \frac{9}{4}(x - 1).$$  □
Problem 6. Consider the following functions on the given interval. Find the inverse function, express it as a function of $x$, and find the derivative of the inverse function.

$$f(x) = \frac{x}{x + 5}.$$ 

solution.

$$y = \frac{5x}{x + 1},$$

implies that

$$x = \frac{y}{1 - y}.$$ 

So

$$f^{-1}(x) = \frac{5x}{1 - x}$$

and

$$(f^{-1}(x))' = \frac{5 - 5x + 5x}{(1 - x)^2} = \frac{5}{(1 - x)^2}.$$ 

□

Problem 7. The following limits represent the derivative of a function $f$ at a point $a$. Find a possible $f$ and $a$, and then evaluate the limit.

$$\lim_{x \to 5} \frac{\tan(\pi \sqrt{3x - 11})}{x - 5}$$

solution. Let $f(x) = \tan(\pi \sqrt{3x - 11})$

$$\lim_{x \to 5} \frac{\tan(\pi \sqrt{3x - 11})}{x - 5} = \lim_{x \to 5} \frac{\tan(\pi \sqrt{3x - 11}) - 0}{x - 5} = \lim_{x \to 5} \frac{\tan(\pi \sqrt{3x - 11}) - \tan(\pi \sqrt{3 \cdot 5 - 11})}{x - 5} = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = f'(5)$$

Now

$$f'(x) = \sec^2(\pi \sqrt{3x - 11}) \cdot \frac{3\pi}{2\sqrt{3x - 11}}$$

and

$$f'(5) = \frac{3\pi}{4}.$$ 

□

Problem 8. A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with the maximum area? What is the area?.

solution. Let us assume the coordinate of the four vertices of the rectangle is given by $(-x, y), (x, y), (-x, 0), (x, 0)$ with the vertices $(\pm x, y)$ is on the upper semicircle of radius 5. Then the height of the rectangle is $y$ and $(\pm x, y)$ and $x^2 + y^2 = 25$.

We want to maximise the area $A = 2xy = 2x\sqrt{25 - x^2}$. To do that, we solve

$$A'(x) = 2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25 - x^2}} = 0$$

which implies that $x^2 = 25/2$ that is, $x = y = 5/\sqrt{2}$ and area is 25. 

□
Problem 9 (points). Let \( f(x) = 2x^3 - 15x^2 + 24x \)

1. Is \( f(x) \) even, odd or neither? Explain why.

\textit{solution.} It is neither.

2. Find \( f'(x) \) and \( f''(x) \).

\textit{solution.} \( f'(x) = 6(x - 4)(x - 1) \) and \( f''(x) = 6(2x - 5) \).

3. Find critical points and possible inflection points.

\textit{solution.} Critical point \( x = 1, 4 \) and inflection point \( x = 5/2 \) since \( f''(x) \) changes sign at \( 5/2 \).

4. Find the intervals on which the function is increasing/decreasing and concave up/down.

\textit{solution.} \( f(x) \) concave down on \((-\infty, 5/2]\) and concave up on \([5/2, +\infty)\).

5. Identifying local maximum, local minimum and inflection points.

\textit{solution.} By 2nd derivative test, we have \( x = 1 \) is local maximum since \( f''(1) = -18 < 0 \) and \( x = 4 \) is local maximum since \( f''(4) = 18 > 0 \).

6. Find \( x \)-intercept and \( y \)-intercept of the graph of \( f(x) \).

\textit{solution.} \( f(x) = x(2x^2 - 15x + 24) \) then the roots for \( f(x) = 0 \) is given by \( x = 0 \) and \( 15 \pm \sqrt{33} \).

7. Graph the function \( f(x) \).