Problem 1. (18 points) Mark each of the following TRUE or FALSE (no explanation is required)

1. Every subgroup of an Abelian group is Abelian.
2. Every element of a group generates a cyclic subgroup of the group.
3. Symmetric group $S_{10}$ has 10 elements.
4. $S_n$ is not cyclic for any $n$.
5. Every one-to-one function from a set to itself is a permutation.
6. $A_4$ has 12 elements.
Problem 2. (20 points) Find all subgroups of the given group and calculate the order of each subgroup.

1. $\mathbb{Z}_{12}$.

2. $\mathbb{Z}_{18}$. 
Problem 3. (20 points) Let $S_n$ denote the symmetric group that are permutations of \{1,2,\cdots,n\}.

1. Express the permutation

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix} \in S_6$$

as a product of disjoint cycles, and then as a product of transpositions.

Solution. (1,2)(3,4,5,6).

2. Compute the product of cycles $(1,4,5)(2,5,7) \in S_7$.

Solution. (1,4,5,7,2).
Problem 4. (20 points) Let $A$ be a finite set, $B \subset A$ be a subset, and let $b \in B$ be one particular element of $B$. Determine whether the given set is a subgroup of $S_A$ under induced operation.

1. $\{ \sigma \in S_A \mid \sigma(b) \in B \}$.

   Solution. This set in general is NOT a subgroup. Here is an example, say

   $A = \{a, b, c\}$

   and

   $B = \{a, b\} \subset A$.

   Now we define \( \sigma(b) = a, \sigma(a) = c \) and \( \sigma(c) = b \), then it is clear that

   \[ \sigma \in \{ \sigma \in S_A \mid \sigma(b) \in B \} \]

   but \( \sigma^{-1}(b) = c \notin B \). Hence \( \sigma^{-1} \notin \{ \sigma \in S_A \mid \sigma(b) \in B \} \).

2. $\{ \sigma \in S_A \mid \sigma(b) = b \}$.

   Proof. To show this is a group, all we need if \( \sigma, \tau \in S_A \) satisfying \( \sigma(b) = b = \tau(b) \) then \( \sigma \circ \tau^{-1}(b) = b \) but this follows from the fact that \( \tau \) is one-to-one.
Problem 5. (14 points) Prove the following about $S_n$.

1. Prove that $(1, 2, 3) \in S_3$ can be written as a product of 2 transposition.

   \[ (1, 2, 3) = (1, 3)(1, 2) \] see page 90. In particular, we have $(a_1, a_2, a_3) = (a_1, a_3)(a_1, a_2)$. \hfill \Box

2. Prove that every permutation in $S_3$ can be written as a product of at most 2 transposition.

   Proof. Every permutation in $S_n$ can be written as disjoint cycles. For $S_3$, there only three cases:
   
   • one disjoint cycle $(a_1, a_2, a_3)$, a cycle.
   • $(a_1, a_2)(a_3)$, a transposition.
   • $(1)(2)(3)$, the identity.

   This combines with part one prove the claim. \hfill \Box

3. Prove that every nontrivial permutation $\sigma \in S_3$ that is not a cycle must be a transpositions.

   Proof. It follows from the proof above that it must be the second possiblity which is a tranposition. \hfill \Box

4. Prove that every permutation in $S_n$ that is not a cycle can be written as a product of at most $n - 2$ transpositions.

   Proof. Any $\sigma \in S_n$ we can write it as a product of disjoint cycles,

   \[ \sigma = (a_1, \ldots, a_{n_1})(a_{n_1+1}, \ldots, a_{n_2}) \cdots (a_{n_{r-1}+1}, \ldots, a_{n_r}) \]

   with $n_r \leq n$. Now $(a_1, \ldots, a_{n_1}) = (a_1, a_{n_1}) \cdots (a_1, a_2)$, i.e. it can be written as $n_1 - 1$ transpositions. So if we do the same to the rest of the cycles above all we need are at most

   \[ n_1 - 1 + (n_2 - n_1 - 1) + \cdots + n_r - n_{r-1} - 1 = n_r - r \leq n - r \]

   transpositions. By our assumption $\sigma$ is not a cycle, we must have $r \geq 2$. So we have $n - r \leq n - 2$. The proof is thus completed. \hfill \Box
Problem 6. (8 points) Show that for every subgroup $H$ of $S_n$ for $n \geq 2$, either all the permutations in $H$ are even or exactly half of them are even.

Proof. Let $H < S_n$ be any subgroup. We define $\bar{H} := \{ \sigma \in H \mid \sigma \text{ is even} \}$. We claim that $\bar{H}$ is a subgroup of $H$. To see that, let $f, g \in \bar{H}$, since $g$ are even, so is $g^{-1}$. Since the product of even permutations are still even, we have $f \circ g^{-1}$ is even.

So there are only two possibilities: either $\bar{H} = H$ or $\bar{H} \leq H$. So we need to show if $\bar{H} \neq H$ then $|\bar{H}| = |H|/2$. To see that, we notice $\bar{H} \neq H$ implies that there exists at least one odd permutation $\sigma \in H$. Now consider the function $f : \bar{H} \to H \setminus \bar{H}$ defined by $f(h) = \sigma \cdot h$ for any $h \in \bar{H}$. Then $\sigma \cdot h$ is odd hence $\sigma \cdot h \in H \setminus \bar{H}$. Also it is easy to see that $f$ is 1-1 and onto, since $f^{-1} : H \setminus \bar{H} \to \bar{H}$ is given by $f^{-1}(h') = \sigma^{-1} \cdot h'$ for every $h' \in H \setminus \bar{H}$, from which we conclude that $|\bar{H}| = |H \setminus \bar{H}|$, hence $|\bar{H}| = |H|/2$. \qed